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Large N_c in Nonperturbative QCD

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ABSTRACT

We comment on large N_c within the context of nonperturbative QCD. In particular, we consider the QCD-based bag model, the instanton gas approximation, and the η' mass.

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I. INTRODUCTION

Considering the number of colors to be large in QCD[1] may be a reasonable approximation to the ordinary QCD ($N_c = 3$). The theory simplifies in this limit and can be interesting on its own right. Recently, the interest in the large N_c was greatly stimulated by the Witten's papers[2] which put forward some new ideas concerning the role of instantons, η' mass, etc.

In this note we will present some observations concerning large N_c within the framework of (phenomenological) nonperturbative QCD. By the latter we understand attempts[3] to relate observable quantities such as resonance masses and widths to vacuum expectation value of various operators, e.g.,

$$<\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}> \quad , \quad <0|\bar{q}q|0>$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, and q is a light quark field. These expectation values vanish by definition in the perturbation theory but seem to be important in the real world.[4]

Naive counting for the vacuum expectation value gives

$$<0|\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0> \sim N_c$$

and we confirm this guess by means of the QCD sum rules. As the next step, we will apply this relation to several problems. In particular, we consider

a) QCD-based bag model and

b) validity of the instanton approximation.

We will comment also on

c) vanishing η' mass in the $N_c \to \infty$ limit and

d) role of the "e^{-N}" terms.

II. MATRIX ELEMENTS $<0|\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0>$

The gluon matrix element, as already was mentioned, plays the central role. In perturbation theory,

$$\alpha_s(\rho) \sim \frac{2\pi}{\frac{N_c}{3}\, 11 \ln \Lambda\rho} \quad , \quad G^a_{\mu\nu} G^a_{\mu\nu} \sim N^2 - 1 \quad , \quad \alpha_s G^a_{\mu\nu} G^a_{\mu\nu} \sim N$$

so that the naive N dependence is known. However, one may worry whether such considerations are justified since $<\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}>$, as defined here, vanishes in perturbation theory. It might be worth deriving the same result in an independent way.

The matrix elements in question enter through the so-called QCD sum rules.[3] As an example, write down sum rules for the e^+e^- annihilation into hadrons with total isotopic spin I = 1:

$$\left(\frac{N_c}{2} M^2\right)^{-1} \int \exp\left(\frac{-s}{M^2}\right) s\sigma(e^+e^- \to \text{hadrons}, I = 1) ds =$$

$$= 1 + \frac{(N_c^2 - 1)3}{8N_c} \frac{\alpha_s(M)}{\pi} + \frac{1}{3}\pi^2 \frac{<0|\frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0>}{N_c \cdot M^4} + O\left(\frac{1}{M^6}\right) \quad . \quad (1)$$

Once the parameter M^2 approaches m_ρ^2 the l.h.s. varies strongly because of the presence of the peak in the physical cross section. The variation of the r.h.s. is due to the power corrections and clearly enough, $m_\rho^2 \sim \sqrt{<\alpha G^2>/N}$.* The detailed analysis (for $N_c = 3$, of course) confirms the guess. If the mass spectrum remains stable for $N_c \to \infty$ then

$$<\frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}> \sim \frac{N_c}{3}\, 0.01\ \text{GeV}^4 \quad (2)$$

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*For the sake of brevity we simplify a bit the actual case.

where the numerical value for $N_c = 3$ is known from the phenomenological analysis.[3]

III. THE QCD BAG MODEL

The existence of the vacuum gluon condensate, i.e., nonvanishing $\langle\alpha_s G^2\rangle$, can be considered the origin of the bag model (for a review of the model see Ref. 5). The point is that quark color field inside hadrons most probably suppresses instantons and other nonperturbative fluctuations. Indeed, the probability to find a fluctuation is proportional to $\exp(-S_{cl}/g^2)$ and the effective coupling constant in an external color field cannot be too large, barring in this way large scale fluctuations present in the vacuum.

The idea was suggested independently by several authors[3,6] and elaborated in more detail in Ref. 6a. We will be interested in the N_c behavior and for this limited purpose the approach of Ref. 3 turns out to be most convenient.

If the quark field inside hadron suppresses the nonperturbative fluctuations more or less entirely then the difference between the energy density in the physical vacuum outside the hadron and "perturbative" vacuum inside is proportional to $\langle 0|\alpha_s G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle$ (it follows immediately from the expression for the trace of the energy-momentum tensor in QCD, $\theta_{\mu\mu} \sim \alpha_s G^2$ and from the obvious relation $\langle 0|\theta_{\mu\mu}|0\rangle = 4\,\varepsilon_{vac}$).[4] On the other hand, just the same difference in the energy density is given by the bag constant B. In this way one comes to the estimate:

$$B \simeq -\frac{b}{32}\langle 0|\frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle \qquad (3)$$

(b is the coefficient in the Gell-Mann-Low function). It is worth emphasizing that the estimate is based on the assumption that inside hadrons the nonperturbative fluctuations are highly suppressed.

Now we are in a position to judge whether this assumption is consistent with the large N_c limit. The conclusion is that it is certainly not. Indeed, from Eq. (3) we would conclude

$$B \sim N_c^2$$

and that would imply that both the masses and the wave functions are unstable against increase in N_c ($m \sim N_c^{1/2}$, $|\Psi(0)|^2 \sim N^{3/2}$).

Therefore, if one believes in a smooth large N_c limit it is more appropriate to assume that the quark color fields only slightly modify the vacuum fluctuations inside the hadrons.

If one compares directly the prediction for $N_c = 3$ with the value of the B extracted from the bag models fits, then it seems certain that the B given by the phenomenological model is lower than the "QCD bag value" determined by Eq. (3). The difference amounts to a factor of ten to twenty. However, the phenomenological value of B varies from one fit to another and is sensitive to hadron radius. Moreover, the bag model in its present form does not account in full for the spontaneous breaking of the chiral symmetry while the phenomenological value of $\langle \alpha G^2 \rangle$ does. Thus, there is some inherent uncertainty which is difficult to resolve. Consideration of large N_c favors the picture of a hadron as a shallow structure on the vacuum energy sea.

IV. VALIDITY OF INSTANTON DILUTE GAS APPROXIMATION

The simplest nonperturbative excitation is described by the well-known instanton solution. It has the smallest classical action and therefore is expected to dominate—over other nonperturbative effects—at the short distances.

For large N the instanton density was found first in Ref. 7:

$$d_o(\rho) \sim \left(\text{const. } g^{-4}(\rho) \exp(-8\pi^2/g^2(\rho))\right)^{N_c} \quad . \tag{4}$$

Therefore, it vanishes exponentially if ρ is small enough. Moreover, naively one would expect that corrections to this earliest manifestation of the nonperturbative effects would vanish as e^{-N}. It is this latter expectation that certainly fails.

Consider instanton of a small size ρ. It interacts with large-scale vacuum fluctuations which are responsible for the non-vanishing $\langle\bar{q}q\rangle$, $\langle\alpha_s G^2\rangle$. This interaction puts limitations on the use of the bare instanton density. The effect was calculated first in Ref. 8 for the realistic case of $N_c = 3$. Generalizing the result obtained to the case of arbitrary N_c we find

$$d_{eff}(\rho) \sim d_o(\rho) \left(\rho^3 \frac{\langle\bar{q}q\rangle}{N}\right)^3 \exp\left\{\frac{\pi^4\rho^4}{2(N^2-1)\alpha_s(\rho)} \langle 0|\frac{\alpha_s}{\pi} G^2|0\rangle\right\} \tag{5}$$

(three massless quarks). This relation holds so long as the correction due to $\langle\alpha G^2\rangle \neq 0$ is small compared to the original instanton density classical action, $2\pi/\alpha_s$. From this condition we determine a bound on ρ:

$$\rho < \rho_{crit} \sim \left[\frac{2(N^2-1)\alpha_s(\rho)}{\pi^3 \langle 0|\frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle}\right]^{1/4} \tag{6}$$

which is clearly N independent. Thus, interaction with external vacuum fields becomes important when the instanton density is exponentially small.

Equation (5) cannot be relied upon once the correction exceeds the main term. It might be worth mentioning, however, that stimulation of small size fluctuations by the vacuum Euclidean fluctuating fields is of very general nature and can well persist beyond the ρ_{crit} specified by Eq. (6). Therefore, one cannot rule out that even in the limit $N_c \to \infty$ the effective density of nonperturbative fluctuations as a function of their size ρ remains independent of N_c.

V. MASS OF η'

As is emphasized by Witten the η' mass vanishes in the $N_c \to \infty$ limit. Nonperturbative QCD shares this conclusion (see, however, the next section). Indeed in the large N_c limit only one-loop terms survive and in this approximation there is no difference between the π and η' channels. Thus, there is nothing new in this respect. Nonperturbative QCD provides more precise estimate of the smallness of the η' mass, however.

First, a few words on the η' residue. In large N_c limit it coincides (up to a Clebsch-Gordon coefficient) with f_π:

$$<0|\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s|\eta'> \equiv if_{\eta'}\kappa_\mu$$

$$f_{\eta'} = \sqrt{N}f_\pi\,(N_c \to \infty) \quad <0|\frac{3\alpha_s}{4\pi}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}|\eta'> = f_{\eta'}m^2_{\eta'} \quad . \tag{7}$$

Now we can judge experimentally on the validity of this relation. Indeed, the radiative decay $J/\psi \to \eta'\gamma$ measures the coupling of η' to the $\alpha_s G\tilde{G}$ current. An analysis of the sum rules provides[9] the damping factor ~ 0.5 for the residue comparing to the large N_c limit (7). This conclusion is of limited numerical accuracy, however, and it is better to turn to the experimental data. Theoretical estimates for the ratio $\Gamma(J/\psi \to \eta'\gamma)/\Gamma(J/\psi \to \eta\gamma)$ would give approximately 3 while experimentally it is around 5. Thus $f_{\eta'} \sim \sqrt{5/3}\, 0.5(f_\pi\sqrt{3})$. Therefore, we conclude that the large N_c result is valid for the residue at least within the factor of 2.

The conclusion on the vanishing of the η' mass appears disturbing at first sight since it is difficult to imagine that 1 GeV as a hadronic mass is small by any standard. Still, there is a natural "standard" for the η' mass and experimentally it is small in its natural scale.

By "natural"scale we understand here the value of M^2 for which asymptotic freedom breaks down (an example of the sum rules is given in Eq. (1)). To clarify the point, consider sum rules in the vector (ρ) an axial-vector (π, A_1) channels. Then in the vector channel, the power corrections to the asymptotic freedom become important at $M^2 \sim m_\rho^2$ (m_ρ^2 is just a number here related to $\langle\alpha_s G^2\rangle$, $\langle\bar{q}q\rangle$ in a well-defined way). Mass of the ρ, m_ρ, is not small in this scale and sum rules are quite sensitive to the ρ mass. Thus, the mass is of the order one in the natural scale. On the other hand, mass of the pion–the lightest in the axial-vector channel–is small in this scale (for axial vector-channel the power corrections are of the same order as for the ρ channel). One cannot determine the m_π^2 by means of the sum rules since the scale is larger ($m_\rho^2 \gg m_\pi^2$). (Of course, in this particular case the presence of nearly massless pion can be determined from general grounds alone.)

It is amusing to observe that the $m_{\eta'}^2$ is, indeed, numerically small compared to its natural scale. The sum rules for the current $\alpha G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a$ (where the η' dominates) are given by[9]

$$T = i\int dx \exp(iqx) \langle 0 \,|T\{\partial_\mu a_\mu(x), \partial_\nu a_\nu(0)\}|\, 0 \rangle$$

$$\frac{1}{\pi}\int \frac{ds}{s} \exp\left(-\frac{S}{M^2}\right) \mathrm{Im}\,T(s) \simeq 10^{-3} M^4 \left[1 + \left(\frac{1.3 \text{ GeV}}{M}\right)^4 + O\left(\frac{1}{M^6}\right)\right] \tag{8}$$

(we give the numbers for $N_c = 3$). It is seen that the power correction to asymptotic freedom becomes 20% at $M^2 \sim 3.5$ GeV^2 while for the ρ channel the same happens at $M^2 \simeq m_\rho^2$. We see that the natural scale for the η' mass is much larger than for the ρ mass.

In other words, η' is dual to the bare graph smeared over $M^2 \sim 6$ GeV^2 and $m^2_{\eta'}$ is small as compared to this duality interval. As a result, the QCD sum rules are insensitive to the exact value of $m^2_{\eta'}$ which is small in this respect.

The difference between the quark (ρ like) (and gluon (η'-like) currents) in their duality intervals was noticed first in Ref. 9. We see that large N_c provide an explanation to this observation.

VI. "e^{-N}" CONTRIBUTIONS

In conclusion we would like to make a few remarks on the role of instantons and other nonperturbative contributions in the large N_c limit. First, one must differentiate between "large" and "small" size nonperturbative fluctuations. If one considers the polarization operator at some Q^2 then by large scale we understand fluctuations whose scale size is independent of Q and is of order of the confinement radius while "small" size is of order (at least formally) 1/Q.

It is beyond any doubt that the large scale nonperturbative fluctuations do exist in the large N_c limit. They provide nonvanishing $\langle \alpha_s G^2 \rangle$ which is crucial at any N_c if the mass spectrum is not to change drastically with N_c. Conclusions concerning the role of the 1/Q instantons is less definite. Still there is good reason to believe that they are important in the resonance region ($Q \sim 1$ GeV) at least for pseudoscalar and scalar channels* and we will argue that there is no reason for their contribution to go away in this region with $N_c \to \infty$.

The typical contribution of small-size instantons to the polarization operator induced by some local current is given by (see, e.g., Ref. 3)

$$\Pi_{inst}(Q^2) \sim \int [K_{-1}(Q\rho)]^2 d(\rho) d\rho \tag{9}$$

where $K_{-1}(Q\rho)$ is the McDonald function and $d(\rho)$ is the density of instanton of size ρ, $d(\rho) \sim \rho^{b+9}$. Note that $K_{-1}(Q\rho) \sim \exp(-Q\rho)$ and provides a cut-off on the integration over the instanton size ρ.

Formally, expression (9) is proportional to a high inverse power of Q^2:

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*M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, unpublished.

$$\Pi_{inst}(Q^2) \sim [\rho d(\rho)]_{\rho \sim 1/Q} \sim (\frac{\Lambda}{Q})^{b+10} \tag{10}$$

(three massless quarks). Since one is inclined to believe that this contribution is negligible once $Q > \Lambda$ and N_c is large enough ($b \sim N_c$). Since 1 GeV is certainly larger than Λ, there is little doubt, at first sight, that there is no small-size instanton contribution to the η' mass, etc.

This conclusion fails, however, explicitly because N is large. The point is that actually the integral of the type is dominated by instantons of the size:

$$\rho_{dominant} \sim (b + 10)/Q \sim N_c/Q \quad . \tag{11}$$

We see that for N_c large and Q fixed $\rho_{dominant}$ becomes absurdly large. Clearly enough, there are no instantons of large size in the vacuum and the integral over "small-size instantons" is actually determined by infra-red cut-off. For the simplest cut-off it is just proportional to $\exp(-2\rho_{cut-off}Q)$. There is no reason to believe in this particular form of the cut-off, of course, but what is important is that the form of this function of Q does not actually depend on N_c and the conclusion that the integral is proportional to $(\Lambda/Q)^n$ is superfluous starting from Q_0, $Q_0 \to \infty$ with $N_c \to \infty$. To be more precise, there is always such Q^2 for which asymptotic form $\Pi_{inst} \sim (\Lambda/Q)^{N_c}$ holds, but for large N_c this region is pushed to such large Q^2 that they certainly have nothing to do with the resonance region and where the small instanton contribution is, by far, inferior to the ordinary quark loop contribution.

Thus, there is no general objections to $1/Q$ instantons to become important in the 1 GeV region and to remain so in the limit $N_c \to \infty$. Moreover, the instanton-like contribution does not necessarily obey the general conclusion on the N_c

dependence since extra factors α_s are eaten up by the instanton field, $A_\mu^{inst} \sim 1/g_s$. For example, amplitude $T(q^2)$ defined in Eq. (8) could have pieces which formally are of order $N_c\ e^{-N_c}$ but actually (at $Q \sim$ several GeV) are of order N. There is no indication, however, that such terms are important for $N_c = 3$ and the possibility can hopefully be disregarded.

To summarize, numerical analysis indicated earlier that contribution which is formally determined by fluctuations of small size $\sim 1/Q$ actually depends heavily on infra-red cut-off at $Q \sim 1$ GeV.[9] Large N_c provides a natural explanation to this observation.

CONCLUSION

Large N_c limit allows us to organize and explain in a simple way many numerical observations made in the process of the analysis of the QCD sum rules. In one particular case, that is of the QCD bag model, it allows to make a judgment which is otherwise obscure. There arises a possibility that the density of the instanton-like fluctuations does not vanish for fixed ρ and $N_c \to \infty$.

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