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Quantum Inverse Method for Two-dimensional Ice and Ferroelectric Lattice Models

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ABSTRACT

The quantum inverse scattering transform method previously developed for continuum field theories is applied to the exactly soluble symmetric six-vertex (ice or ferroelectric) lattice model. Operators analogous to those which appear in the quantum inverse treatment of the nonlinear Schrödinger and sine-Gordon equations are constructed on the lattice by forming strings of vertices contracted over horizontal arrows. From the commutation relations for these operators, exact formulas for the eigenstates and eigenvalues of the transfer matrix are obtained without making an explicit ansatz for the wave functions. These results illustrate the connection between the quantum inverse method and the transfer matrix formalism for lattice models.

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I. INTRODUCTION

The inverse scattering method was developed as a means of solving certain classical nonlinear field equations.[1] The possibility that this technique might be generalized to provide a method for solving quantum field theory was suggested by studies of the nonlinear Schrödinger equation.[2,3] In its classical form, this equation had been solved via the 2×2 matrix inverse problem of Zakharov and Shabat.[4] The quantum nonlinear Schrödinger equation (also known as the delta-function gas) had also been solved by the Bethe ansatz of Lieb and Liniger.[5] The connection between these two methods was established by constructing quantum operators analogous to the classical Jost functions and scattering data of the Zakharov-Shabat eigenvalue problem.[3,6-8] An operator $B(k)$ thus constructed was found to create the Bethe ansatz eigenstates of the delta-function gas. Recently, the quantum inverse method has been applied to the sine-Gordon equation[9] and shown to reproduce the results of the Bethe ansatz solution of the massive Thirring model.[10] The elegant formulation of this method by Faddeev, Skylanin, and Takhtajan[9] exhibits a striking connection with the transfer matrix formalism developed in the treatment of solvable lattice statistical models.[11] In this paper we explore this connection by applying the quantum inverse method to the ice and ferroelectric lattice models of Lieb and Baxter[12,13] which were originally solved by writing a Bethe ansatz for the eigenvectors of the transfer matrix.[14]

We find a very compact derivation of the known results by constructing operators on the lattice which are analogous to the A and B operators used in the quantum nonlinear Schrödinger[6-8] and sine-Gordon[9] equations. This formulation illustrates a profound connection between the 2×2 matrix structure of the inverse scattering eigenvalue problem used in continuum field theories, and the matrix structure represented by the horizontal arrows of the lattice theory. The vertical arrows are associated with the operators of the field theory. The transfer matrix T is related to the A operator, while the B operator creates eigenstates of T. The path-ordered exponential expression which describes solutions of the eigenvalue problem in the inverse method arises on the lattice as a string of vertices contracted over horizontal indices. It is remarkable that the inverse method, which originated in classical field theory, is so closely related (in its quantum field version) to the transfer matrix formalism for lattice models.

The general ice or ferroelectric model (symmetric six-vertex model) is constructed by placing arrows on the bonds of a square lattice in all possible ways which obey the "ice rule," i.e., that there are two arrows in and two arrows out at each vertex. It is a special case of the Baxter eight-vertex model[15] with Baxter's parameter $d=0$. This eliminates the two vertices with four arrows in or four arrows out. The symmetric model is then described by three vertex weights, a,b, and c in Baxter's notation. The elementary vertex can be written as

$$L(\alpha,\beta;\lambda,\mu) = \sum_{i=1}^{4} w_i \sigma^i_{\alpha\beta}\sigma^i_{\lambda\mu} \tag{1}$$

where σ^i, $i=1,2,3$, are Pauli matrices, $\sigma^4=1$, and the indices α,β and λ,μ refer to horizontal and vertical arrows, respectively. The parameters w_i are related to the vertex weights by

$$w_1 = w_2 = \tfrac{1}{2}c \quad , \tag{2a}$$

$$w_3 = \tfrac{1}{2}(a-b) \quad , \tag{2b}$$

$$w_4 = \tfrac{1}{2}(a+b) \quad . \tag{2c}$$

For our considerations, it is convenient to regard the vertex (1) as an explicit 2×2 matrix in the horizontal indices, each element of which is a spin operator in the space of vertical indices. Thus, we write

$$L_n = \begin{pmatrix} w_3\sigma^3_n + w_4\sigma^4_n & 2w_1\sigma^-_n \\ 2w_1\sigma^+_n & -w_3\sigma^3_n + w_4\sigma^4_n \end{pmatrix} \tag{3}$$

where $\sigma^{\pm} = \tfrac{1}{2}(\sigma^1 \pm i\sigma^2)$ and the subscript n indicates that the σ-matrices act on the vertical arrow at site n.

In the usual quantum inverse method for continuum field theories,[6-9] one considers solutions to a linear problem of the form

$$\left[\frac{\partial}{\partial x} + iQ\right]\psi = 0 \quad , \tag{4}$$

where ψ is a 2-component column vector, and Q(x) is a 2×2 matrix, each element of which is a function of the field (e.g., nonlinear Schrödinger or sine-Gordon) at the point x.

A solution to eq. (4) can be written as a path-ordered exponential,

$$\psi(y) = P \exp\left\{-i\int_x^y Q(x')dx'\right\}\psi(x) \quad . \tag{5}$$

The observation which leads to the present application of the inverse method is that the path-ordered exponential in (5) has a precise analog in the lattice theory. It is a string of elementary vertices formed by contracting on the horizontal arrows, i.e., by multiplying matrices of the form (3) along adjacent sites in a row.

For a lattice with N sites in a row, the quantities which correspond to the scattering data in the continuum inverse method are obtained by multiplying over the whole row, leaving the end arrows uncontracted,

$$\mathcal{T} = L_1 L_2 \ldots L_N \tag{6}$$

Henceforth, we will adopt Baxter's parametrization of the vertex weights[14] (specialized to the six-vertex case),

$$w_1 = w_2 = \rho \sin 2\eta \quad , \tag{7a}$$

$$w_3 = \rho \sin \eta \cos v \quad , \tag{7b}$$

$$w_4 = \rho \cos \eta \sin v \quad . \tag{7c}$$

For the discussion to follow, η is regarded as a real constant and v as a variable. (They are related to coupling constant and rapidity, respectively, in field theory.[10]) Without loss of generality, we can take the overall normalization $\rho=1$.

The elements of $\mathscr{T}$ given by (6) are the "scattering data" operators of the theory,

$$\mathscr{T}(v) = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix} . \qquad (8)$$

The transfer matrix is just the trace of (8),

$$T(v) = \mathrm{Tr}\,\mathscr{T}(v) = A(v) + D(v) \quad . \qquad (9)$$

Let us define the direct product of two matrices as follows:

$$M \otimes N = \begin{pmatrix} M_{11}N_{11} & M_{11}N_{12} & M_{12}N_{11} & M_{12}N_{12} \\ M_{11}N_{21} & M_{11}N_{22} & M_{12}N_{21} & M_{12}N_{22} \\ M_{21}N_{11} & M_{21}N_{12} & M_{22}N_{11} & M_{22}N_{12} \\ M_{21}N_{21} & M_{21}N_{22} & M_{22}N_{21} & M_{22}N_{22} \end{pmatrix} . \qquad (10)$$

Here, each element is a product of operators, and must be written in the specified order. As in other applications of the quantum inverse method,[6,9] we find that the direct products of two elementary vertices $L_n(v)$ and $L_n(v')$, taken in different order, are related by a similarity transformation,

$$L(v') \otimes L(v) = R\, L(v) \otimes L(v')\, R^{-1} \quad . \qquad (11)$$

Here R is a c-number matrix of the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad (12)$$

where

$$\alpha(v,v') = \frac{\sin(v-v')}{\sin(v-v'-2\eta)} \quad , \tag{13a}$$

$$\beta(v,v') = \frac{-\sin 2\eta}{\sin(v-v'-2\eta)} \quad . \tag{13b}$$

Eq. (11) may be verified by direct calculation. Formation of the direct products in eq. (11) may be visualized as the contraction of two vertices along a vertical arrow (represented by an operator product in field theory). The matrix R in eq. (12) is the same as one constructed for the sine-Gordon theory by Faddeev, et al. It is also the d=0 limit of a matrix constructed by Baxter, who used an equation of the form (11) in his derivation of commuting transfer matrices for the eight-vertex model.[14]

The fundamental relation (11) provides all the commutation relations needed to construct the eigenvectors of the transfer matrix and to calculate its eigenvalues. The scattering data matrix $\mathcal{T}(v)$, by its definition, eq. (6), satisfies a similar equation,

$$\mathcal{T}(v') \otimes \mathcal{T}(v) = R\left[\mathcal{T}(v) \otimes \mathcal{T}(v')\right]R^{-1} \tag{14}$$

which specifies the commutation relations among the operators A, B, C and D. Just as in the sine-Gordon case, eq. (14) leads to the following results:

$$\left[A(v),\ A(v')\right] = \left[B(v),\ B(v')\right] = 0 \quad , \tag{15a}$$

$$A(v)B(v') = \frac{1}{\alpha(v',v)}\, B(v')A(v) - \frac{\beta(v',v)}{\alpha(v',v)}\, B(v)A(v') \;, \tag{15b}$$

$$D(v)B(v') = \frac{1}{\alpha(v,v')}\, B(v')D(v) + \frac{\beta(v,v')}{\alpha(v,v')}\, B(v)D(v') \;, \tag{15c}$$

$$\left[A(v)+D(v), A(v')+D(v')\right] = 0 \quad . \tag{15d}$$

As in the usual Bethe ansatz formulation,[12,13] the eigenstates of the transfer matrix $T(v) = A(v) + D(v)$ are constructed upon one of the two direct product eigenstates, e.g., the state with all spins up,

$$|\Omega_0\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \ldots \otimes |\uparrow\rangle_N \tag{16}$$

Notice that L_n, eq. (3), when acting on a down spin at site n, becomes a triangular matrix,

$$L_n|\uparrow\rangle_n = \begin{pmatrix} \sin(v+\eta) & \sin 2\eta\ \sigma^- \\ 0 & \sin(v-\eta) \end{pmatrix} |\uparrow\rangle_n \quad . \tag{17}$$

From (16), (17), and (6), we conclude that $|\Omega_0\rangle$ is an eigenstate of $A(v)$ and $D(v)$ separately,

$$A(v)|\Omega_0\rangle = \left[\sin(v+\eta)\right]^N |\Omega_0\rangle \quad , \tag{18a}$$

$$D(v)|\Omega_0\rangle = \left[\sin(v-\eta)\right]^N |\Omega_0\rangle \quad . \tag{18b}$$

Eigenstates of $T(v)$ with n reversed arrows are constructed by applying operators $B(v_i)$, $i=1, \ldots, n$ (where $B(v)$ is defined by (3), (6), and (8)) to the state $|\Omega_0\rangle$,

$$|v_1,\ldots,v_n\rangle = \prod_{i=1}^{n} B(v_i)|\Omega_0\rangle \quad . \tag{19}$$

Conditions on the v_i's emerge in the course of verifying that (19) is an eigenstate of $T(v)$. Using the relations (18) and (15), the following result can be shown:

$$T(v)|v_1,\ldots,v_n\rangle = \Lambda(v\,;\,v_1,\ldots,v_n)|v_1,\ldots,v_n\rangle \quad , \tag{20}$$

where

$$\Lambda(v;v_1\ldots,v_n) = \left[\sin(v+\eta)\right]^N \prod_{i=1}^{n}\left[\frac{\sin(v-v_i-2\eta)}{\sin(v-v_i)}\right] - $$
$$+ \left[\sin(v-\eta)\right]^N \prod_{i=1}^{n}\left[\frac{\sin(v-v_i+2\eta)}{\sin(v-v_i)}\right] \quad . \tag{21}$$

To show (20), we write T(v) = A(v) + D(v) and commute A and D past all of the B operators in (19) using (15b) and (15c). When such a procedure is carried out, for example, on A(v), it produces 2^n terms. One of these terms comes entirely from the first term in (15b) and, along with the corresponding term from D(v), yields directly the right-hand side of (20) with the eigenvalue (21). The remaining terms involve states in which one of the v_i's is replaced by v, and these terms must be made to cancel if eq. (20) is to be satisfied. The first such term, where v_1 is replaced by v, is easily found to be

$$\frac{\beta(v,v_1)}{\alpha(v,v_1)}\left\{\left[\sin(v_1+\eta)\right]^N \prod_{\ell=2}^{n} \frac{1}{\alpha(v_\ell,v_1)} - \left[\sin(v_1-\eta)\right]^N \prod_{\ell=2}^{n} \frac{1}{\alpha(v_1,v_\ell)}\right\} |v,v_2\ldots,v_n> \quad . \tag{22}$$

Other terms involving the states in which v_j is replaced by v, with j>1, may also be calculated directly, but such a calculation is unnecessary, From the symmetry of the state (19), which follows from the second commutator in (15a), we see that each of the remaining terms may be obtained from (22) simply by interchanging v_1 and v_j . The requirement that all such terms vanish leads to the conditions

$$\left[\sin(v_j-\eta)\right]^N \prod_{\substack{\ell=1\\ \ell\neq j}}^{n} \left[\sin(v_j-v_\ell+2\eta)\right] = \left[\sin(v_j+\eta)\right]^N \prod_{\substack{\ell=1\\ \ell\neq j}}^{n} \left[\sin(v_j-v_\ell-2\eta)\right] \qquad (23)$$

Equations (21) and (23) are the familiar transfer matrix eigenvalues and periodic boundary conditions for the ice models.[12,13] Thus, we have constructed the eigenstates and eigenvalues of the transfer matrix by a method which is considerably more transparent than the original Bethe ansatz treatment and which clearly demonstrates the connection between soluble lattice models and the quantum inverse formalism.

From the examples of the nonlinear Schrödinger equation, the sine-Gordon/massive Thirring model, and the ice models discussed here, it is apparent that the quantum generalization of the classical inverse scattering technique provides an elegant formulation of exact results for soluble quantum field theories and lattice statistical models. Further refinement and extension of this method may provide additional insight into the nature of conservation laws and exact integrability in quantum field theory.

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