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WHY A SECOND-ORDER MAGNETIC OPTICAL ACHROMAT WORKS

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ABSTRACT

A design procedure for a charged particle optical system where all second order matrix elements can be made to vanish simultaneously has been described by Brown.[1] A proof is given that the system works as advertised.

I. Introduction

Design criteria for a second-order magnetic optical achromat have been described in a previous publication by Brown.[1] Such a system will provide a transverse phase space configuration at the final point which is a faithful reproduction of the beam at the entrance to the system. The transformation matrix for the system will be the identity matrix and all second order transverse terms will be identically zero.

The beam line described consists of a number of identical cells. Each cell contains focusing and defocusing quadrupole components and one or more dipoles to provide dispersion. Two sextupoles are inserted into each cell. One sextupole will couple more strongly to second order terms in the horizontal phase space, while the other will couple more strongly to vertical terms. Corresponding sextupoles in all cells have identical magnetic fields. Thus two independent sextupole strengths are sufficient to make all second order terms vanish.

The demonstration of the vanishing of all geometric second order terms is quite straightforward. It is a direct consequence of the symmetry inherent in the cellular structure of the beam line. The proof is provided in Brown's original paper. To show that all chromatic second order terms also vanish is more subtle. Here we provide that demonstration. The consequences on the transformation matrix of the cellular structure are first determined. We then derive the relationships among the various second order terms and show that with only two independent sextupole strengths they may all be made to vanish simultaneously. The second order matrix elements are then related to their driving terms and we explore in greater depth the relation among the various terms. Finally, a simple example is used to illustrate the mathematical formalism.

II. Cell Structure and the Transformation Matrix

We assume that the beam line is comprised of a number of identical cells such that the net transormation matrix to first order is the identity. We further assume that the number of cells is equal to or greater than four, and that at no intermediate point in the beam line is a first order transformation equal to the identity matrix realised.

The eigenvalues of the total transformation matrix of the beam line will then be just powers of those of a single cell. Since the eigenvalues of the total transformation matrix are unity, those of a single cell will lie on the unit circle in the complex plane. In either transverse plane, the transformation matrix for a single cell may be written as[2]

$$R = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix}$$

Here the parameters α, β, γ, and μ are used only to parametrize the transfer matrix, and no relation to any transmitted beam matrix is implied. Because of the assumptions made above, it follows that we must have $\sin\mu \neq 0$. It is also necessary that

$$\beta\gamma - \alpha^2 = 1$$

A beam line of n cells may now be equally well regarded as either a single cell followed by a beam line of n-1 cells or as a beam line of n-1 cells followed by a single cell. Since all cells are identical, the first cell has the same transformation matrix as the last n-1 cells. We identify a transformation matrix element by placing after it in parenthesis the number of cells it represents. We may then write the following relationships for the total first and second-order transformation matrices, indicated by the letters R and T respectively:

$$R(n) = R(1)R(n-1) = R(n-1)R(n) \tag{3}$$

$$T(n) = R(n-1)T(1) + T(n-1)R(1)R(1) \tag{4}$$

$$T(n) = R(1)T(n-1) + T(1)R(n-1)R(n-1) \tag{5}$$

Since the letter T indicates a second order matrix, we must define what we mean by matrix multiplication involving T. To do so, we give an example by rewriting equation (s) as:

$$T_{ijk}(n) = \sum_{\ell} R_{i\ell}(1)\ T_{\ell jk}(n-1) + \sum_{\ell m} T_{i\ell m}(1) R_{\ell j}(n-1) R_{mk}(n-1) \tag{6}$$

If we solve both (4) and (5) for T(n-1), equate the two results, and reformulate the new equation using (3), we derive

$$T(n) = R(1)T(n)R^{-1}(1)R^{-1}(1) \tag{7}$$

Thus the symmetry of the beam line requires certain algebraic relations among the second-order matrix elements.

III. Simultaneous Vanishing of Second Order Terms

We now insert two sextupoles in each cell and adjust their strengths so that two of the second order matrix elements are made to vanish, one in the horizontal plane and one in the vertical. We maintain the symmetry of the cells by giving identical excitations to corresponding sextupoles in different cells.

In maintaining the symmetry of the cells, all the second order geometric aberrations are kept identically equal to zero, as shown by Brown.[1] Therefore, we need consider only the consequences on the remaining chromatic terms. If we write out equation (7) for the chromatic terms, expressing the elements of the inverse R matrix in terms of those of the R matrix itself, we arrive at:

$$T_{116} = R_{11}(1)R_{22}(1)T_{116} - R_{11}(1)R_{21}(1)T_{126} + R_{12}(1)R_{22}(1)T_{216} - R_{12}(1)R_{21}(1)T_{226} \tag{8}$$

$$T_{126} = -R_{11}(1)R_{12}(1)T_{116} + R_{11}(1)R_{11}(1)T_{126} - R_{12}(1)R_{12}(1)T_{216} + R_{12}(1)R_{11}(1)T_{226} \tag{9}$$

$$T_{216} = R_{21}(1)R_{22}(1)T_{116} - R_{21}(1)R_{21}(1)T_{126} + R_{22}(1)R_{22}(1)T_{216} - R_{22}(1)R_{21}(1)T_{226} \tag{10}$$

$$T_{226} = -R_{21}(1)R_{12}(1)T_{116} + R_{21}(1)R_{11}(1)T_{126} - R_{22}(1)R_{12}(1)T_{216} + R_{22}(1)R_{11}(1)T_{226} \tag{11}$$

Because of Liouville's theorem, we have

$$R_{11}(1)R_{22}(1) - R_{12}(1)R_{21}(1) = 1 \tag{12}$$

so that equations (8) and (11) may be seen to be identical. Applying Liouville's theorem to the second order terms for the entire beam line, we can also derive

$$T_{116} + T_{226} = 0 \tag{13}$$

We choose the sextupole strength so that the term T_{126} is made to vanish, then from equations (8) - (13) we can derive

$$2\,R_{12}(1)R_{21}(1)T_{116} + R_{12}(1)R_{22}(1)T_{216} = 0 \tag{14}$$

$$2\,R_{21}(1)R_{22}(1)T_{116} + \left[R_{22}^{2}(1) - 1\right]T_{216} = 0 \tag{15}$$

We now have two simultaneous homogeneous linear equations for the terms T_{116} and T_{216}. The determinant of the pair of equations is given by

$$D = 2\left\{R_{12}(1)R_{21}(1)\left[R_{22}^{2}(1) - 1\right] - R_{12}(1)R_{21}(1)R_{22}^{2}(1)\right\} \tag{16}$$

$$= -\,2\,R_{12}(1)R_{21}(1)$$

In terms of the variables introduced in equation (1), we have

$$D = 2\beta\gamma\,\sin^2\mu$$

$$= 2(\alpha^2+1)\,\sin^2\mu \tag{17}$$

By the assumptions stated in Section II, this expression cannot equal zero. Therefore, if the term T_{126} is made to vanish, then the matrix elements T_{116}, T_{216}, and T_{226} will necessarily vanish simultaneously.

This same procedure may be used to show that if T_{346} is set to zero, then T_{336}, T_{436}, and T_{446} will also vanish. The only terms which remain unexamined are T_{166} and T_{226}. If we write out equation (6) for these terms, deleting all terms which are known to vanish we derive

$$T_{166} = R_{11}(1)\;T_{166} + R_{12}(1)\;T_{266} \tag{18}$$

$$T_{266} = R_{21}(1)\ T_{166} + R_{22}(1)\ T_{266} \tag{19}$$

The determinant of this pair of linear homogeneous equations is given by

$$\begin{aligned} D &= 2 - R_{11}(1) - R_{22}(1) \\ &= 2\ (1 - \cos\mu) \end{aligned} \tag{20}$$

which is also non-vanishing according to our assumptions.

We have now proved the remaining two second-order matrix elements to vanish. Therefore, all second-order matrix elements may be made to vanish by the adjustment of two parameters.

IV. Relation to Driving Terms

An alternate interpretation of equations (4) and (5) can be illuminating. Consider an augmented beam line of n+1 cells obtained by adding a single cell to the original beam line of n cells. The matrix T(n) in equation (4) represents the transformation first through a single cell then one through n-1 cells. The matrix T(n) in equation (5) represents the transformation from the end of the first cell to the end of the last cell of the augmented beam line. It is regarded as a transformation first through the n-1 cells numbered 2 through n, then through the single remaining cell. In this comparison the terms R(n-1) and T(n-1) represent a transformation through the same set of n-1 cells in equations (4) and (5). The two second order matrices representing n cells will have identical values for all elements because the two beam lines they describe are alike in all respects.

We may now define cosinelike, sinelike, and dispersion rays with the beginning of cell 2 as the point of origin. Henceforth they will be donoted by c^*, s^*, and d^*. In terms of the corresponding rays as defined from the beginning of the beam line we have

$$c^* = R_{22}(1)\, c - R_{21}(1)\, s \tag{21}$$

$$s^* = -R_{12}(1) c + R_{11}(1)\, s \tag{22}$$

$$d^* = d + \alpha c + \beta s \tag{23}$$

where

$$\alpha = R_{12}(1)\; R_{26}(1) - R_{22}(1)\; R_{16}(1) \tag{24}$$

$$\beta = R_{21}(1)\; R_{16}(1) - R_{11}(1)\; R_{21}(1) \tag{25}$$

We may also express the second order matrix elements in terms of their driving terms. For purposes of illustration we use the expressions for the high energy limit as given by Brown.[3] We include only those contributions from quadrupoles and sextupoles. In our configuration, the expressions for the horizontal chromatic terms are given by

$$T_{116} = 2\sum_j S_j c_x s_x d_x - \sum_q \frac{c_x s_x}{f_q} \tag{26}$$

$$T_{126} = 2\sum_j S_j s_x^2 d_x - \sum_q \frac{s_x^2}{f_q} \tag{27}$$

$$T_{216} = -2\sum_j S_j c_x^2 d_x + \sum_q \frac{c_x^2}{f_q} \tag{28}$$

$$T_{226} = -2\sum_j S_j c_x s_x d_x + \sum_q \frac{c_x s_x}{f_q} \tag{29}$$

$$T_{166} = \sum_j S_j s_x d_x^2 - \sum_q \frac{s_x d_x}{f_q} \tag{30}$$

$$T_{266} = - \sum_j S_j c_x d_x^{\,2} + \sum_q \frac{c_x d_x}{f_q} \qquad (31)$$

Here the S_j indicate the normalized sextupole strengths

$$S = \left(\frac{B_o}{a^2}\right) \left(\frac{1}{B\rho}\right) L \qquad (32)$$

where B_o is the pole-tip field, a is the aperature, L the length, and $B\rho$ is the rigidity at the beam, given by its momentum divided by its charge. The f_q are the focal lengths of the quadrupoles where

$$\frac{1}{f_q} = \left(\frac{B_o}{a}\right) \left(\frac{1}{B\rho}\right) L \qquad (33)$$

The sums may be made either over the first n cells using the characteristic rays c , s, and d or over cells 2 through n + 1 using the rays c^*, s^*, and d^*. Since the behavior of the characteristic rays is completely determined by first-order considerations, they will have identical values in cells 1 and n + 1. Therefore, in the expressions for the T matrix elements in terms of the starred functions, we may replace the contributions from cell n + 1 by equivalent contributions from cell 1. The aberrations are then given by identical sums through the same beam line of expressions using either starred or unstarred characteristic rays.

Since the starred rays are linear combinations of the unstarred rays, the T matrix terms must then be linear combinations of each other. From this point of view we could again derive equations (8) - (11), (14) and (15), and once again prove the properties of simultaneous vanishing of all the terms.

However, greater insight is obtained by considering an illustrative example. We select one in which the linear relationships among the T terms are particularly simple. As our illustration, we take the first example from Brown's paper. His beam line consists of four identical cells each of which is symmetric front to back.

The diagonal terms in the transfer matrix for one cell are then equal to zero. Therefore, after one cell, the sinelike and cosinelike rays simply exchange roles, except for some scale factors. We now have

$$c^* = R_{12}(1)\ s \tag{34}$$

$$s^* = -R_{12}(1)\, c \tag{35}$$

$$d^* = d + \alpha c + \beta s \tag{36}$$

If we write, for example, the expression for T_{126} in terms of c^*, s^*, and d^*, we may then by substitution derive a new expression in terms of c, s, and d. Part of this new expression can be recognized as a constant times T_{216}. Specifically, we have

$$T_{126} = -\,R_{12}{}^2(1) T_{126} + \alpha R_{12}{}^2(1) \sum_j S_j c^{\ 2} s \ + \beta R_{12}{}^2(1) \sum_j S_j c^{\ 3} \tag{37}$$

The two last terms arise from the transformation of the dispersion ray. The expressions for the geometric terms T_{111} and T_{211} are given by

$$T_{111} = \sum_j S_j c_x{}^2 s_x \tag{38}$$

$$T_{211} = -\sum_j S_j c_x{}^3 \tag{39}$$

These are but two of the geometric terms, all of which vanish identically due to the symmetry of the beam. Therefore, we have

$$T_{126} = - R_{12}^{\,2}(1)\; T_{216} \tag{40}$$

so that if one of these terms vanishes, the other must also. We may also show that

$$T_{116} = T_{226} \tag{41}$$

But, by inspection we have

$$T_{116} = - T_{226} \tag{42}$$

Therefore, these must also both vanish identically. Similarly, we can demonstrate that the terms T_{166} and T_{266} must transform into each other and therefore vanish. Also, if one of the vertical second order terms T_{336}, T_{346}, T_{436}, and T_{446} vanishes, the other three must vanish also.

V. Summary

In any system with the required symmetry, the characteristic rays, i.e., the sinelike, cosinelike, and dispersion rays, are equivalent to linear combinations of themselves. Therefore, the second-order terms are also equivalent to linear combinations of themselves. If one horizontal term vanishes, they they must all necessarily vanish. Similarly, if one term in the vertical plane vanishes, all vertical terms will necessarily vanish. Therefore, two independent sextupole strengths are sufficient to make all second order terms vanish.

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