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1/N Expansion and the Theory of Composite Particles

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ABSTRACT

Dynamical structures of nonlinear spinor and bosonic theories are studied in the framework of the 1/N expansion. It is shown that a wide class of four-fermion theories contains composite particles and that they can be cast into equivalent field theories with the so-called compositeness condition. With the formation of composite particles, the ultraviolet behavior of a large class of four-fermion theories is improved so that they become renormalizable and well-defined field theories beyond two (but less than four) dimensions in spite of their apparent nonrenormalizability in the conventional perturbation expansions.

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I. INTRODUCTION

Nonlinear spinor theories, as considered by Nambu and Jona-Lasinio,[1] have various composite particles or collective excitations of fundamental spinor fields. The chiral symmetric model of Nambu and Jona-Lasinio, which is based on an analogy with superconductivity, was originally proposed as a dynamical model of nucleons and pions. In this model pions are massless composite bosons associated with dynamical spontaneous breakdown of the chiral symmetry. Following the same idea, Bjorken[2] and others[3,4] studied the possibility that quantum electrodynamics can be constructed out of a four-fermion vector interaction.

All composite particles are created as a result of interactions. The existence of composite particles is not manifest at the level of fundamental Lagrangians. This fact makes the study of the dynamical structure of theories involving composite particles difficult.[5] Fortunately, there is a useful and practical treatment of composite particles, which has been developed in connection with the study of the compositeness criteria of particles and which makes the dynamics of composite particles manifest at the Lagrangian level: Earlier works of Jouvet,[6] Salam[7], Weinberg[8] and others[9] showed that composite particles of certain field theories can be described in terms of equivalent field theories whose Lagrangians involve elementary fields corresponding to the above composite particles. In these equivalent theories some physical parameters are constrained so that the wave-function and vertex renormalization constants associated with these elementary fields vanish; this constraint $Z = 0$ (the conpositeness condition) effectively turns the elementary particles into composite particles. This equivalence relation, however, has so far been studied only in certain (Hartree-Fock type) approximations[10-12] since standard perturbation expansions (developed in powers of the coupling constant) fail to produce bound states to any finite orders.

Recently, the $1/N$ expansion scheme[13] has been extensively applied to the study of phase transitions and some other nonperturbative aspects[14-16] of various field theories. For nonlinear spinor theories, in particular, the $1/N$ expansion begins with a summation of bubble diagrams and is expected to preserve richer nonlinear features of the exact theory than standard perturbation expansions. Furthermore, as noted by Parisi[17] and Gross[18], in the $1/N$ expansion some nonlinear spinor theories become renormalizable by power-counting beyond two dimensions where they are nonrenormalizable in standard perturbation expansions. It is therefore of interest to study the equivalence relation based on the compositeness condition in the framework of the $1/N$ expansion.

The purpose of this paper is to show that a wide class of nonlinear spinor (and bosonic) theories can be converted into equivalent composite-particle theories to each order in the $1/N$ expansion. The limit $Z \rightarrow 0$

in equivalent field theories serves to regularize the short-distance structure of original nonlinear field theories. This limit affects the renormalization counterterms, but not the renormalized Green's functions when the original theories are renormalizable. It will be shown that in the $1/N$ expansion scheme the particle spectrum and the renormalizability of nonlinear field theories can be studied by the construction of equivalent composite-particle theories. In the framework of the $1/N$ expansion many of four-fermion theories turn out to be renormalizable and well-defined field theories under four dimensions.

In Sec. II we consider, as an illustration, a simple O(N)-symmetric four-fermion theory and show that it is equivalent to a Yukawa theory with a composite boson in the $1/N$ expansion scheme. In Sec. III we extend a similar analysis to other interactions: Four-fermion scalar or pseudoscalar interactions (both Abelian and non-Abelian) as well as Abelian four-fermion vector interactions produce bound states and become renormalizable under four dimensions within the $1/N$ expansion scheme. Non-Abelian versions of vector four-fermion theories turn out to be equivalent to massive Yang-Mills theories with composite Yang-Mills fields in two dimensions in the $1/N$ expansion. A bosonic theory (the CP^{N-1} model[19]) is also briefly discussed. Section IV is devoted to concluding remarks.

II. FOUR-FERMION INTERACTIONS AND EQUIVALENT YUKAWA INTERACTIONS

In this section we show the equivalence of a four-fermion theory and a Yukawa theory via the compositeness condition in the $1/N$ expansion scheme.

Let us consider a set of N fermion fields $\psi(x) = \{\psi^a(x)\}$ $(a = 1, \ldots, N)$ self-coupled through the O(N)-symmetric interaction:

$$\mathscr{L}_F[\psi, \overline{\psi}] = \overline{\psi}^a (i \not\partial - M) \psi^a + \frac{1}{2} G (\overline{\psi}^a \psi^a)^2 , \tag{2.1}$$

where summation over a $(a = 1, \ldots, N)$ is understood. The Gross-Neveu model[14] corresponds to the two-dimensional version of the above model with $M = 0$. In the conventional perturbation theory, we develop perturbation series expansions in powers of the four-fermion coupling constant G. On the other hand, in the $1/N$ expansion scheme, we regard G to be of order $1/N$ and arrange perturbation series expansions in powers of $1/N$. For $G > 0$ the force between a fermion and an antifermion is attractive, and is therefore expected to give rise to a bound state composed of a fermion-antifermion pair. As is well-known in the Gross-Neveu model[14] and as we shall see later, there in fact arises an O(N)-singlet scalar bound-state within the $1/N$ expansion.

In general, the composite operator $\overline{\psi}^a \psi^a$ develops a vacuum expectation value $\langle \overline{\psi}^a \psi^a \rangle_0$, which can simply be absorbed into the fermion bare mass $M \to M - G\langle\bar{\psi}^a\psi^a\rangle_0$. Correspondingly, for convenience (though not essential[20]), we shall henceforth regard $\bar{\psi}^a\psi^a$ in Eq. (2.1) as the normal ordered product $:\bar{\psi}^a\psi^a: = \bar{\psi}^a\psi^a - \langle\bar{\psi}^a\psi^a\rangle_0$, though not indicated explicitly.

In order to generate the $1/N$ expansion in a systematic way, let us introduce an O(N)-singlet scalar field $\sigma(x)$ and convert the Lagrangian (2.1) into an equivalent form[14]

$$\begin{aligned}\mathscr{L}_{YF}[\psi,\bar{\psi};\sigma] &= \mathscr{L}_F[\psi,\bar{\psi}] - \frac{1}{2}\{m\sigma - (g/m)\bar{\psi}^a\psi^a\}^2 \\ &= \bar{\psi}^a(i\not{\partial} - M + g\sigma)\psi^a - \frac{1}{2}m^2\sigma^2 \,, \end{aligned} \tag{2.2}$$

where G has been rewritten as $G = g^2/m^2$. For the $1/N$ expansion g and m are regarded to be of order $(1/N)^{1/2}$ and $(1/N)^0$, respectively. We note that the $1/N$ expansion for this model is generated successively by power-series expansions in g. The simplest way to see this is to contract ψ and $\bar{\psi}$ in (2.2). We then obtain the action expressed in terms of the σ field alone

$$S[\sigma] = -iN\,\mathrm{Tr}\,\ell n(i\not{\partial} - M + g\sigma) - \int dx\left(\frac{1}{2}m^2\sigma^2\right), \tag{2.3}$$

where the logarithmic term is to be expanded in powers of $g\sigma$. [Since we regard $\bar{\psi}^a\psi^a$ as $:\bar{\psi}^a\psi^a:$, the term linear in σ is missing in the expansion.] The σ^2 term is zeroth order in $1/N$ while σ^k terms

$(k \geq 3)$ are higher order in $1/N$. This structure implies that the successive perturbation expansion in g gives the successive $1/N$ expansion.

The four-fermion theory (2.2) is renormalizable in two dimensions within the standard perturbation expansion. In the $1/N$ expansion scheme it becomes renormalizable by power-counting[17,18] beyond two (but less than four) dimensions. On the other hand, the Yukawa theory based on the Lagrangian

$$\mathscr{L}_Y[\psi, \overline{\psi}; \sigma] = \overline{\psi}^a(i\not\partial - M + g\sigma)\psi^a - \frac{1}{2} m^2\sigma^2 + \frac{1}{2}(\partial_\mu \sigma)^2 , \tag{2.4}$$

is super-renormalizable under four dimensions. This theory looks quite identical to the four-fermion theory (2.2), except that the σ-field kinetic-energy is missing in the latter. Our task now is to relate this Yukawa theory to the four-fermion theory (2.2).

The renormalization program for the Yukawa theory (2.4) is well-known. Renormalized fields and parameters are defined by the renormalization transformation

$$\psi = (Z_\psi)^{1/2}\psi_R , \quad \sigma = (Z_\sigma)^{1/2}\sigma_R , \quad g = Z_g \, Z_\sigma^{-1/2} Z_\psi^{-1} \, g_R$$

$$MZ_\psi = M_R + \delta M, \quad m^2 Z_\sigma = m_R^2 + \delta m^2 \quad . \tag{2.5}$$

In terms of these renormalized quantities, the Lagrangian (2.4) is written as

$$\mathscr{L}_Y[\psi,\overline{\psi};\sigma] = \overline{\psi}^a_R \{Z_\psi(i\not\partial) - M_R - \delta M + Z_g\, g_R\, \sigma_R\}\, \psi^a_R$$

$$- \frac{1}{2}\,(m^2_R + \delta m^2)\,\sigma^2_R + \frac{1}{2}\, Z_\sigma\,(\partial_\mu \sigma_R)^2\ . \tag{2.6}$$

Under four dimensions only the mass counterterms δM and δm^2 are ultraviolet divergent while the renormalization constants Z_ψ, Z_σ and Z_g are convergent to each order in perturbation theory in g_R. As usual, they are determined according to some appropriate renormalization conditions. Typically, one can perform the mass and wave-function renormalizations of the σ field by adjusting δm^2 and Z_σ so that the renormalized inverse σ propagator $\Gamma(p^2)$ satisfies the normalization condition

$$\Gamma(p^2) = p^2 - m^2_R + O((p^2 - \mu^2)^2)\ , \tag{2.7}$$

around an arbitrary subtraction point $p^2 = \mu^2$. In general, $\Gamma(p^2)$ has the structure

$$\Gamma(p^2) = p^2 - m^2_R + \Pi(p^2) + (Z_\sigma - 1)\, p^2 - \delta m^2\ , \tag{2.8}$$

where $\Pi(p^2)$ stands for the quantum-loop corrections and $(Z_\sigma - 1)p^2 - \delta m^2$ is the renormalization counterterm. In terms of $\Pi(p^2)$ the renormalization condition (2.7) reads

$$\delta m^2 = (Z_\sigma - 1)\,\mu^2 + \Pi(\mu^2)\,,$$

$$Z_\sigma = 1 - (\partial/\partial\mu^2)\,\Pi(\mu^2)\,. \tag{2.9}$$

As in the foregoing example, this Yukawa theory also has a calculable $1/N$ expansion scheme if g is regarded to be of order $(1/N)^{1/2}$.

Analogously, for the four-fermion theory (2.2) we introduce the same renormalization transformation as (2.5). After this rescaling, the Lagrangian (2.2) is rewritten as (2.6) with the last term $\frac{1}{2} Z_\sigma (\partial_\mu \sigma_R)^2$ missing. For this theory we define renormalizations by subtraction procedures corresponding to the renormalization counterterms generated by the rescaling and employ the same renormalization conditions as for the Yukawa theory (2.4).

The Yukawa theory (2.4) contains three independent parameters g_R, m_R and M_R, apart from the degrees of freedom related to renormalizations, the subtraction point μ^2 and the ultraviolet cutoff Λ^2. (If we use the dimensional regularization,[21] Λ^2 is replaced by the space-time dimension n.) The four-fermion theory (2.1), on the other hand, contains only two independent parameters $G = g^2/m^2$ and M. Now suppose that the three independent parameters of the Yukawa theory are constrained so that

$$Z_\sigma(g_R, m_R, M_R; \mu^2, \Lambda^2) = 0 . \tag{2.10}$$

Under four dimensions, Z_σ is finite and independent of the ultraviolet cutoff Λ^2. In the renormalized Yukawa theory, g_R, m_R, M_R and $\sigma_R(x)$ are finite quantities. With the constraint $Z_\sigma = 0$ on renormalized parameters (the compositeness condition), the bare field $\sigma(x)$ vanishes while the bare parameters g and m become infinite. However, terms like $m^2\sigma^2 = (m_R^2 + \delta m^2)\sigma_R^2$ and $g\sigma = (Z_g/Z_\psi)\, g_R\sigma_R$ remain as they are and the renormalized Yukawa Lagrangian (2.6) turns into the four-fermion Lagrangian (2.2) re-expressed in terms of renormalized quantities. This argument for the equivalence of both theories is only heuristic. To prove this we shall first analyze the lowest order of the 1/N expansion and then go to higher orders.

To zeroth order in 1/N, the self-energy $\Pi(p^2)$ defined by (2.8) is calculated from a diagram shown in Fig. 1. In n dimensions[21] it is given by

$$\Pi(p^2) = N g_R^2 n_F (2\sqrt{\pi})^{-n}(n-1)\Gamma\left(1-\frac{1}{2}n\right)\int_0^1 dx \left[M_R^2 - x(1-x)p^2 - i\epsilon\right]^{\frac{1}{2}n-1} , \tag{2.11}$$

where n_F is defined by $Tr(\gamma_\mu \gamma_\nu) = n_F g_{\mu\nu}$; i.e. $n_F = k$ for k-component spinor fields. This $\Pi(p^2)$ is ultraviolet divergent for $n \geq 2$ and behaves like $(-p^2)^{\frac{1}{2}n-1}$ for $p^2 \to -\infty$. To the same order, Z_σ is given by

$$Z_\sigma = 1 - N g_R^2 C(M_R; \mu^2) , \qquad (2.12)$$

where

$$N g_R^2 C(M_R; \mu^2) = (\partial / \partial \mu^2) \Pi(\mu^2) .$$

In accordance with the Källén-Lehmann bound[22] $0 \leq Z_\sigma < 1$, $C(M_R; \mu^2) > 0$. Therefore the compositeness condition $Z_\sigma = 0$ is solvable for g_R to this order; i.e. $g_R^2 = 1/[N C(M_R; \mu^2)] > 0$. As verified easily, imposing the normalization condition (2.7) in the original four-fermion theory plays the same role as solving $Z_\sigma = 0$ in the equivalent Yukawa theory. This establishes the equivalence of both theories to lowest order in $1/N$.

The renormalized inverse σ propagator $\Gamma(p^2)$, defined by Eq. (2.8), grows like p^2 for $p^2 \to -\infty$ when $Z_\sigma \neq 0$. However, when $Z_\sigma = 0$, it behaves, for large p_μ, like $(-p^2)^{\frac{1}{2}n-1}$ for $2 \leq n < 4$ and like a constant $(\sim -m^2)$ for $n < 2$. Correspondingly, in higher orders in the $1/N$ expansion, the divergence structure of the Yukawa theory is different for $Z_\sigma = 0$ (the four-fermion theory (2.2)) and for $Z_\sigma \to 0$ (the Yukawa theory with the compositeness condition). For $Z_\sigma \neq 0$ the theory is super-renormalizable under four dimensions. On the other hand, when

$Z_\sigma = 0$, the theory (with the lowest-order σ propagator behaving asymptotically like $1/p^{n-2}$ for $2 \le n < 4$) is renormalizable[17,23] for $2 \le n < 4$ and is super-renormalizable for $1 \le n < 2$. Note that the limiting procedure $Z_\sigma \to 0$ serves to regularize ultraviolet divergences. In general, the limit $Z_\sigma \to 0$ and the renormalization procedure (or equivalently, performing Feynman integrals) do not commute in the presence of ultraviolet divergences. Under one dimension ($n < 1$) both the Yukawa theory ($Z_\sigma \neq 0$) and the four-fermion theory are finite theories (free of ultraviolet divergences); correspondingly, the Yukawa theory is not singular in the limit $Z_\sigma \to 0$ and reduces to the four-fermion theory. Suppose that, after renormalizations done under one dimension, we analytically continue both theories into higher dimensions. Then, as $Z_\sigma \to 0$, some renormalization counterterms of the Yukawa theory become singular for $1 \le n < 4$, because the corresponding counterterms of the four-fermion theory are ultraviolet divergent. [As $Z_\sigma \to 0$, Z_ψ, Z_g, δM and δm^2 become singular for $2 \le n < 4$ while only δM becomes singular for $1 \le n < 2$.] The renormalized Green's functions of the Yukawa theory, on the other hand, are not singular in the limit $Z_\sigma \to 0$ and are reduced to those of the four-fermion theory. This is because the latter theory is a renormalizable theory having convergent renormalized Green's functions. Hence, terms singular as $Z_\sigma \to 0$ are absorbed into the renormalization counterterms. It will be clear that going to sufficiently lower dimensions in the above argument is replaced by keeping the ultraviolet cutoff Λ^2 finite in the cutoff language.

We have already seen that $Z_\sigma = 0$ is solvable to zeroth order in $1/N$. It follows from this that the $Z_\sigma = 0$ condition is solvable to each order in $1/N$ since the higher-order corrections to Z_σ are power-series in $1/N$; hence g_R^2 is uniquely determined by a power-series in $1/N$. The solvability of the $Z_\sigma = 0$ condition, together with the above argument, shows that the Yukawa theory (2.4) becomes, via the compositeness condition $Z_\sigma = 0$, identical to the four-fermion theory under four dimensions $(n<4)$ to each order in the $1/N$ expansion scheme.

It will be clear from the above argument that, if a (renormalized) equivalent field theory is not singular in the compositeness limit $Z \to 0$, the original theory is renormalizable. The renormalizability of the original theory can thus be studied by the construction of its equivalent field theories. In most examples to be discussed in Sec. III, in particular, the coupling constant determined from the compositeness condition and the Green's functions are power-series in $1/N$. In such cases, the renormalizability of original theories is seen simply from the solvability of the compositeness condition.

The fact that $Z_\sigma = 0$ is solvable implies the existence of a bound state; in the original four-fermion theory, the propagator for the composite field $m\sigma \sim (g/m)\bar{\psi}^a \psi^a$ develops a single pole, or equivalently, the σ-field kinetic-energy term $\frac{1}{2}(\partial_\mu \sigma_R)^2$ is effectively created by the quantum fluctuations of fundamental fermions. According to our renormalization condition (2.7), the mass m_σ of this bound state is determined by setting $\mu^2 = m_R^2 = m_\sigma^2$ (i.e. on-mass-shell renormalization) in $Z_\sigma = 0$ [Eq. (2.10)]. This bound state is created to zeroth order in $1/N$. As is explicit in the equivalent Yukawa theory, the remaining higher-order corrections (in $1/N$) describe the interaction between this bound state and fundamental fermions. No additional bound states are created to finite orders in $1/N$. In connection with the bound-state formation, it is important to check the signs of the bare four-fermion coupling constant $G = g^2/m^2$ and the renormalized one $G_R \equiv g_R^2/m_R^2$. The attractive force between a fermion and an antifermion at the tree level ($G_R > 0$) should remain attractive ($G > 0$) even when the quantum corrections are included according to the $1/N$ expansion. From (2.5) we learn that

$$G = g^2/m^2 = (Z_g/Z_\psi)^2(1/Z_m)(g_R^2/m_R^2), \tag{2.13}$$

where $Z_m = (m_R^2 + \delta m^2)/m_R^2$. Hence G and G_R are of the same sign only when $Z_m > 0$, which is indeed the case for the attractive interaction $G > 0$: The renormalization condition (2.9) implies that

$$Z_m = Z_\sigma + (\Pi(m_\sigma^2)/m_\sigma^2), \tag{2.14}$$

where we have used the on-mass-shell renormalization convention $\mu^2 = m_R^2 = m_\sigma^2$. Therefore, when $Z_\sigma = 0$, $Z_m > 0$ if $\Pi(m_\sigma^2) > 0$. In the present model, if we use the ultraviolet cutoff Λ, $\Pi(p^2) \propto N g_R^2 \Lambda^{(n-2)} > 0$ (apart from $\ln \Lambda^2$) to order $(1/N)^0$ in n dimensions ($n \geq 2$), as verified easily.[24]

On the other hand, since $Z_m > 0$ for $m_\sigma^2 > 0$, the repulsive interaction $G = g^2/m^2 < 0$ (i.e. $g^2 > 0$ and $m^2 < 0$) is inconsistent with $m_\sigma^2 > 0$ (although the $Z_\sigma = 0$ condition is solvable for $g_R^2 > 0$ to zeroth order in $1/N$). This simply means that the repulsive interaction does not produce a bound state. For the repulsive interaction the equivalence relation based on the $Z_\sigma = 0$ condition does not make sense.

Let us now study the four-fermion theory (2.1) in more detail on the basis of the equivalence relation. In three dimensions (where we set[25] $n = 3$ and $n_F = 2$ in Eq. (2.11)) the compositeness condition $Z_\sigma = 0$ determines, to zeroth order in $1/N$, the bound-state mass m_σ in terms of g_R:

$$2\pi/(N g_R^2) = \int_0^1 dx\, x(1-x)/\left[M_R^2 - x(1-x)\, m_\sigma^2 - i\epsilon\right]^{1/2},$$

$$= \frac{1}{16} M_R^{-1} \int_1^\infty du \left[u - \{m_\sigma/(2M_R)\}^2 - i\epsilon\right]^{-1} (1+u)u^{-3/2}, \qquad (2.15)$$

where in the last line we have performed a change of variable $4M_R^2 u - m_\sigma^2 = (4M_R^2 - m_\sigma^2)/(2x-1)^2$ along with an integration by parts. It is clear from (2.15) that the bound-state mass m_σ decreases monotonically from $2M_R$ to zero as g_R^2 varies from zero to $12\pi M_R$; in particular, for weak coupling $N g_R^2 << M_R$,

$$m_\sigma^2 = 4 M_R^2 \left[1 - 4 \exp\{-(16\pi M_R/(g_R^2 N)) + 1\} \right] , \tag{2.16}$$

where in the exponent we have neglected terms that vanish as $N g_R^2/M_R \to 0$. In two dimensions, the $Z_\sigma = 0$ condition leads to

$$16\pi M_R^2/(N g_R^2) = \int_1^\infty du \left[u - \{m_\sigma/(2M_R)\}^2 - i\epsilon \right]^{-1} u^{-3/2} (u-1)^{-1/2}, \tag{2.17}$$

where we have set $n_F = n = 2$ in (2.11) and made a change of variable $4x(1-x)u = 1$. With increasing g_R^2 from zero to $12\pi M_R^2$, the bound-state mass m_σ decreases from $2M_R$ to zero; for weak coupling $N g_R^2 << M_R^2$,

$$m_\sigma^2 = 4 M_R^2 - (N g_R^2)/(2\pi) . \tag{2.18}$$

Let us finally consider the four-dimensional case. In the presence of the ultraviolet cutoff Λ^2 (or under one dimension) the four-fermion theory (2.1) is equivalent to the following Yukawa theory

$$\mathscr{L}_Y[\psi, \overline{\psi}; \sigma] = \overline{\psi}^a (i \not\partial - M + g\sigma)\psi^a - \frac{1}{2} m^2 \sigma^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{3!} \lambda \sigma^3 - \frac{1}{4!} \xi \sigma^4 , \tag{2.19}$$

with the compositeness condition

$$Z_\sigma = Z_\lambda = Z_\xi = 0 . \tag{2.20}$$

These renormalization constants are defined in the usual way (e.g. $\lambda = Z_\lambda Z_\sigma^{-3/2} \lambda_R$ and $\xi = Z_\xi Z_\sigma^{-2} \xi_R$, etc). Here λ_R and ξ_R are regarded to be of order $(1/N)^{\frac{1}{2}}$ and $(1/N)$, respectively. Without the compositeness condition (2.20) this Yukawa theory is renormalizable in four dimensions. In four dimensions, g_R, λ_R and ξ_R determined from the compositeness condition are ultraviolet-cutoff dependent; the Λ^2 determines g_R for fixed m_σ, i.e. $N g_R^2 \propto 1/\ln(\Lambda^2)$. This means that the original four-fermion theory is still nonrenormalizable in the $1/N$ expansion scheme. In this scheme, however, the ultraviolet divergences remaining in the renormalized Green's functions are only logarithmic and are far milder than they are in the conventional perturbation expansion in G.

III. OTHER EXAMPLES

So far we have studied a simple four-fermion theory and its equivalent Yukawa theory in detail. We have especially noted (1) the existence of calculable $1/N$ expansion schemes for both theories and (2) the solvability of the compositeness condition. These two points rely on the O(N) nature of the symmetry and are generally met in theories with an O(N) symmetry. In most cases point (2) is guaranteed by the Källén-Lehmann spectral representation.

It is straightforward to include some internal symmetries other than the O(N) symmetry into four-fermion theories. As long as the additional internal symmetry commutes with the O(N) symmetry, we can construct equivalent composite-particle theories in much the same way as in Sec. II. A non-Abelian version of the previous model is given by the following $O(N) \times SU(2)$ theory

$$\mathscr{L} = \bar{\Psi}^a \cdot (i\not{\partial} - M)\Psi^a + \frac{1}{2} G(\bar{\Psi}^a \cdot \tau^{(k)}\Psi^a)^2 , \tag{3.1}$$

where $\Psi^a(x) = \{\psi_i^a(x)\}$ is an O(N)- vector $(a = 1,\ldots,N)$ SU(2)-doublet $(i = 1,2)$. The group matrices $\tau^{(k)}$ $(k = 1,2,3)$ are the Pauli spin matrices and $\bar{\Psi}^a \cdot \tau^{(k)}\Psi^a \equiv \bar{\psi}_i^a (\tau^{(k)})_{ij} \psi_j^a$ is an O(N)-singlet SU(2)-triplet. As before, within the $1/N$ expansion, this theory is equivalent to the following $O(N) \times SU(2)$ Yukawa theory

$$\mathscr{L} = \bar{\Psi}^a \cdot (i\not\partial - M + g\,\sigma^{(k)}\tau^{(k)})\Psi^a + \frac{1}{2}(\partial_\mu \sigma^{(k)})^2 - \frac{1}{2} m^2 (\sigma^{(k)})^2, \qquad (3.2)$$

with the compositeness condition $Z_\sigma = 0$, where $G = g^2/m^2$ and Z_σ is the wave-function renormalization constant for the O(N)-singlet, SU(2)-triplet scalar field $\sigma^{(k)}(x)$ $(k = 1,2,3)$. To zeroth order in $1/N$, the mass m_σ of the O(N)-singlet, SU(2)-triplet scalar bound-state is given by (2.15) or (2.17) with g_R^2 replaced by $2g_R^2$.

It is possible to include a chiral symmetry as well. For example, the $SU(2)_L \times SU(2)_R$ four-fermion theory

$$\mathscr{L} = \bar{\Psi}^a \cdot i\not\partial\Psi^a + \frac{1}{2} G \left\{ (\bar{\Psi}^a \cdot \Psi^a)^2 + (\bar{\Psi}^a \cdot i\gamma_5 \tau^{(k)} \Psi^a)^2 \right\} , \qquad (3.3)$$

produces, within the $1/N$ expansion, four O(N)-singlet bound-states corresponding to the composite operators $\bar{\Psi}^a \cdot \Psi^a$ and $\bar{\Psi}^a \cdot i\gamma_5 \tau^{(k)}\Psi^a$ and becomes equivalent to an $SU(2)_L \times SU(2)_R$ liner σ model (based on the $(\frac{1}{2}, \frac{1}{2})$ representation). In this way, four-fermion scalar or pseudoscalar interaction theories (both Abelian and non-Abelian) become renormalizable under four dimensions within the $1/N$ expansions, as long as they can be cast into equivalent super-renormalizable Yukawa-like theories.

Let us next proceed to vector or axial-vector interactions. As a first example, we consider the O(N)-symmetric vector interaction

$$\mathscr{L} = \overline{\psi}^a (i\not\partial - M)\psi^a - \frac{1}{2} E(\overline{\psi}^a \gamma_\mu \psi^a)^2 , \tag{3.4}$$

where a runs from one to N. We regard the coupling constant E to be of order 1/N and write $E = e^2/\kappa^2$ with $e = O((1/N)^{1/2})$. For $E > 0$, a fermion attracts an antifermion. The Lagrangian (3.4) can be cast into an equivalent form

$$\mathscr{L}[\psi, \overline{\psi}; A] = \overline{\psi}^a (i\gamma\cdot\partial - M + eA^\mu\gamma_\mu)\psi^a + \frac{1}{2}\kappa^2 A_\mu^2 , \tag{3.5}$$

where $A_\mu(x)$ is an O(N)-singlet vector field. Since the $\overline{\psi}^a(i\not\partial - M + e\not A)\psi^a$ term has a gauge-invariant structure, the quantum fluctuations of ψ and $\overline{\psi}$ can effectively create a gauge-invariant kinetic-energy term $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ for the vector field in the 1/N expansion. Correspondingly, we construct an equivalent field theory by adding a kinetic-energy term $-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ to (3.5),

$$\mathscr{L}[\psi, \overline{\psi}, A] = \overline{\psi}^a(i\not\partial - M + e\not A)\psi^a + \frac{1}{2}\kappa^2 A_\mu^2 - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 , \tag{3.6}$$

and by imposing the compositeness condition

$$Z_A = 0 , \tag{3.7}$$

on the renormalized parameters e_R, κ_R and M_R. Renormalized quantities are defined by

$$\psi = (Z_\psi)^{1/2}\psi_R,\ A^\mu = (Z_A)^{1/2}A^\mu_R,\ e = (Z_e/Z_\psi)Z_A^{-1/2}e_R$$

$$\kappa = Z_A^{-1/2}\kappa_R,\ \text{etc.} \tag{3.8}$$

As is well-known, $Z_e = Z_\psi$, and the last expression $\kappa = Z_A^{-1/2}\kappa_R$ implies the absence of the vector-meson mass renormalization; this follows from the Ward-Takahashi (WT) identities associated with the neutral vector-meson theory[26] (3.6). In the so-called Proca form[26] (3.6), the free vector-meson propagator is given by $\{-g_{\mu\nu}+(1/\kappa_R^2)p_\mu p_\nu\}/(p^2-\kappa_R^2)$ and the theory is renormalizable only in two dimensions; according to power-counting, $Z_e = Z_\psi$ is logarithmically divergent while Z_A is finite. It is, however, known that the neutral vector-meson theory is renormalizable up to four dimensions. In fact, with the Stückelberg formalism,[26] the theory can be cast into a less divergent form

$$\mathscr{L}[\psi', \overline{\psi}', A'] = \overline{\psi}'^a(i\not\partial - M + e\not A')\psi'^a + \frac{1}{2}\kappa^2(A'_\mu)^2 - \frac{1}{4}(\partial_\mu A'_\nu - \partial_\nu A'_\mu)^2$$

$$- \frac{1}{2\alpha}(\partial^\mu A'_\mu)^2\ , \tag{3.9}$$

where α is an arbitrary parameter characterizing the gauge. Within the Stückelberg formalism a massive vector field is regarded as a genuine gauge field and the Lagrangians (3.6) and (3.9) simply correspond to two different choices of the gauge condition. One can pass from (A_μ, ψ)

to (A'_μ, ψ') by certain field redefinition or nonlinear gauge transformations.[26] The renormalization transformation for the theory of the form (3.9) is given by

$$\psi'^a = (X_\psi)^{\frac{1}{2}} \psi'^a_R , \quad A'^\mu = (X_A)^{\frac{1}{2}} A'^\mu_R , \quad e = (X_e/X_\psi) X_A^{-\frac{1}{2}} e_R ,$$

$$\kappa^2 = X_A^{-1} \kappa_R^2 , \qquad \alpha = X_A \, \alpha_R , \text{ etc.} \tag{3.10}$$

Here again the WT identity $X_e = X_\psi$ holds. The bare parameters e, κ, M are common to both Lagrangians (3.6) and (3.9). In the on-mass-shell renormalization scheme, the renormalized parameters e_R, κ_R and M_R also are common to both Lagrangians (because they are physical parameters defined on the mass shell). Therefore $Z_A = X_A$ and, the compositeness condition $Z_A = 0$ in the Proca form of the theory is replaced by

$$X_A = 0 , \tag{3.11}$$

in the new form (3.9) of the theory. In this new form the neutral vector-meson theory is super-renormalizable under four dimensions; X_A and $X_e = X_\psi$ are finite. The Källén-Lehmann bound $0 \leq X_A < 1$ holds[22] in quantum electrodynamics as well as the neutral vector-meson theory; hence, the $X_A = 0$ condition is solvable to each order in $1/N$. To lowest order in $1/N$, $X_A = 0$ leads to expressions (2.15) ~ (2.18) with g_R^2 replaced by $2e_R^2$. Hence, within the $1/N$ expansion scheme, the Abelian vector four-fermion theory (3.4) is equivalent to the neutral vector-meson theory with a composite vector meson and is a renormalizable and well-defined field theory under four dimensions. Note that, unlike the previous case, the four-fermion coupling constant $E = e^2/\kappa^2 = e_R^2/\kappa_R^2$ remains unrenormalized.

We have arrived at the final form (3.9) via the Proca form. Let us examine whether we can get to (3.9) directly from (3.5). After the renormalization transformation (3.10), the last two terms of the Lagrangian (3.9) are written as

$$\mathscr{L}[\psi', \overline{\psi}', A'] = \ldots - \frac{1}{4} X_A (\partial_\mu A'_\nu - \partial_\nu A'_\mu)_R^2 - X_\alpha \frac{1}{2\alpha_R} (\partial_\mu A'^\mu_R)^2 , \qquad (3.12)$$

where we have defined α_R by the more general rescaling $\alpha = X_\alpha^{-1} X_A \alpha_R$ instead of the last one in (3.10). If one can restrict renormalized parameters α_R, g_R, m_R etc. so that $X_A = X_\alpha = 0$, this theory is reduced directly to the four-fermion theory. However, $X_\alpha = 0$ cannot be a constraint on α_R. In fact, the zeroth-order $1/N$ quantum fluctuations come from a single-fermion-loop diagram which is independent of α_R; hence, $X_\alpha = 1$ to this order and one cannot set $X_\alpha = 0$. (In higher orders, $X_\alpha = 0$ may be solvable but then α_R is no longer zeroth order in $1/N$.) This implies that the direct transition from (3.12) to the four-fermion theory is impossible. This is the reason why one has to pass from the Proca form to the final form (3.9) by means of the field redefinition.

It is simple to include an internal symmetry into the vector four-fermion theory. For definiteness, we consider the following $O(N)\times SU(2)$ vector interaction

$$\mathscr{L}[\Psi,\overline{\Psi}] = \overline{\Psi}^a\cdot(i\not\partial - M)\Psi^a - \frac{1}{2}E(\overline{\Psi}^a\cdot\gamma_\mu\tau^{(k)}\Psi^a)^2 , \tag{3.13}$$

where $\Psi^a(x) = \{\psi_i^a(x)\}$ $(a=1,\ldots,N$ and $i=1,2)$ is, as before, an SU(2) doublet, and $E=e^2/\kappa^2$ is of order $1/N$. As in the Abelian case, within the $1/N$ expansion, an equivalent composite-particle theory is given by the following $O(N)\times SU(2)$ massive Yang-Mills theory

$$\mathscr{L}[\Psi,\overline{\Psi},A] = \overline{\Psi}^a\cdot(i\not\partial - M + e\not A^{(k)}\tau^{(k)})\Psi^a + \frac{1}{2}\kappa^2\vec{A}_\mu^2 - \frac{1}{4}\left(\vec{F}_{\mu\nu}[A]\right)^2 , \tag{3.14}$$

with $\vec{F}_{\mu\nu}[A] = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + e\vec{A}_\mu\times\vec{A}_\nu$, where $\vec{A}_\mu(x) = \{A_\mu^{(k)}(x)\}$ is an O(N)-singlet, SU(2)-triplet vector field $(k=1,2,3)$. The gauge-invariant kinetic term $(\vec{F}_{\mu\nu}[A])^2$ is to be produced by the quantum fluctuations of Ψ and $\overline{\Psi}$ of the gauge-invariant form $\overline{\Psi}\cdot(i\not\partial - M + e\not A^{(k)}\tau^{(k)})\Psi$. The compositeness condition is

$$Z_A = 0 . \tag{3.15}$$

Renormalized fields and parameters are defined by the same rescaling as in (3.8). The absence of the vector-meson mass renormalization and the relation $Z_e = Z_\psi$ in the on-mass-shell renormalization scheme again follow from the WT identities. Unfortunately, in the Proca form the massive Yang-Mills theory appears nonrenormalizable by power-counting in two dimensions.

The massive Yang-Mills theory can be regarded as a spontaneously-broken gauge theory where the Higgs field is realized nonlinearly.[27] To see this let us consider the Yang-Mills field coupled to the SU(2) nonlinear σ model

$$\mathscr{L}_0[\Psi, \overline{\Psi}, A, \pi] = \overline{\Psi}^a \cdot (i \not{\partial} - M + e \not{A}^{(k)} \tau^{(k)}) \Psi^a - \frac{1}{4}\left(\vec{F}_{\mu\nu}[A]\right)^2$$

$$+ \frac{1}{4} \operatorname{Tr}\left[(\mathscr{D}_\mu[A] \mathscr{M})^+ (\mathscr{D}^\mu[A] \mathscr{M}) \right], \tag{3.16}$$

where $\mathscr{M}$ is the (2×2)-matrix field $\mathscr{M}(x) = \sigma(x)\, 1 + i\, \pi^{(k)}(x)\, \tau^{(k)}$ $(k = 1, 2, 3)$ subject to the constraint $\mathscr{M}^+\mathscr{M} = [\sigma^2 + (\pi^{(k)})^2]\, 1 = (4\kappa^2/e^2)\, 1$, i.e. $\sigma(x) = (2\kappa/e)[1 - (e/2\kappa)^2 (\pi^{(k)}(x))^2]^{\frac{1}{2}}$ and $\mathscr{D}_\mu[A]$ is the covariant derivative $\mathscr{D}_\mu[A] = \partial_\mu + \frac{1}{2}\, i e \tau^{(k)} A_\mu^{(k)}$. Here the SU(2)-triplet field $\pi^{(k)}(x)$ is the Higgs field. If we choose the gauge condition $\pi^{(k)} = 0$ $(k = 1, 2, 3)$, this theory is reduced to the Proca theory (3.14). In the standard gauge characterized by the effective Lagrangian

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_0[\Psi, \overline{\Psi}, A, \pi] - \frac{1}{2\alpha} (\partial^\mu \vec{A}_\mu)^2 + (\text{ghost term}), \tag{3.17}$$

where the last term contains the Faddeev-Popov ghost fields, the theory turns out[27] renormalizable in two dimensions (although the Lagrangian (3.17) is nonpolynomial in $\pi^{(k)}$). In this gauge, we use the following rescaling

$$(\sigma, \pi^{(k)}) = (X_\pi)^{\frac{1}{2}} (\sigma, \pi^{(k)})_R, \ \Psi = (X_\Psi)^{\frac{1}{2}} \Psi_R, \ A^\mu = (X_A)^{\frac{1}{2}} A^\mu_R$$

$$e = (X_e/X_\Psi) X_A^{-\frac{1}{2}} e_R, \ \kappa^2 = X_\kappa X_\pi X_A^{-1} \kappa^2_R, \text{ etc.} \tag{3.18}$$

Only X_π and X_κ are ultraviolet divergent in two dimensions. By the same argument as before, the compositeness condition $Z_A = 0$ in the Proca gauge (3.14) is translated into

$$Z_A = X_A (X_\Psi/X_e)^2 = 0 \ , \tag{3.19}$$

in the renormalizable gauge (3.17). [Due to the ghost-field contribution, $X_\Psi \neq X_e$ in general.] To zeroth order in $1/N$, $X_e = X_\Psi = 1$ and X_A is calculated from the same one-fermion-loop diagram as in the Abelian case, apart from the overall group factor. Hence the compositeness condition (3.19) is solvable in the $1/N$ expansion; to zeroth order in $1/N$, (3.19) leads to expressions (2.15) ~ (2.18) with g_R^2 replaced by $4e_R^2$. Thus, within the $1/N$ expansion, the non-Ableian vector four-fermion theory (3.13) is identical to the massive Yang-Mills theory with composite massive Yang-Mills fields and becomes renormalizable in two dimensions.

Dynamical structures of some bosonic theories are also made explicit by the construction of equivalent composite-particle theories. For example, the CP^{N-1} model, which has recently been studied[19] in connection with instanton effects, can be converted into an $O(N)\times U(1)$ nonlinear σ model coupled to a $U(1)$ composite vector-meson in the $1/N$ expansion. It is known[16] that the $O(N)$ nonlinear σ model has two different phases in $n = 2+\epsilon$ dimensions $(\epsilon > 0)$ in the large-N limit: a weak-coupling massless phase where the $O(N)$ symmetry is spontaneously broken down to $O(N-1)$ and a strong-coupling massive phase where the full $O(N)$ symmetry is restored with the occurrence of a scalar bound state; only the latter symmetric phase exists in two dimensions. Correspondingly, in the symmetric phase, the composite vector-meson of the CP^{N-1} model becomes a massless bound-state[28] (when the instanton effects are neglected). In the broken-symmetry phase, however, the compositeness condition is not solvable on the mass shell; this is because the vector meson, which is massive in this phase, is unstable, decaying into $(N-1)$ Goldstone bosons.

IV. CONCLUDING REMARKS

We have studied, for a wide class of nonlinear field theories, the construction of equivalent field theories in the framework of the $1/N$ expansion. The particle spectrum and the renormalizability of those theories are made explicit in their equivalent field theories. It is interesting to observe that with the formation of bound states nonlinear field theories become better-behaved at short distances.

The construction of the theory of composite particles through the compositeness condition will be a general theoretical framework, not restricted to the $1/N$ expansion or some other particular calculational schemes. In the presence of some phenomenological ultraviolet cutoff, even four-dimensional four-fermion interactions can effectively be converted into equivalent Yukawa-like interactions.

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[24] The self-energy $\Pi(p^2)$, given by Eq. (2.11), is convergent and positive under two dimensions ($n < 2$). This sign $\Pi(p^2) > 0$ will persist for $n \geq 2$ although it is obscured in the dimensional regularization because of the analytic continuation in n.

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[28] The masslessness of this vector bound-state is a consequence of the local U(1) symmetry of the CP^{N-1} model.

FIGURE CAPTION

Fig. 1 Quantum correction to the σ-field propagator to zeroth order in 1/N.

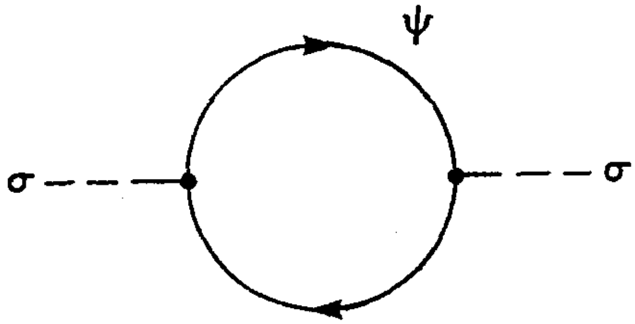

Fig. 1