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Phase Transitions in the Abelian Higgs Model

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ABSTRACT

A lattice version of the Abelian Higgs model is studied in arbitrary Euclidean dimension. Two different representations of the theory, one in terms of the Higgs and gauge fields and the other in terms of the topological excitations, are used to understand what phases exist for the system. In addition to limiting cases there is, in two dimensions, a plasma phase of vortex excitations. The vortices (instantons) in this phase cause confinement (in the sense of Wilson) of fractional, but not integer charges. In three and more dimensions, there is a plasma phase similar to the one in two dimensions as well as another phase which does not confine any charge. We argue that the confinement due to topological excitations in the plasma phase has the same physical basis as the usual large coupling constant (high temperature) confinement of the lattice gauge theory. Effects of a background field in two dimensions are also described.

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I. INTRODUCTION

In this paper we will study the Abelian Higgs model in various dimensions primarily with a view toward understanding, at least qualitatively, what phases may occur in the model. The physical motivations for studying this model have been discussed in Ref. 1 and will not be repeated here.

We formulate the Abelian Higgs model on a d-dimension, Euclidean hypercubical lattice. The partition function (generating functional) for the theory is

$$Z = \int_{-\pi}^{\pi} \delta\theta_{\mu}(j)\delta\chi(j)e^{\mathscr{L}}$$

$$= \int_{-\pi}^{\pi} \delta\theta_{\mu}(j)\delta\chi(j)\mathrm{Exp}\left[\kappa\sum_{\ell}\cos\left(\Delta_{\mu}\chi(j) - \theta_{\mu}(j)\right)\right. \tag{1.1}$$

$$\left. + \frac{\beta}{2}\sum_{p}\cos\left(\frac{1}{(d-2)!}\epsilon_{\mu\nu,\beta,\ldots\beta_{\alpha-2}}\epsilon_{\beta_1\ldots\beta_{\alpha-2},\rho\sigma}\Delta_{\rho}\theta_{\sigma}(j)\right)\right].$$

The sum over ℓ is a sum over all links of the lattice, and the sum over p is a sum over all elementary two-dimensional squares, or plaquettes of the lattice. It was shown in Ref. 1 that when factors of the lattice spacing, a, are properly included, (1.1) becomes the generating functional of the continuum Abelian Higgs theory in the naive limit $a \to 0$. $\chi(j)$ is the phase angle of the Higgs field and $\theta_{\mu}(j) = aA_{\mu}(j)$ where $A_{\mu}(j)$ becomes the gauge vector potential in the continuum. In (1.1) the radial degree of freedom of the Higgs field is completely frozen.

For large β and κ (low temperatures), a very good approximation to (1.1) is[1]

$$Z = \widetilde{\sum_{\{a_\mu,\, b_{\mu\nu}\}}} \int_{-\infty}^{\infty}{}' \delta\chi\, \delta\theta_\mu \exp\left[\Sigma - \frac{\kappa}{2}(\Delta_\mu \chi - \theta_\mu + 2\pi a_\mu)^2\right.$$

$$\left. - \frac{\beta}{4}\left(\frac{1}{(d-2)!}\, \epsilon_{\mu,\nu,\beta,\ldots\beta_{\alpha-2}}\, \epsilon_{\beta,\ldots\beta_{\alpha-2},\, \rho_\sigma} \Delta_\rho \theta_\sigma + 2\pi b_{\mu\nu}\right)^2\right]. \qquad (1.2)$$

In this expression, unlike equation (1.1), to avoid infinities we must choose a gauge when integrating over χ and θ_μ (this is indicated by the prime). The integers $-\infty < a_\mu,\ b_{\mu\nu} < \infty$ are included in Z so that this Lagrangian has a periodic structure like that of (1.1). The tilde over the sum reminds us that it is redundant to sum independently over all integer values of a_μ and $b_{\mu\nu}$. As discussed in Ref. 1, one must also "choose a gauge" for these integer fields so that Z is finite.

Using an exact duality transformation,[1] the partition function (1.1) can be rewritten in terms of the topological excitations of the angles χ and θ_μ. In Ref. 1 we showed that the topological excitations of this model in d-dimensions are closed "vortex" surfaces of dimension d-2, and open vortex surfaces of dimension d-2 bounded by monopole surfaces of dimension d-3. A similar duality transformation applied to (1.1) results in an expression which contains the same topological excitations as the dual form of (1.1), and coincides with it when $\beta, \kappa >> 1$.

Now the dual form of (1.1) is an exact representation of the theory. In this paper we shall use both (1.1) (or its approximate form (1.2)) and its dual form to understand the different phases of the theory. It turns out that a very simple picture of the nature of the different phases emerges from a consideration of the topological excitations. In some phases the topological excitations are very large and influential, and in others they are small and relatively unimportant. In addition to examining the partition function, we will discuss the expectation value, Γ, of a large electric gauge loop integral, sometimes called the Wilson loop integral.[2] The large distance behavior of this object is also determined by the presence or absence of certain topological excitations. We show that the asymptotic behavior of the gauge loop integral can also be used to discriminate between certain phases of the theory.

Section V contains a review of our results, but we will briefly summarize the most important points here. In addition to phases which are naively associated with the limits $\beta, \kappa \to 0$ or ∞, we find in two dimensions only one other phase. This is a plasma phase of vortex points which have only short range interactions. The gauge loop integral behaves like e^{-A} for fractional charges and e^{-P} for integer charges where A is the area enclosed by the loop and P is its perimeter. We also study the effects of a background field in two dimensions and find that in the presence of such a field the qualitative behavior of Γ can be changed drastically.

In three or more dimensions (again, aside from limiting cases) there are two phases. One phase is characterized by a massive vector boson; the topological excitations (e.g. in three dimensions open vortex strings with monopoles on the ends and closed vortex loops) are small and not too important. $\Gamma \sim e^{-P}$ for all charges. Another phase, a plasma phase analogous to the phase in two dimensions described above has very large open and closed topological excitations, and in this phase $\Gamma \sim e^{-A}$ for non-integer charges and $\Gamma \sim e^{-P}$ for integer charges.

In addition to these phases, other phases exist as limits of the coupling constants κ and β. Of particular interest is the limit $\beta \to \infty$ in which the model becomes equivalent to the globally invariant x-y model.[3] As we shall discuss, there is some reason to believe that these phases also exist for large but finite values of β.

The rest of the paper is organized as follows. In the next section we discuss the model in two dimensions. We give a complete discussion of the various limiting cases including the x-y limit mentioned above. We then argue that there is one and only one additional phase, and we approximate the behavior of Γ in that phase. Finally we describe what happens to the theory in the presence of a background field. Section III deals with the model in three dimensions. We describe the different phases in terms of the topological excitations and discuss the behavior of Γ in these phases. Section IV generalizes the arguments of section III to four (and more) dimensions. Finally, some remarks and a summary comprise section V.

II. THE TWO-DIMENSIONAL CASE

A. Description of the Phases in Terms of the Topological Excitations

In this section we will describe the phases we expect to occur for our model in two dimensions. We shall usually deal with the periodic quadratic form of the Abelian Higgs model. As discussed in Ref. 1, this may be thought of as an approximation to the full, compact theory. Both the full theory with cosine interactions, and the periodic quadratic theory have the same topological singularities, and are therefore expected to have qualitatively similar phase transitions. Thus, our considerations should apply equally well to both theories.

We will first discuss the model assuming periodic (spherical) boundary conditions. At the end of the section we will describe what happens when we impose certain other boundary conditions which correspond, in instanton language, to different θ-vacuua.[4]

We begin by recalling that in two dimensions the partition function (1.2) which is the periodic quadratic form of (1.1) can be written[1]

$$Z = Z_0 \sum_{\{\mathbb{P}_j\}} e^{-\kappa 4\pi^2 \sum_{j,k} \mathbb{P}(j) D(j-k;m^2) \mathbb{P}(k)} \tag{2.1}$$

where

$$Z_0 = \int_{-\infty}^{\infty} \delta\phi\, e^{-\frac{1}{4\kappa}\sum\left[\Delta_\mu \phi(j)\right]^2 + m^2\phi^2(j)} \tag{2.2}$$

and $D(j - k; m^2)$ is the two-dimension lattice Green's function satisfying

$$(-\Delta_\mu{}^2(j) + m^2)D(j - k; m^2) = \delta_{jk} \tag{2.3}$$

with $m^2 = \kappa/\beta$. Z_0 is the partition function of a free massive spin-wave (massive scalar field). The integers $\{\mathbb{P}(j)\}$ are the vortex excitations of the original χ and θ_μ fields and range from $-\infty$ to ∞. The position vectors, $\underset{\sim}{j}$, refer to the sites of the dual lattice. The dual lattice is obtained from the original lattice by shifting the lattice by half a lattice spacing in each direction.

To get a feeling for the possible phases of the model, it is useful to consider various limiting cases. Only the vortex contribution is relevant since, at finite m^2, the contribution from spin waves, Z_0, to Z is analytic. First of all, in the very large m^2 limit, it is easy to see explicitly that there is only one phase. To be precise, consider κ getting larger with fixed β. Expanding the lattice Green's function in powers of $(m^2)^{-1}$, we have for the leading term

$$D(\underset{\sim}{j} - \underset{\sim}{k}; m^2) = \int_{-\pi}^{\pi} \frac{d^2 q \, e^{iq(\underset{\sim}{j} - \underset{\sim}{k})}}{2\sum_{\mu} (1 - \cos q_{\mu}) + m^2} \tag{2.4}$$

$$\xrightarrow[m^2 \to \infty]{} \frac{1}{m^2} \delta_{jk} + \mathcal{O}\left(\frac{1}{m^4}\right) \quad .$$

Hence the partition function (2.1) becomes

$$Z \xrightarrow[\substack{m \to \infty \\ \beta, \text{ fixed}}]{} (\pi\beta)^{N/2} \sum_{\{p(j)\} = -\infty}^{\infty} e^{-\beta(2\pi)^2 \sum_j p^2(j)} \quad , \tag{2.5}$$

which is a theory of non-interacting vortices. (N is the number of lattice sites.)

The last factor is a product of Jacobi theta functions[5] which are analytic for $\beta > 0$ and so the free energy, $F = 1/N \ln Z$, has no singularities in this limit. It is also easy to show using (1.1) that the free energy of the full Abelian Higgs model is analytic in this limit and is proportional to $\ln \beta + c \ln I_0(\beta)$, c being a constant.

As a second limiting form, consider the behavior as $\beta \to \infty$ for fixed k. This limit corresponds to the familiar x-y model.[3] To see this, look at equation (1.1). As $\beta \to \infty$, the only values of $\theta_\mu(j)$ which contribute to Z are those for which $F_{\mu\nu} = 0$. Hence $\theta_\mu(j)$ can be written as $\theta_\mu(j) = \Delta_\mu \Lambda(j)$, and so the combination $\chi(j) - \Lambda(j)$ can be thought of as the angle of the x-y model spin. [Note that $\Delta_\mu(\chi(j) - \Lambda(j))$ is gauge invariant.] When $m^2 = 0$, the Green's function $D(\underset{\sim}{j} - \underset{\sim}{k};\, 0) \propto \ln |\underset{\sim}{j} - \underset{\sim}{k}|$ when $|\underset{\sim}{j} - \underset{\sim}{k}| >> 1$. Furthermore, a careful analysis of this limit reveals that with spherical boundary conditions there is a neutrality condition on the total vorticity: the only configurations allowed in the sum of equation (2.1) are those for which $\sum_j p(j) = 0$.

It is generally accepted[3] that the d=2 x-y model undergoes a topological phase transition at some temperature $\kappa = \kappa_c$. Because $D(j - k;\, 0)$ grows logarithmically and $\sum_j p(j) = 0$, the low temperature phase of the theory ($\kappa > \kappa_c$) is dominated by a few tightly bound vortex-antivortex pairs, in addition to the spin waves described by Z_0 (with $m^2 = 0$) in equation (2.1). But the entropy for finding a vortex-antivortex pair a distance r apart is also proportional to ln r, and so for κ less than some κ_c, the entropy dominates the free energy of a vortex-antivortex pair, and it becomes highly probable to find pairs whose members are an arbitrarily large distance apart. This unbinding causes certain correlation functions which had been power behaved for $\kappa > \kappa_c$ to fall exponentially, and can be thought of as signalling a phase transition. Note that the x-y model has only a global U(1) symmetry as opposed to the local U(1) symmetry of the Higgs model. This breakdown of local gauge symmetry as $m \to 0$ will be important for distinguishing phases of our system, as we shall discuss below.

With this picture in mind, let us now consider the case of finite, non-zero m^2. In this case we have no strict neutrality condition (although for small m^2 there is some suppression of configurations with $\sum_j p(j) \neq 0$). Furthermore, $D(\underset{\sim}{j} - \underset{\sim}{k}, m^2) \sim (e^{-m|\underset{\sim}{j} - \underset{\sim}{k}|})/\sqrt{m|\underset{\sim}{j} - \underset{\sim}{k}|}$ for large $|\underset{\sim}{j} - \underset{\sim}{k}|$, so the attractive force between vortex-antivortex pairs is short ranged. Since the entropy is still proportional to $\ln |\underset{\sim}{j} - \underset{\sim}{k}|$ it will be very likely to find isolated vortices rather than just tightly bound dipoles at any non-zero temperature for any non-zero m^2. Hence, our <u>naive</u> expectation is that the theory is always in the plasma phase and there is no phase transition at finite temperature. As we shall see in the next subsection, this situation is peculiar to two dimensions.

We have described the finite m^2 case as well as the limits $\kappa \to \infty$ with β fixed and $\beta \to \infty$ with κ fixed. Now consider the infinitely massive limit $\beta \to 0$, κ fixed. This limit generates a trivial theory. From (1.1), we see that if $\beta = 0$, the only term in the theory is the Higgs interaction. But since we must still integrate over θ_μ as well as χ, we still have the usual local gauge symmetry. Hence, no matter what configuration of $\{\chi, \theta_\mu\}$ we are given, we can always gauge transform the theory to a state with all $\chi(j) = 0$. The theory is then a theory of non-interacting gauge fields, or links, θ_μ, and contains no dynamics.[6]

Finally, the limit $\kappa \to 0$, β fixed describes the pure compact gauge theory. In two dimensions this is also a trivial theory (in the absence of external sources) since the gauge fields have no dynamical degrees of freedom.

Our naive expectation that the theory has no phase transition for finite m^2 may have to be modified. A careful discussion requires analyzing the large distance behavior of the theory, i.e., correlations over distances large compared to the lattice spacing. In the language of field theory, m^2 plays the role of a bare mass when the theory is defined with an ultraviolet cutoff $1/a$, where a is the lattice spacing. It is quite possible that there is some positive value of m^2, m_c^2 (which could be a function of κ) such that for $m^2 < m_c^2$ the renormalized mass vanishes as $a \to 0$. If so, then, for $m^2 < m_c^2$, the large distance behavior of the lattice theory will be that of a theory with $m^2 = 0$, viz. the x-y model.

We now want to give a summary of the various phases we expect this theory to have. The discussion of the next few paragraphs will be heuristic; nevertheless, it is a good path to follow to get some feeling for the structure of the theory. The description will be couched in terms of the behavior of the vortices. Later we will be more specific and compute correlation functions in the different phases.

In Fig. 1 we have sketched what we believe is a schematically correct phase diagram for this model. A distinct phase of the model is defined by a range of the couplings β and κ for which the large distance behavior of the theory (to be determined, for example, by a renormalization group calculation) is qualitatively the same. The dashed lines are lines of constant m^2. Phase I fulfills our naive expectation for finite m^2, and is a phase which has a massive spin wave as well as a plasma of vortices interacting through a short range potential. Phases II and III are the high and low temperature x-y model phases, respectively. Phases IV, V, and VI are the trivial limiting theories described above with VI being the pure gauge theory. It is not clear whether these phases are only limiting cases or whether they have finite two-dimensional support in the diagram, although we think it more likely that they only exist as limits. Furthermore, the behavior of the theory at the corners of the diagram is somewhat problematical and probably depends on how the corner is approached. We shall not dwell on that here.

The three interesting phases for finite, non-zero β and κ are delineated by the separatrices AB, BC, and DB. We do not know the precise shape of these lines, but their general features can be understood as follows: Point A marks the critical point of the x-y model. For $\beta = \infty$ and $(\kappa + 1)^{-1} < A$, the system is described by tightly bound vortex-antivortex pairs. These pairs become effectively unbound by the thermal motion when we raise the temperature so that $(\kappa + 1)^{-1} > A$. Now, if we make the intervortex interaction weaker the vortex pairs will unbind at a lower temperature. Adding a mass to the vortex-vortex interaction certainly weakens its large distance effects. Therefore, for those values of m^2 and κ for which we will be driven by the renormalization group to the left-hand side of the diagram, it follows that the larger m^2 is the larger κ must be in order that we stay in phase III. This accounts for the general downward slope of the lines AB and BC. The line BD is drawn vertically for the following reason: Both phases I and II are vortex plasma phases. They are distinguished by the fact that in phase II the gauge fields are effectively frozen in the large. (As we shall see below, this has implications for the behavior of the Wilson loop integral.) But this effect, naively, seems to be controlled only by the size of β, hence BD is expected to be vertical.

These speculations could be wrong in several ways. First, it is possible (though doubtful) that AB is horizontal, or that point B coincides with point A, or that point C is really at the origin. (This could happen either smoothly or discontinuously--i.e. the separatrix BC could have a discontinuity at the x-axis of Fig. 1.) The line DB could also have a different shape; it could even collapse onto the left axis so that phase II would only exist for $m^2 = 0$. Finally, it is also possible that phase III exists only for $m^2 = 0$.

None of these possibilities can be ruled out without doing a renormalization group calculation for this model (and we would not be too supprised if some of them turned out to be correct). Nevertheless, there is some support for the general picture painted in Fig. 1 from calculations done on similar models. A self-consistent Hartree-Fock calculation for the d=2 O(n) Higgs model was carried out by Bander and Bardeen[7] to leading order in $1/n$. They found two phases which roughly correspond to the phases I and II in Fig. 1. Phase III is not expected to exist in two-dimensions for the O(n) sigma model with $n > 2$, so it is not suprising that it did not appear in their calculation. Of even more direct interest is the approximate renormalization group calculation of Kosterlitz and Thouless[8] discussed also by Kadanoff.[9] This calculation was done on a version of the x-y model which was modified to incorporate a kind of local gauge symmetry. Roughly speaking, it corresponds to taking the periodic quadratic model (1.2) and setting all $\theta_\mu = 0$. The local gauge symmetry is then expressed through the integers, $\{a_\mu, b_{\mu\nu}\}$. Their computation showed very clearly the existence of two-dimensional support for phases I and III, and in particular showed very nice renormalization group flow lines leading into the line of fixed points between 0 and A on the left-hand axis from region III. On the other hand, phase II was relegated to the left-hand axis of the diagram in their calculation. In addition, there was some evidence for an additional phase sitting where our phase II sits. But the nature of this phase and even its existence in the sense of having long range behavior distinct from phase I is in doubt.[9]

B. The Gauge Loop Integral

We have qualitatively described the phases of Fig. 1. We now want to describe the behavior of the Wilson loop integral[2] $< e^{iq\oint \theta_\mu dx^\mu} >$ for both integer and fractional charges, q. We have computed some other physically interesting correlation functions such as the vortex-vortex correlation function and the Higgs-Higgs correlation function. We will not discuss them in detail, but will refer to them when appropriate.

In computing correlation functions of fractionally charged objects, one must face an important technical problem.[10] It is easy to see that using the Lagrangian (1.1) and simply computing $\langle e^{iq\oint \theta_\mu dx_\mu}\rangle$ will give nonsensical results. The point is that the U(1) gauge fields must be periodic with respect to the smallest charge in the theory. Hence if quarks of charge, say, $1/\lambda$ (λ an integer) are introduced as external sources, the gauge fields must be able to couple to them in a U(1) invariant way. This can be accomplished by defining the unit charge to be $1/\lambda$ and coupling a Higgs particle of charge λ to the gauge field. The Lagrangian one uses is then

$$\mathscr{L} = \overline{\kappa} \sum_{\ell} \cos\,(\Delta_\mu \chi - \lambda\,\theta_\mu) + \frac{\overline{\beta}}{2} \sum_{p} \cos\left[\frac{1}{(d-2)!}\,\epsilon_{\mu\,\nu,\beta,\ldots\beta_{d-2}}\,\epsilon_{\beta,\ldots\beta_{d-2},\rho\,\sigma}\Delta_\rho\theta_\sigma\right]. \tag{2.6}$$

One can now compute $\langle e^{ic\oint \theta_\mu dx_\mu}\rangle$ with this Lagrangian where c is an integer, $0 \leq c \leq \lambda$. This is the correct way of computing an electric gauge loop of charge c/λ in the presence of an integer charged Higgs, if the gauge group is U(1).

Now, following the approach of Ref. 1 we can determine the periodic quadratic approximation to (2.6). In any dimension it is

$$\mathscr{L} = -\frac{\overline{\kappa}}{2} \sum_{\ell} (\Delta_\mu \chi - \tau_\mu + 2\pi a_\mu)^2$$

$$-\frac{\overline{\beta}}{\lambda^2} \sum_{p} \left(\frac{1}{(d-2)!}\,\epsilon_{\mu,\nu,\alpha,\ldots\alpha_{d-2}}\,\epsilon_{\alpha,\ldots\alpha_{d-2}\rho\sigma}\Delta_\rho\tau_\sigma + 2\pi B_{\mu\nu}\right)^2 \tag{2.7a}$$

where $\tau_\mu \equiv \lambda\theta_\mu$, $-\infty < \chi$, $\tau_\mu < \infty$, a_μ takes on all integer values and $B_{\mu\nu}$ takes on values λn, n an integer, and where certain restrictions apply to the sums over a_μ and $B_{\mu\nu}$.[1] In particular, in two dimensions, we know that the vorticity is $p = \epsilon_{\mu\nu}\Delta_\mu a_\nu + B_{\mu\nu}$ so that restricting $B_{\mu\nu}$ to values of λn does not change the allowed vortex configurations. (This, however, is not true above two dimensions. See section III.) Hence, to compute the expectation value of a gauge loop of charge $q = c/\lambda$ in the periodic quadratic approximation in any dimension we can use the Lagrangian

$$\mathscr{L} = -\frac{\kappa}{2}\sum_{\ell}(\Delta_\mu\chi - \theta_\mu + 2\pi a_\mu)^2$$

$$-\frac{\beta}{4}\sum_{p}\left(\frac{1}{(d-2)!}\epsilon_{\mu,\nu,\alpha_1\ldots\alpha_{d-2}}\epsilon_{\alpha_1\ldots\alpha_{d-2}\rho\sigma}\Delta_\rho\theta_\sigma + 2\pi\lambda b_{\mu\nu}\right)^2 \qquad (2.7b)$$

and calculate $<e^{i\frac{c}{\lambda}\oint\theta_\mu dx_\mu}>$. This is the periodic quadratic approximation to the calculation of $<e^{ic\oint\theta_\mu dx_\mu}>$ using (2.6) with $\kappa = \overline{\kappa}$ and $\beta = \overline{\beta}/\lambda^2$.

Consider now (2.6) and suppose we compute the discrete form of Wilson's loop integral

$$\Gamma_q \equiv <e^{ic\oint\theta_\mu dx_\mu}> = \frac{1}{Z}\int_{-\pi}^{\pi}\delta\theta_\mu\delta\chi\; e^{\mathscr{L}+i\sum\theta_\mu q_\mu}$$

where the loop integral covers an area large with respect to the lattice spacing squared. Here, q_μ is the "tangent vector" along the gauge loop. We have absorbed the charge into its definition so its non-zero components have magnitude c. Because the gauge loop is closed, we have $\Delta_\mu q_\mu = 0$. Suppose $\overline{\beta}, \overline{\kappa} \lesssim 1$. We can then expand $e^{\mathscr{L}}$ in powers of $\overline{\beta}$ and $\overline{\kappa}$. If $c = n\lambda$ (for integer n), it is clear that as the gauge loop gets very large the leading contribution to Γ_q will be a term of order

$(\bar{\kappa})^P$ where P is the perimeter of the gauge loop. This indicates a relatively weak long ranged force between the "integer charged quarks" represented by the external sources in this computation. Their charge is completely screened by the Higgs particles in the vacuum with which they form neutral bound states. This can be seen graphically by noting that the terms in this high temperature expansion which give this leading contribution just correspond to stringing factors of $\cos(\Delta_\mu \chi - \lambda\theta_\mu)$ along the perimeter of the gauge loop. Suppose now that $c < \lambda$. In this case it is easy to see that the coefficient of $(\bar{\kappa})^P$ is zero (being proportional to factors like $\int_{-\pi}^{\pi} d\theta_\mu e^{in\theta_\mu}$, n a non-zero integer.) The leading term in the limit of large gauge loops comes instead from terms proportional to $\bar{\beta}$, and is of order $(\bar{\beta})^{cA}$, where A is the area enclosed by the gauge loop. In this case the Higgs particle cannot completely screen the charge of the external quark. To get a non-zero contribution to Γ_c we must fill up the interior of the gauge loop with cA factors of $\cos F_{\mu\nu}$. This generates a linear potential between the quarks and gives us the area law behavior,

$$\Gamma_q \sim e^{Ac \ln \bar{\beta}} \quad q = \frac{c}{\lambda} \neq \text{integer} \ . \tag{2.8}$$

(Note the dependence on the charge of the quark, c.)

It is now instructive to compute the same quantity in a manner which displays explicitly the influence of the vortices. For that purpose it is convenient to use (2.7). As we mentioned before, this is a good low temperature approximation to (2.6) and in addition is expected to have features which correctly represent the qualitative behavior of the theory. (If we computed Γ using (2.6), the results would agree with what follows at the quadratic level.) Using (2.7b) we have

$$\Gamma_q = \frac{1}{Z} \widetilde{\sum_{a_\mu, B}} \int_{-\infty}^{\infty} \delta\chi\, \delta\theta_\mu \exp\Big[\Sigma - \frac{\kappa}{2}(\Delta_\mu \chi - \theta_\mu + 2\pi a_\mu)^2$$

$$- \frac{\beta}{2}\,(\epsilon_{\mu\nu}\Delta_\mu \theta_\nu + 2\pi B)^2 + i \sum q_\mu \theta_\mu\Big] \tag{2.9}$$

where we have rescaled the gauge field so that now the non-zero components of q_μ have magnitude c/λ. B takes on values which are integer multiples of λ and a_μ takes on all integer values. Since (2.9) is Gaussian, the integrals can easily be performed. It is simplest to work in the gauge $\kappa = 0$ and to shift $\theta_\mu \to \theta_\mu + 2\pi a_\mu$ before integrating. Then one obtains

$$\Gamma = \frac{1}{Z} \widetilde{\sum_{\{a_\mu, B\}}} \exp\Big[\frac{1}{2\beta}\Sigma(iq_\nu(j) - 2\pi\beta\epsilon_{\nu\mu}\Delta_\mu \mathbb{P}(j))\, D_{\nu\nu'}(j-k; m^2)$$

$$(i\mathbb{P}_{\nu'}(k) - 2\pi\beta\,\epsilon_{\nu'\mu'}\Delta_{\mu'}\mathbb{P}(k))\Big] \exp\Big[\Sigma(2\pi i q_\nu(j) a_\nu(j) - 2\pi^2\beta\mathbb{P}(j)^2)\Big] \tag{2.10}$$

where the vorticity $\mathbb{P}(j) = B(j) + \epsilon_{\mu\nu}\Delta_\mu a_\nu(j)$ and $D_{\mu\nu}$ is the two dimensional Green's function:

$$D_{\mu\nu}(j - k; m^2) = \left(\delta_{\mu\nu} - \frac{\Delta_\mu \Delta_\nu}{m^2}\right) D(j - k; m^2)$$

$$= (-\Delta^2 + m^2) D(j - k; m^2) = \delta_{jk} . \tag{2.11}$$

This expression can be simplified by noting that the gradient terms do not contribute. After some algebra and summation by parts, we find

$$\Gamma = \frac{1}{Z} \exp\left[-\frac{1}{2\beta}\Sigma q_\nu(j) q_\nu(k) D(j - k; m^2)\right]$$

$$\widetilde{\sum_{\{a_\mu, B\}}} \exp\left[-\Sigma D(j - k; m^2)\left\{2\pi^2 \kappa p(j) p(k) - 2\pi i q_\nu(j)\left(m^2 a_\nu(k) + \epsilon_{\nu\mu}\Delta_\mu B(k)\right)\right\}\right]. \tag{2.12}$$

Since $\Delta_\nu q_\nu = 0$, we may write q_ν as a curl

$$q_\nu = \epsilon_{\nu\mu}\Delta_\mu Q$$

where Q is a scalar associated with sites of the dual lattice. It is equal to c/λ for each plaquette enclosed by the gauge loop and zero elsewhere. The last term may equally well be written in terms of Q after summation by parts (Stokes' Theorem),

$$q_\nu(j) a_\nu(k) \to Q(j) \epsilon_{\mu\nu}\Delta_\mu a_\nu(k) \quad .$$

Thus (2.12) may also be written as a summation over the area enclosed by the gauge loop

$$\Gamma = \frac{1}{Z} \exp\left[-\frac{1}{2\beta}\Sigma q_\nu(j) q_\nu(k) D(j-k;\, m^2)\right] \tag{2.13}$$

$$\widetilde{\sum_{\{a_\mu, B\}}} \exp\left[-\Sigma\, D(j-k; m^2)\left\{2\pi^2 \kappa p(j) p(k) - 2\pi i (m^2 Q(j)\, \epsilon_{\mu\nu} \Delta_\nu a_\mu(k) + \Delta_\mu Q(j) \Delta_\mu B(k))\right\}\right]$$

To understand qualitatively the behavior of (2.13), consider m^2 to be large, so $D(j-k;\, m^2) \mapsto \frac{1}{m^2}\delta_{jk}$. Then we can write

$$\Gamma = \frac{1}{Z} \exp\left[-\frac{1}{2\kappa}\Sigma q_\nu(j)^2\right] \widetilde{\sum_{\{a_\mu, B\}}} \exp\left[-\Sigma\left\{2\pi^2 \beta p(j)^2 - 2\pi i Q(j)\, \epsilon_{\mu\nu} \Delta_\mu a_\nu(j)\right\}\right]. \tag{2.14}$$

The first factor is simple,

$$\exp\left[-\frac{1}{2\kappa}\sum_j q_\nu(j)^2\right] = \exp\left[-\frac{1}{2\kappa}\sum_j (\Delta_\nu Q(j))^2\right] = \exp\left[-\frac{1}{2\kappa}\left(\frac{c}{\lambda}\right)^2 P\right]$$

where P is the length of the perimeter of the gauge loop. Now, in the limit we are considering, there is no interaction between lattice sites, so

$$\Gamma = \exp\left[-\frac{1}{2\kappa} q^2 P\right] \left(\frac{\sum_p \exp\left[-2\pi^2 \beta p^2 - 2\pi i q p\right]}{\sum_p \exp\left[-2\pi^2 \beta p^2\right]}\right)^A \tag{2.15}$$

where A is the area enclosed by the loop. Note, however, that if q is an integer, then there is no area term so that ln Γ is proportional to the perimeter.

The area law for non-integer charges means that there is a long range linear potential, (modulo logarithms) i.e., arbitrary charges are not completely screened. In this sense, there is no Higgs phenomenon for fractional charge. On the other hand, this is precisely due to the vortices of the Higgs field, since, if we set $a_\nu(j) = 0$ in (2.13), for example, we would get a perimeter law for any q_μ (for non-zero m^2). (Note, though, that eliminating the $a_\nu(j)$ is not the same as eliminating the complete Higgs field which occurs in the limit $\kappa \to 0$, β finite. See below.) If, however, the external charge is an integer multiple of the Higgs charge, then we find a perimeter law even in the presence of vortices.

This discussion closely parallels the calculation of Callan, Dashen, and Gross[11] for the continuum Abelian Higgs model. But these results also agree with those of the high temperature expansion (cf. (2.8)). Here then is a specific example of a case in which strong coupling lattice confinement has the same physical origin as confinement by instantons in the continuum. Both follow from the compact nature of the symmetry.

Next, we would like to examine the behavior of Γ_q in the limits where κ or β approach zero or infinity. Consider first the limit corresponding to the x-y model, $\beta \to \infty$, κ finite. This limit is easily implemented in

(2.12) where we see that as $\beta \to \infty$, $\Gamma_q \to 1$ independent of κ. We do not even have any residual perimeter effects. This is understandable; as $\beta \to \infty$, the gauge fields are frozen and we can write $\theta_\mu = \Delta_\mu \Lambda$. Hence, $\oint \theta_\mu dx_\mu = 0$. (Note that there is no contradiction with the existence of vortices in this limit; the physical (and gauge invariant) x-y spin angle is $\chi - \Lambda$, not just Λ.) This naive limiting behavior of Γ_q may well be modified if we actually do a renormalization group analysis and if phases II and/or III exist for finite β as in Fig. 1. The precise behavior of Γ_q depends on the behavior of $\beta_{eff}(L)$, the running coupling constant as a function of distance, but Γ_q is not expected to fall as strongly as e^{-P}. This qualitatively different behavior of Γ_q can be used to distinguish phases II and III from phase I; in particular it discriminates between the two plasma phases, I and II.

The pure gauge theory limit, $\kappa \to 0$, β finite, is also simple to analyze. The behavior of Γ_q can be deduced from (2.9) (or from (2.6)) with the result that $\Gamma_q \sim e^{-A}$ for any q. Of course this confinement has nothing to do with compactness of the gauge group or with vortices. It is simply due to the fact that the Coulomb potential in one space dimension is linear. Finally, we can consider the two $m \to \infty$ limits. For $\kappa \to \infty$, β fixed, the leading behavior of Γ_q is given by (2.15), while for $\beta \to 0$, κ fixed, $\Gamma_q \to 0$ for non-integer q.

C. The Background Field

Experience with the Schwinger model[12] and the continuum Abelian Higgs model[11] suggests that, in two dimensions, there are different, orthogonal universes corresponding to different constant background fields. (These are referred to as different θ-vacuua.) To realize this possibility in our lattice formulation, we must depart from spherical boundary conditions to allow for changes of phase as the lattice is traversed. To this end, we suppose our lattice is a square plane and we choose "free" boundary conditions; that is, we will integrate independently over all the variables, θ_μ, on the boundary of our lattice.

To induce a background field, we place an external current, J_μ, around the boundary of the lattice. Choose a closed current loop of magnitude y_0. The partition function for this system can be derived by beginning with the usual Lagrangian (say, (2.7)) and adding the term $i\Sigma J_\mu \cdot \theta_\mu$. At this point it is clear that, just as in the calculation of the Wilson gauge loop, Z_0 must be restricted to a value c/λ, with c an integer, in order to retain consistently the U(1) character of the gauge group. With this in mind, we may use (2.10) to Fourier expand Z and derive the partition function in terms of the vortices. Choose the gauge $\chi = 0$. We then have

$$Z = \sum_{\{a_\mu, B\}} \int_{-\infty}^{\infty} \delta\ell_\mu \, \delta z \, \delta\theta_\mu \exp\left[\Sigma - \frac{1}{2\kappa}\ell_\mu^{\,2} + i\ell_\mu(-\theta_\mu + 2\pi a_\mu)\right.$$
$$\left. - \frac{1}{2\beta} z^2 + iz(\epsilon_{\mu\nu}\Delta_\mu\theta_\nu + 2\pi B) + iz_0\epsilon_{\mu\nu}\Delta_\mu\theta_\nu\right] \qquad (2.16)$$

where we have used Stokes theorem to re-express the last term.

After integrating over $\{\theta_\mu\}$ and $\{\ell_\mu\}$ we can write (2.16) as

$$Z = \sum_{\{p\}} \int_{-\infty}^{\infty} dz \exp\left[\Sigma - \frac{1}{2\kappa}\left(\epsilon_{\mu\nu}\Delta_\nu z - J_\mu\right)^2 - \frac{1}{2\beta} z^2 + i2\pi z p - i2\pi z_0 p\right] . \quad (2.17)$$

As in the calculation of the Wilson loop integral, the extra J_μ term in (2.17) will give rise to a vortex independent term, proportional to the perimeter of the space, in the free energy, which will cancel in the calculation of correlation functions. Aside from this term we see that we have a constant background magnetic field z_0 coupled to the vortices, which corresponds to a $\theta \neq 0$ vacuum. This derivation clearly shows that spherical boundary conditions imply a $\theta = 0$ vacuum.

To help understand the effect of the background field on the physics we can compute Γ_q with these boundary conditions. The calculation is analogous to that leading to (2.15), and we find (in phase I of Fig. 1)

$$\Gamma_q \propto \left[\frac{1+2e^{-2\pi^2\beta}\cos 2\pi(z_0+Q)}{1+2e^{-2\pi^2\beta}\cos 2\pi z_0}\right]^A . \tag{2.18}$$

For values of Q such that $\cos 2\pi(z_0+Q) < \cos 2\pi z_0$, Γ_q falls exponentially with increasing area and we have confinement. For values of Q such that $\cos 2\pi(z_0+Q) = \cos 2\pi z_0$, perimeter terms will dominate and we have freedom. If Q is such that $\cos 2\pi(z_0+Q) > \cos 2\pi z_0$, Γ_q grows exponentially with A, the quarks are forced to the edge of space, and we have exile. The situation is summarized in Fig. 2. Freedom is evidently a rather special condition.

III. THREE DIMENSIONS

In three (and greater) dimensions qualitatively new features appear which are absent in two dimensions. We begin our discussion by considering the theory with charge $\lambda = 1$. (The theory with $\lambda > 1$ is quite similar, although there is one additional complication. This will be discussed fully below.)

Recall the dual form of (1.2) in three dimensions[1] (we assume spherical boundary conditions):

$$Z = \sum_{\{J_\lambda\}} \int_{-\infty}^{\infty} \delta A_\lambda \exp\left[\Sigma - \frac{1}{4\kappa}(\epsilon_{\sigma\rho\lambda}\Delta_\rho A_\lambda)^2 - \frac{1}{2\beta}A_\lambda^{\ 2} + i2\pi J_\lambda A_\lambda\right]$$

$$= Z_0 \sum_{\{J_\lambda\}} \exp\left[\Sigma - 4\pi^2\kappa J_\mu(j)D_{\mu\nu}(j-k;\, m^2)J_\nu(k)\right] . \tag{3.1}$$

$D_{\mu\nu}$ is the three dimensional lattice Green's function defined by

$$D_{\mu\nu}(j - k; m^2) \equiv \left[\delta_{\mu\nu} - \frac{\Delta_\mu \Delta_\nu}{m^2}\right]_j D(j - k; m^2)$$

with (3.2)

$$(-\Delta_\mu^{\ 2} + m^2)D(j - k; m^2) = \delta_{jk}$$

and $m^2 = \kappa / \beta$. This form was derived from (1.2) by choosing the gauge $\chi = 0$. The J_λ are associated with the links of the dual lattice and represent the vortex strings of our model. The sum runs over all possible configurations of currents whose components are integer valued. Some of these configurations have $Q \equiv \Delta_\lambda J_\lambda \neq 0$, which may be interpreted as a monopole density.[1] Point monopoles exist because the gauge fields are compact.[1,13] Thus, the topological excitations in three dimensions are closed vortex rings and vortex strings which end on monopoles.

A. Description of the Phases in Terms of the Topological Excitations

Suppose that m^2 is finite and let us examine the behavior of the system as a function of κ. Since $D((j - k); m^2) \propto e^{-m|j - k|}$ for $|j - k| \gtrsim 1$, we may, for this discussion, approximate D by retaining only its diagonal term. Let us suppose also that the temperature of the system is low enough that we may neglect values of $|J_\mu| \geq 2$. In that case the sum over $\{J_\lambda\}$ in (3.1) may be thought of as a sum over closed vortex strings and open strings ending on monopoles. (When there is a significant probability of strings overlapping (i.e. $|J_\mu| \geq 2$) a sum over all possible string configurations is not the same as the sum over $\{J_\lambda\}$.) Now, when $m^2 = \infty$, the partition function (3.1) is trivial and there is clearly no phase transition. But when m^2 is finite there are non-trivial interactions (for example, from the $\Delta_\mu J_\mu$ term) and hence the possibility of a phase transition. To understand what to expect qualitatively, we note that we may associate a pseudoenergy

$$4\pi^2 \kappa D(0, m^2) L \tag{3.3a}$$

with a closed vortex loop of total length L, and

$$4\pi^2 \kappa D(0; m^2) \left[L + \frac{2}{m^2} \right] \tag{3.3b}$$

with an open string of length L which ends on monopoles. We now ask whether it is likely or unlikely to find a closed or an open string of a given length in the system. The entropy for such an object is just the logarithm of the number of different possible configurations. For an open string one end of which is fixed at some point in the (dual) lattice, this is just the number of non-repeating random walks of length L. By non-repeating we mean that once a link has been traversed it must not be stepped along again. Note that this is somewhat less restrictive than a self-avoiding walk--we allow a site of the dual lattice which has previously been stepped on to be stepped on again, but the traveller must proceed in a hitherto unexplored direction.[14] For a closed loop of given length, fixing some point of the circumference, the number of configurations is equal to the number of non-repeating random walks which return to this point. Unfortunately, very little seems to be known about non-repeating walks. However, the very closely related problems of self-avoiding and closed self-avoiding walks have been much studied. Since for large L the leading behavior of non-repeating and self-avoiding walks is likely to be similar, we will use results on the latter as a guide in what follows.

The number of possible self-avoiding random walks of L steps, is known to behave like $\mu_1^L f_1(L)$ where $\left[f_1(L)\right]^{1/L} \to 1$ as $L \to \infty$.[15] μ_1 and f_1 both depend on dimension and lattice type. The number of possible self-avoiding random walks of length L which return to the origin has the behavior $\mu_2^L f_2(L)$ where again $\left[f_2(L)\right]^{1/L} \to 1$ as $L \to \infty$.[15] (Domb[16] conjectures that $f_1(L)(f_2(L))$ is power behaved with a positive (negative) exponent as $L \to \infty$.) For a fixed number of dimensions and lattice type it can be proved that $\mu_1 = \mu_2$.[15] (For a three-dimensional simple cubic lattice, μ_1 ($= \mu_2$) has been estimated to be 4.6826.[15]) From these results it is quite likely that the leading behavior for the number of non-repeating walks and closed non-repeating walks is the same: $e^{L \ln \mu + \mathcal{O}(\ln L)}$ with μ being close to the self-avoiding value. The ln L corrections should differ for open and closed walks.

We can now calculate the free energy for open and closed strings of length L. Up to an overall factor, it is

$$\mathcal{F}_c = \left(4\pi^2 \kappa D(0; m^2) - \mu\right) L + \mathcal{O}(\ln L); \quad \text{closed loops} \tag{3.4a}$$

$$\mathcal{F}_o = \left(4\pi^2 \kappa D(0; m^2) - \mu\right) L - 8\pi^2 \beta D(0; m^2) + \mathcal{O}(\ln L); \quad \text{open strings.} \tag{3.4b}$$

For κ large (low temperatures), $\mathcal{F}$ has its minimum at $L = 0$. For small enough κ (high temperatures), the minimum of $\mathcal{F}$ occurs at $L = \infty$.

The transition takes place suddenly at a temperature determined by $\mu = 4\pi^2 \kappa_c D(0; m^2)$. This is a new phase transition at finite m^2 which does not occur in two dimensions. Physically, the low temperature phase described here consists of massive spin waves (Z_0 in (3.1)), and, in addition, small vortex rings and elementary dumbells, i.e., monopole-anti-monopole pairs with one (or a few) vortex links joining them. Larger topological structures have an exponentially smaller probability of existing. At some $\kappa = \kappa_c$, the entropy term in (3.4) dominates, and it suddenly becomes likely to have arbitrarily large vortex rings and strings. Note that because of the result $\mu_1 = \mu_2$, the transition temperature is the same for both open and closed strings. This transition is similar to the vortex dissociation transition of the d = 2 x-y model. However, in two dimensions for large enough m^2 (more precisely, in the region of phase I, Fig. 1) only the analogue of the high temperature (spaghetti) phase exists. The low temperature (alphabet soup) phase is absent.[17]

In Fig. 3 we plot the expected phase structure for this model. Phases I through VI are analogous to the phases with the same numbers in Fig. 1 for the d = 2 case. The new phase described above is phase VII. Notice that the relative numbers of small vortex loops to elementary dumbells in this phase decreases as m^2 is increased for fixed κ (see 3.3). Phases II and III correspond to the high and low temperature phases of the d = 3 x-y model. In phase III we expect to find massless spin waves plus small vortex loops. Phase II is characterized by massless spin waves plus arbitrarily large vortex loops. Note that in the limit $\beta \to \infty$ there are no monopoles. An estimate for the position of the critical line AB may be obtained using a formula analogous to (3.4). The long range interactions between the vortex string bits in the x-y phase (i.e. the fact that $D(j-k; 0) \sim \frac{1}{|j-k|}$) contributes an additional power-behaved term to the energy of a vortex loop of length L which is proportional to $L^{\mathcal{P}}$. But $\mathcal{P}$ is expected to be less than one,[18] and so, in the context of our crude approximation, will not affect the value of κ_c. For this reason B is a quadracritical point. (As usual, this naive picture may be refined by a more careful renormalization group analysis. It could happen, for example, that B is actually split into two tricritical points with the left terminus of the line BE displaced above the right terminus of AB, and resting on the line BD.) As in two dimensions, the $m^2 = \infty$ phases IV and V are trivial. Phase VI is again a pure compact gauge theory phase, but in three dimensions it is not trivial. In this phase[13, 18, 19, 20] the topological excitations are free monopoles without strings. From (3.2), we see that as κ decreases for fixed β, strings cost less and less energy to make relative to monopoles. Ultimately, in the limit that $\kappa \to 0$, the vacuum becomes filled with strings and the only excitations we see are the monopoles. Remember, though, that as in two dimensions, this is a singular limit of (3.2) and requires an extra gauge choice in the integral of (3.1).

B. The Gauge Loop Integral

We now compute the behavior of fractional- and integer-charged gauge loops in this model. The comments made in the last section about the qualitative similarity of the periodic quadratic and full compact theories apply here as well. Moreover, to compute fractionally charged gauge loops, it is again important to work with the Higgs theory with $\lambda > 1$.

Let us first compute Γ_c using the three-dimensional version of (2.6). For $\bar{\kappa}$, $\bar{\beta}$ small enough and $\bar{\kappa}/\bar{\beta}$ sufficiently large, we will be in phase I. Using a high temperature expansion as in section II, we will have $\Gamma_q \sim e^{-A}$ for $c < \lambda$ and $\Gamma_\lambda \sim e^{-P}$, where A is the minimum area enclosed by the gauge loop, and P is its perimeter. Thus, as in two dimensions, we have confinement for fractional charges (c/λ) and freedom for integer charges. In the limit $\bar{\kappa} = 0$, we have the pure gauge theory (phase VI), and, as discussed elsewhere,[13, 20] confinement for all charges.

To understand these results in terms of the topological excitations, and to compute Γ_q when $\bar{\kappa}$ and $\bar{\beta}$ are not very small, it is useful to use the dual form of the partition function. We start with the periodic quadratic form (2.7b). In three dimensions we have

$$\Gamma_q = \frac{1}{Z} \widetilde{\sum_{a_\mu, B_{\mu\nu}}} \int_{-\infty}^{\infty} \delta\chi\, \delta\theta_\mu \exp\Big[\Sigma - \frac{\kappa}{2}(\Delta_\mu \chi - \theta_\mu + 2\pi a_\mu)^2 - \frac{\beta}{4}(\epsilon_{\mu\nu\lambda}\epsilon_{\lambda\rho\sigma}\Delta_\rho \theta_\sigma + 2\pi B_{\mu\nu})^2 + i\sum q_\mu \theta_\mu\Big] \tag{3.5}$$

where a_μ takes on integer values, $B_{\mu\nu}$ takes on values which are integer multiples of λ, and the tangent vector, q_μ, is defined as in the two-dimensional case. Its non-zero components have the value c/λ. It will simplify the tensor algebra to define B_λ by $B_{\mu\nu} = \epsilon_{\mu\nu\lambda} B_\lambda$. The Gaussian integral may be performed as before, leading to an expression analogous to (2.10):

$$\Gamma_q = \frac{Z_0}{Z} \widetilde{\sum_{a_\mu, B_{\mu\nu}}} \exp\left[\frac{1}{2\beta} \Sigma \left(iq_\nu(j) - 2\pi\beta\epsilon_{\nu\lambda\mu}\Delta_\lambda J_\mu(j)\right) D_{\nu\nu'}(j-k;\, m^2)\right.$$

$$\left.\left(iq_{\nu'}(j) - 2\pi\beta\epsilon_{\nu'\rho\sigma}\Delta_\rho J_\sigma(j)\right)\right] \exp\left[\Sigma\left(2\pi i q_\nu(j) a_\nu(j) - 2\pi^2\beta J_\nu(j)^2\right)\right] \tag{3.6}$$

where we define the topological current $J_\lambda \equiv B_\lambda + \epsilon_{\lambda\mu\nu}\Delta_\mu a_\nu$. The Green's function $D_{\mu\nu}$ is defined by the obvious extension to three dimensions of Eq. (2.11). Since the current J_λ is not divergence free, the gradient terms in $D_{\mu\nu}$ must be retained. Bearing this in mind, we arrive at an expression similar to (2.12),

$$\Gamma_q = \frac{Z_0}{Z} \exp\left[-\frac{1}{2\beta}\Sigma D(j-k;\, m^2) q_\nu(j) q_\nu(k)\right]$$

$$\widetilde{\sum_{\{a_\lambda, B_\nu\}}} \exp\left[-2\pi^2\kappa\, \Sigma J_\mu(j) D_{\mu\nu}(j-k;\, m^2) J_\nu(k)\right] \tag{3.7}$$

$$\exp\left[-2\pi i \Sigma q_\lambda(j)\left(m^2 a_\lambda(k) + \epsilon_{\lambda\mu\nu}\Delta_\mu B_\nu(k)\right) D(j-k;\, m^2)\right] .$$

Before proceeding with the evaluation of (3.7), it is appropriate to mention the differences between the three-dimensional Higgs theory with $\lambda = 1$ and the theory with $\lambda > 1$. The partition function is obtained from the numerator (3.7) by setting all q_λ to zero. The resulting expression has in addition to the factor Z_0, the usual (quadratic) factor describing the interactions of the topological singularities. The only difference is that since $B_{\mu\nu}$ is an integer multiple of λ, B_μ is λ times the corresponding B_μ in the $\lambda = 1$ system. Since $\Delta_\mu J_\mu = \Delta_\mu B_\mu$ it is clear that the monopoles in the $\lambda > 1$ theory have a charge λ relative to the possible values of flux (all integers) contained in the full current. Hence, the basic topological excitations in this case are closed vortex rings of (any) integer vorticity and open vortex strings whose flux is an integer multiple of λ and which terminate on monopoles whose charge is an integer multiple of λ.

Now, at the level of discussion associated with equations (3.4), we might expect to have an extra phase when $\lambda > 1$. The reasoning would be that since a minimum flux of λ is required to produce an open, string equation (3.4b) becomes modified to read

$$\mathscr{F}_o = 4\pi^2\lambda^2\kappa D(0;\, m^2) - \mu L - 8\pi^2\lambda^2\beta D(0;\, m^2) + \mathscr{O}(\ln L)$$

while (3.4a) remains the same. Thus the transition to large open strings would seem to occur at a higher temperature than the transition to large closed strings. One might therefore expect an intermediate phase, lying between regions I and VII in Fig. 3 which would have arbitrarily large closed strings but small open strings. Such a phase would be distinguished from phase I by the fact that the only important contributions to Z would be those in which one had a local balance of monopole charge.

That such a phase is not likely to exist may be understood by remembering that all the flux from the monopole need not pass along a single dual lattice link. At any temperature where large strings of a single flux are likely, configurations such as those of Fig. 4 will allow monopole-antimonopole pairs to become widely separated with good probability. Such a configuration may be viewed as a superposition of a single open string of flux λ and $\lambda-1$ closed strings of flux one. This configuration has a lower free energy (as well as a lower energy) than would a single string of flux λ joining the same monopole-antimonopole pair. While this argument seems quite convincing, only a renormalization group calculation can decide the issue definitively. One should therefore bear in mind the possibility, however unlikely, of an intermediate phase between I and VII of Fig. 3.

Now let us return to a discussion of Eq. (3.7). Consider the term in the exponent involving $q_\nu(j)q_\nu(k)$. Since, for finite m^2, $D(j - k; m^2)$ is short-ranged, this term contributes to $\ln \Gamma$ a piece proportional to the perimeter. Consequently, if we neglect topological excitations, we find $\ln \Gamma \propto$ length of loop for all values of $q = c/\lambda$. To include the effects of the topological excitations, it is useful to first use Stokes' theorem to rewrite the last term in the expression for Γ. To this end, we note that, since $\Delta_\lambda q_\lambda = 0$, we may write q_ν as a curl

$$q_\lambda = \epsilon_{\lambda\mu\nu} \Delta_\mu Q_\nu$$

where Q_ν is a vector associated with links of the dual lattice. Choose a surface bounded by the gauge loop. Then we may think of Q_ν as a vector normal to each elementary plaquette of this surface. Its value, like the components of q_λ, is c/λ. (To all other plaquettes, we may assign $Q_\nu = 0$.) In the last term of Eq. (3.7), we may sum by parts and make the replacement

$$q_\nu(j)a_\nu(k) \rightarrow -Q_\lambda(j)\epsilon_{\lambda\mu\nu}\Delta_\mu a_\nu(k)$$

$$q_\nu(j)\epsilon_{\nu\rho\sigma}\Delta_\rho B_\sigma(k) \rightarrow Q_\lambda(j)\left(\Delta_\lambda(\Delta_\mu B_\mu(k)) - \Delta^2 B_\lambda(k)\right) \quad .$$

As before, it is useful to consider the behavior of Γ_q for m^2 large. Then (3.7) may be written as

$$\Gamma_q = \frac{Z_0}{Z} \exp\left[\frac{1}{2\kappa} \Sigma q_\nu(j)^2\right] \widetilde{\sum_{\{a_\lambda, B_\nu\}}} \exp\left[-\Sigma\left(2\pi^2\beta J_\mu(j)^2 + 2\pi i Q_\mu(j) J_\mu^{(a)}(j)\right)\right] \quad (3.8)$$

where $J_\mu^{(a)} \equiv \epsilon_{\mu\nu\lambda}\Delta_\nu a_\lambda$.

Let us recall the interpretation of the topological excitations represented by J_μ. The contribution due to $\epsilon_{\lambda\mu\nu}\Delta_\mu a_\nu$ describes closed vortex loops carrying (arbitrary) integer flux. The contribution due to B_λ may be thought of as links between vertices of the dual lattice where there may or may not be located monopoles (depending on whether $\underset{\sim}{\Delta} \cdot \underset{\sim}{B} \neq 0$ or $\underset{\sim}{\Delta} \cdot \underset{\sim}{B} = 0$ at that vertex.) The flux associated with each B_λ is an integral multiple of λ. Thus, the last term in (3.8) measures the net "closed loop" flux which passes through a surface enclosed by the gauge loop. Note that this quantity is invariant under a change in the definition of the surface on which $Q_\lambda \neq 0$. (Notice also that $Q_\lambda(j)J_\lambda^{(B)}(j) = Q_\lambda(j)B_\lambda(j)$ is always an integer so, in fact, we may write $Q_\lambda(j)J^a(j) = Q_\mu(j)J(j)$ in (3.8).)

Now, although (3.8) resembles the two-dimensional result (2.14), it is more difficult to estimate, particularly when β is not large. We can, however, argue qualitatively as follows. Suppose we are in the Higgs phase, phase VII of Fig. 3. In this phase, the flux loops are small. To obtain a non-zero (and non-integral) contribution to $\Sigma\, Q_\lambda(j) J_\lambda^{(a)}(j)$, we need a flux loop which encircles the gauge loop like the links of a chain. Since we have in this phase only small loops, they will contribute to a perimeter effect in the limit of very large P. Consequently, in phase VII, we always will get a perimeter law for ln Γ.

Now suppose we are in phase I. Since vortex loops can have arbitrary size, it will be highly probable to find vortex loops which penetrate any element of the surface on which $Q_\lambda \neq 0$ but which still encircle the perimeter of the gauge loop. This will clearly give rise to an area law behavior for non-integer q. To see this heuristically, we approximate phase I as a phase of essentially uncoupled vortex string bits. (This is certainly not a very good approximation but has the essential feature we want. It is probably not difficult to produce a better, but still tractable approximation.) If we ignore the terms which contribute perimeter effects, then the evaluation of (3.8) leads to a form like (2.15) and we have an area law for non-integer q_μ. When q_μ is an integer, the last term in (3.8) has no effect, and the perimeter terms dominate in the exponent. Thus the different large distance behaviors of Γ_q for non-integer q can be used to discriminate between phase VII and phase I. Notice that the monopoles played no essential role in the preceding argument. (Paradoxically, it is evidently the monopoles which cause confinement in the pure compact gauge theory in three dimensions.[13] We comment on this below.)

We now wish to consider the $m^2 = 0$ limits of (3.7). This will put us in phases II or III (x-y model), or VI (pure gauge theory) depending on how the limit is taken. Consider first the limit $\beta \to \infty$, κ fixed (x-y model limit). It is useful to rewrite the expression (3.7):

$$\Gamma_q = \frac{Z_0}{Z} \exp\left[-\frac{1}{2\beta} \Sigma q_\nu(j) q_\nu(k) D(j-k; m^2)\right] \widetilde{\sum_{\{a_\mu, B_{\mu\nu}\}}} \exp\left\{-2\pi^2 \Sigma\left[\kappa J_\mu(j) J_\mu(k) \right.\right. \tag{3.9}$$

$$\left.\left. + \beta \Delta_\mu J_\mu(j) \Delta_\nu J_\nu(k)\right] D(j-k; m^2) - i2\pi \Sigma \underset{\sim}{\Delta} \cdot \underset{\sim}{Q}(j) D(j-k; m^2) \underset{\sim}{\Delta} \cdot \underset{\sim}{J}(k) + o(m^2)\right\} .$$

The last term $o(m^2)$ in (3.9) disappears in any $m^2 \to 0$ limit and may be ignored. From this expression it is clear that when $\beta \to \infty$ we obtain a non-zero contribution to Γ_q (or Z) only if $\underset{\sim}{\Delta} \cdot \underset{\sim}{B} \equiv \underset{\sim}{B} = \underset{\sim}{\Delta} \cdot \underset{\sim}{J} \equiv J = 0$. This is just the statement that there are no monopoles in the d=3 x-y model. Hence, the numerator of (3.9) becomes independent of Q_λ and $\Gamma_q \to 1$ for all q. As in two dimensions, this is a reflection of the fact that when $\beta \to \infty$ the non-integer part of $F_{\mu\nu}$ is frozen so that $\theta_\mu = \Lambda_\mu \Lambda$.

Next, consider the limit $\kappa \to 0$, β fixed, the pure gauge theory. In this limit the expression (3.7) is not defined, since the Higgs field disappears from the problem. We must go back to (3.5) and make an additional gauge choice to render Γ_q finite. This is easily done, and we find:

$$\Gamma_q = \frac{Z_0}{Z} \exp\left[-\frac{1}{2\beta}\Sigma q_\nu(j)q_\nu(k)D(j-k;\ 0)\right]\sum_{\{m\}} \tag{3.10}$$

$$\exp\left[-2\pi\Sigma D(j-k;\ 0)(\pi\beta m(k)m(j) + i\underset{\sim}{\Delta}\cdot\underset{\sim}{Q}(j)m(k))\right]$$

where $m(j) \equiv \underset{\sim}{\Delta}\cdot\underset{\sim}{B}(j) = \underset{\sim}{\Delta}\cdot\underset{\sim}{J}(j)$ is the monopole density. This expression has been discussed by Polyakov.[13] According to his analysis, we have in this limit $\Gamma_q \sim e^{-A}$ for all finite β and for both integral and non-integral q. This result is due to the monopoles, which do exist in this limit, interacting with the scalar $\underset{\sim}{\Delta}\cdot\underset{\sim}{Q}$ according to the expression (3.10). Note in particular that $D(j-k;\ 0)$ is power-behaved, so that the effect of monopoles through the whole space is important for the confinement.

Now, as we remarked above, this limit is rather singular, since, when $\kappa = 0$, the Higgs term in the Lagrangian disappears. But it is precisely this term which gives rise to $J_\lambda^{(a)}$ and which is therefore responsible for the confinement of fractional charge in phase I. Furthermore, in the pure gauge theory $\Gamma_q \sim e^{-A}$ even for integer charge, a result which is clearly associated with the complete disappearance of the Higgs coupling, since for any nonvanishing κ, no matter how small, $\Gamma_q \sim (\kappa)^P$, according to the high temperature expansion. Nevertheless, we can get some additional insight into the structure of the pure gauge theory vacuum by the following heuristic reasoning: as κ decreases for fixed β, it becomes easier and easier to make both closed and open vortex strings. This is because the oscillations of the Higgs field are less damped. (See e.g. (3.9)). But the easier it becomes to produce vortex strings, the faster Γ_q will decrease, since there will be larger and larger fluctuations in the amount of vortex penetration through the gauge loop. In the limit $\kappa = 0$, we can therefore think of the vacuum as a state which is filled with vortex strings which cost no energy to produce, and which cause confinement so that $\Gamma_q \sim e^{-A}$. The shortcomings of this description are evident, and the reader is cautioned to bear them in mind.

IV. FOUR (AND GREATER) DIMENSIONS

The phase diagram in four and more dimensions is quite similar to Fig. 3, the diagram in three dimensions, but the topological excitations which induce the phase transitions are somewhat different. In four dimensions, the dual form of the partition function in the periodic quadratic approximation was given in reference 1, Eq. 47. After performing the Gaussian integration on this expression, one obtains

$$Z = Z_0 \sum_{\{B_{\rho\sigma}, a_\rho\}} \exp -2\pi^2\kappa \left[J_{\rho\sigma}(j) J_{\rho\sigma}(k) + \frac{2}{m^2} Q_\rho(j) Q_\rho(k) \right] D(j-k; m^2) \qquad (4.1)$$

where $Q_\rho \equiv \Delta_\sigma J_{\rho\sigma}$ and D is the four-dimensional lattice Green's function. As usual, Z_0 is the partition function for a massive spin wave. The topological current density is

$$J_{\rho\sigma} = \epsilon_{\rho\sigma\mu\nu}(B_{\mu\nu} + \Delta_\mu a_\nu - \Delta_\nu a_\mu) \quad . \qquad (4.2)$$

In the theory with a Higgs particle of charge λ, $B_{\mu\nu}$ takes on integer multiple values of λ, while a_μ takes on all integer values. ($B_{\mu\nu}$ and a_μ are simply the generalization to four dimensions of the quantities appearing in Eq. (3.5).) Recall the interpretation[1] of the topological excitations described by $J_{\rho\sigma}$ as closed two-dimensional manifolds and open manifolds bounded by monopole current loops of density Q_ρ. These closed and open surfaces are obvious generalizations of the closed and open strings existing in three dimensions.

For finite m^2, we expect phases analogues to the phases VII and I of Fig. 3. To see this, we need to argue that the number of closed or open two-dimensional surfaces of total area A has for large A the leading behavior $e^{\mu A}$. Consider for instance open connected surfaces. Draw a link from the center of a plaquette to one of its neighbors. Continuing in this way it is possible to associate a connected path with the surface. Sometimes the path will be a single linear chain and sometimes it will have branches. Moreover, it is clear that in general there will be many such paths associated with a given surface. On the other hand, up to overall orientations, there is only one two-dimensional surface associated with each connected path, by the above construction. Now, the number of configurations of a random walk of L steps with q branches is of order $e^{\mu L}$ (modulo powers of L). Assuming that summing over the number of branches does not change this leading behavior (except, perhaps, to change the value of μ) we conclude that the number of configurations of open surfaces with area A, N(A), does not grow faster than $e^{\mu A}$ for large A (modulo powers of A). To get a lower bound on N(A), we consider adding a single plaquette to configurations whose area is A with $A >> 1$. Then N(A + 1) is

$$N(A+1) = N(A) + \overline{p}(A) \tag{4.5}$$

where $\overline{p}(A)$ is the average perimeter for an open surface of total area A. It is obvious that $\overline{p}(A)$ does not decrease as A increases, so that $\overline{p}(A) \geq c > 1$.

Using this in (4.5) we conclude that $N(A) \geq e^{cA}$. Since N(A) is bounded from above and below by an exponential (modulo powers) its leading behavior is exponential. As in three dimensions, restriction to closed or open surfaces is not expected to significantly affect the leading behavior of N(A).

The arguments for phase transitions at finite m^2 now follow those of the last section, balancing entropy and energy and looking for a minimum of the free energy as a function of A. In region VII we expect to find only very small closed or open surfaces, in addition to massive spin waves. Phase I will contain arbitrarily large closed and open surfaces plus the ubiquitous spin waves. From arguments analogous to those of Section III, we do not expect any intermediate between I and VII. The other phases also have properties which are direct generalizations from three dimensions. Moreover, it is clear that this pattern of generalization continues for $d > 4$.

We can now study the behavior of the gauge loop integral, Γ_q. First we note that for finite m^2 and κ very small, $\Gamma_q \sim e^{-A}$ for non-integer q and $\Gamma_q \sim e^{-P}$ for integer q, just as in three dimensions. As before, this result follows from a high temperature expansion using the four dimensional version of (1.1). Next, we express Γ_q in terms of the topological excitations. We start with the periodic Gaussian approximation

$$\Gamma_q = \frac{1}{Z} \widetilde{\sum_{\{a_\mu, B_{\mu\nu}\}}} \int \delta\chi\, \delta\,\theta_\mu \exp\Big[\Sigma - \frac{\kappa}{2}(\Delta_\mu\chi - \theta_\mu + 2\pi a_\mu)^2 - \frac{\beta}{4}\big(\tfrac{1}{2}\epsilon_{\mu\nu\beta_1\beta_2}\epsilon_{\beta_1\beta_2\rho\sigma}\Delta_\rho\theta_\sigma + 2\pi B_{\mu\nu}\big)^2 + i\sum q_\mu\theta_\mu\Big] \tag{4.6}$$

where $B_{\mu\nu} = -B_{\nu\mu}$, an integral multiple of λ, and q_μ, as before, is a "tangent vector" of magnitude c/λ. Proceeding with the integration in a manner analogous to the discussions in two and three dimensions, we find (cf. 3.7)

$$\Gamma_q = \frac{Z_0}{Z} \exp\left[-\frac{1}{2\beta}\Sigma D(j-k;\, m^2) q_\nu(j) q_\nu(k)\right]$$

$$\sum_{\{a_\nu, B_{\mu\nu}\}}^{\sim} \exp\left[-\beta\pi^2 \Sigma J_{\mu\nu}(j)^2 + 2\pi^2\beta\Sigma D(j-k;\, m^2)\Delta_\lambda J_{\lambda\nu}(j)\Delta_\mu J_{\mu\nu}(k)\right] \qquad (4.7)$$

$$\exp\left[-2\pi i \Sigma q_\nu(j)(m^2 a_\nu(k) + \Delta_\mu B_{\mu\nu}(k)) D(j-k;\, m^2)\right]$$

where $J_{\mu\nu} \equiv B_{\mu\nu} + \Delta_\mu a_\nu - \Delta_\nu a_\mu$. As before, for finite m^2, the leading piece of the first exponent (involving $q_\nu(k)q_\nu(j)$) is proportional to the length of the gauge loop. The last exponent may be written as an integral over the area enclosed by the loop. Define an antisymmetric tensor $Q_{\sigma\tau}$ by $q_\mu = \epsilon_{\mu\rho\sigma\tau}\Delta_\rho Q_{\sigma\tau}$. Then in the last term in (4.7) we may sum by parts and make the replacement

$$q_\nu(j)(m^2 a_\nu(k) + \Delta_\mu B_{\mu\nu}(k)) \rightarrow Q_{\sigma\tau}(j)\,\epsilon_{\sigma\tau\rho\nu}\Delta_\rho(m^2 a_\nu(k) + \Delta_\mu B_{\mu\nu}(k)) \quad .$$

As in the discussion of the three dimensional case, it is useful to consider the large m^2 limit where we find,

$$\Gamma_q = \frac{Z_0}{Z} \exp\left[-\frac{1}{2\kappa} \Sigma q_\nu(j)^2\right]$$

$$\widetilde{\sum_{\{a_\nu, B_{\mu\nu}\}}} \exp\left[-\beta\pi^2 \Sigma J_{\mu\nu}(j)^2\right] \exp\left[2\pi i \Sigma Q_{\sigma\tau}(j) \epsilon_{\sigma\tau\rho\nu} \Delta_\rho a_\nu(j)\right] .$$

Now the last exponent represents the net intersection of closed topological surfaces with the gauge loop. Analogous to the three dimensional case, these surfaces contribute at most a perimeter effect except in those phases where surfaces of arbitrarily large extent are likely. Thus, in phase VII (the Higgs phase where topological excitations are small), we expect $\Gamma_q \sim e^{-P}$. However, in phase I where we have a plasma of large closed (and opened) surfaces, $\Gamma_q \propto e^{-A}$ for non-integer q and $\Gamma_q \propto e^{-P}$ for integer q.

V. DISCUSSION

It is perhaps worthwhile to summarize our picture of the different possible phases in three dimensions (Fig. 3). The limiting cases IV, V, and VI correspond, respectively, to theories of non-interacting vortex loops and vortices terminating on monopoles, of infinitely massive, non-interacting spin waves, and of the pure (compact) gauge theory. Phases II and III are analogous to the phases of the xy model.[3] Wilson's loop integral is one in both cases. (But recall the two paragraphs preceding section II. C.) In phase III, topological excitations are suppressed; there are only small vortex loops. There is long range order of the Higgs fields analogous to a ferromagnetic In phase II, there is an explosion of large vortex loops which leads to a breakdown of long range order, with the appearance of a finite correlation length (mass gap).

Phase VII is another low temperature phase in which topological excitations are relatively unimportant for the large distance structure. This phase corresponds in the continuum limit to the so-called Higgs phase and appears to be a theory of a free, massive vector boson with only small topological loops and dumbbells. The Wilson loop integral obeys a perimeter law, i.e., it is proportional to the length of the loop, so there is screening of arbitrary charge. At higher temperatures, there is a transition to phase I in which one finds a plasma of arbitrarily large vortex rings and monopoles with strings which cause a kind of disordering (see below). In this respect it is similar to phase II, but now the Wilson loop integral is proportional to the area enclosed by the loop for the fractionally charged case. For integer charges, it again is proportional to the perimeter, so there is no Higgs mechanism but there is "quark trapping", i.e. confinement of the elementary, fractional charges of the theory. Our arguments indicate that, even in the theory in which the Higgs charge is not equal to the elementary unit charge ($\lambda > 1$) the dissociation of monopole-antimonopole pairs occurs at the same temperature at which very large vortex loops become likely. Hence we do not expect any phase intermediate between I and VII. In higher dimensions, the situation is expected to be analogous to the three dimensional case, and we have discussed the picture in four dimensions in some detail.

The Abelian Higgs model is the Ginzburg-Landau theory of superconductivity in which the Higgs field, ϕ, is the electron pairing field.[21] Segments of the vortex rings of our model can be thought of as penetrations of magnetic flux in a superconductor. It is worthwhile to relate our results in three dimensions to some of the known properties of superconductors. In particular, we would like to know whether our model exhibits properties of a Type I or a Type II superconductor.

Recall that in a Type I superconductor, the pairing field coherence length, ξ, is significantly larger than the magnetic field penetration depth, δ. This has the consequence that when magnetic flux penetrates the medium it prefers to do so in an extended, continuous region. Since $<\phi> = 0$ at the center of a vortex, the entire extended region becomes normal (disordered), and we have a complete breakdown of the Meissner effect. In a Type II superconductor, on the other hand, δ is significantly larger than ξ. As a result, there is a phase of the system as a function of applied magnetic field which exhibits only a partial breakdown of the Meissner effect. For a range of magnetic fields $H_{c1} < H < H_{c2}$, flux penetration occurs in relatively thin well-defined tubes separated by regions of superconductors in which $<\phi> \neq 0$. Only for $H > H_{c2}$ is there complete disordering with $<\phi> = 0$ everywhere.

To decide whether our theory represents a Type I or Type II superconductor one might try taking the naive continuum limit of our theory and identify parameters with the parameters of the Ginzburg-Landau theory. However, this procedure will not result in a correct identification of the physics. In defining our lattice theory, we have formally frozen the radial degree of freedom of the Higgs field on each lattice site. Hence, in the "classical" naive continuum limit, the coherence length, ξ, is infinite since ϕ is never zero. But there is a dynamically generated radial degree of freedom (i.e. $<\phi(i)\,\phi(j)> \neq 1$) and thus, in general, a finite ξ which, like the penetration depth, is temperature dependent. It is these dynamically meaningful quantities which will determine whether we have a Type I or Type II system.

It is possible to determine ξ and δ as a function of the bare parameters of our theory, β and κ, by doing a renormalization group calculation. In the absence of such calculations we can turn to the arguments we have presented as a guide to the physics. Let us fix m^2

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ERRATUM:

The paragraph beginning in the middle of page 49 should be replaced by the following:

Now, in passing from region III to region II we encounter a complete breakdown of the Meissner effect. The phase transition from III to II is, as we have discussed, a topological transition, but it is also the usual Wilson-Fisher phase transition, and thus in phase II we have a complete disordering of the system: $\langle\phi\rangle = 0$ everywhere. This is clearly the kind of behavior expected in a Type I superconductor. Note that in these phases the vortices can have a long-range disordering effect since they are the sources of a massless field (see, for example, the $\beta \to \infty$ limit of (3.1)). In contrast, the transition from phase VII to I does not seem to signal a complete breakdown of the Meissner effect. The dynamics of phase I is not the dynamics of normal scalar QED (for example, there is still a massive vector field) and so it is not totally disordered. Flux penetration does occur, but the flux will penetrate in thin tubes separated by regions where $\langle\phi\rangle \neq 0$. This is exactly what we expect in the mixed phase of a Type II superconductor. Naively, one expects that complete disordering of the Type II system will appear in our model only at infinite "bare" temperatures--somewhere in the elusive upper right-hand corner of Fig. 3. To see this phase emerge clearly evidently requires a renormalization group analysis.

and vary κ. Focus on values of κ near the separatices AB and BE. Furthermore, let us discuss some large but finite region of the material. For values of κ which put us in regions I or II we imagine restricting ourselves to configurations of vortex strings (we may ignore the U(1) monopoles for this discussion) such that there is some fixed (albeit almost arbitrary) net magnetic flux passing through the region under consideration.[22] With these constraints, varying κ so that we pass from region III to II or VII to I is similar to varying the external magnetic field on a superconductor from a value which allows no flux penetration to a value which does allow flux to penetrate.

Now, in passing from region III to region II we encounter a complete breakdown of the Meissner effect. The phase transition from III to II is, as we have discussed, a topological transition, but it is also the usual Wilson-Fisher phase transition, and thus in phase II we have a complete disordering of the system: $<\phi> = 0$ everywhere. This is clearly the kind of behavior expected in a Type I superconductor. Note that in these phases there is a long range attractive interaction between vortex strings with flux in the same direction and so it is easy to see why flux penetration prefers to occur in a finite extended region. In contrast, the transition from phase VII to I does not seem to signal a complete breakdown of the Meissner effect. The dynamics of phase I is not the dynamics of normal scalar QED (for example, there is still a massive vector field) and so it is not totally disordered. Flux penetration does occur, but there is no tendency for the flux lines to congregate (the force between them is exponentially damped) and so the flux will penetrate in more or less randomly distributed thin tubes separated by regions where $<\phi> \neq 0$. This is exactly what we expect in the mixed phase of a Type II superconductor. Naively, one expects that complete disordering of the Type II system will appear in our model only at infinite temperatures--somewhere in the elusive upper right-hand corner of Fig. 3. To see this phase emerge clearly evidently requires a renormalization group analysis.

Finally, it is amusing to note that our analysis suggests that the difference in critical behavior between type I and type II superconductors is just the difference between a system whose long range behavior is described by a globally invariant theory (x-y model) and one described by a locally invariant theory (Abelian Higgs model).

We turn now to a brief remark about the continuum limit of our theory. Unfortunately, it is difficult to be very precise about the correspondence of our theories with the continuum theories in the absence of renormalization group analyses. But we have seen that in two dimensions our lattice discussion is quite similar to that of Callan et al.[11] for the continuum theory. Moreover, as explained in the text, it is quite plausible that at least some of the topological excitations we have found exist in the continuum limit of at least some of the phases which our theories manifest. If these excitations do persist in the continuum, they have very interesting consequences for field theories. For example, our analysis suggests that the continuum compact Abelian Higgs model in four dimensions may have soliton solutions of two types: 1) monopole-antimonopole pairs connected by flux tubes, and 2) vortex rings.

Moreover, in our theory it is clear that the two types of solutions are related--whenever one has solutions of the first type, solutions of the second type also exist. Nambu[23] has recently shown that there are solutions of the Weinberg-Salam $SU(2) \times U(1)$ gauge theory which represent monopole-antimonopole pairs connected by a flux tube. A completely unjustified analogical leap implies that closed vortex rings may also appear in the Weinberg-Salam model. This is discussed elsewhere.[24]

Other problems deserving further consideration include a quantitative derivation of the area law in phase I of the three dimensional theory, and a more precise exposition of the relationship outlined in Ref. 1 between Higgs theories and models of spin-glasses. Of course, a renormalization group analysis of the various phases of our theory would be most interesting.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Fig. 1: Phase diagram for two dimensions (see Sec. II for discussion).

Fig. 2: Behavior of Wilson's correlation function in two dimensions in the presence of a background field (see Eq. 2.18).

Fig. 3: Phase diagram for three and higher dimensions. (See Secs. III and IV for discussion.)

Fig. 4: Typical favored configuration of vortex lines between widely separated monopoles of charge λ (in this case $\lambda = 7$) in phase I in three dimensions.

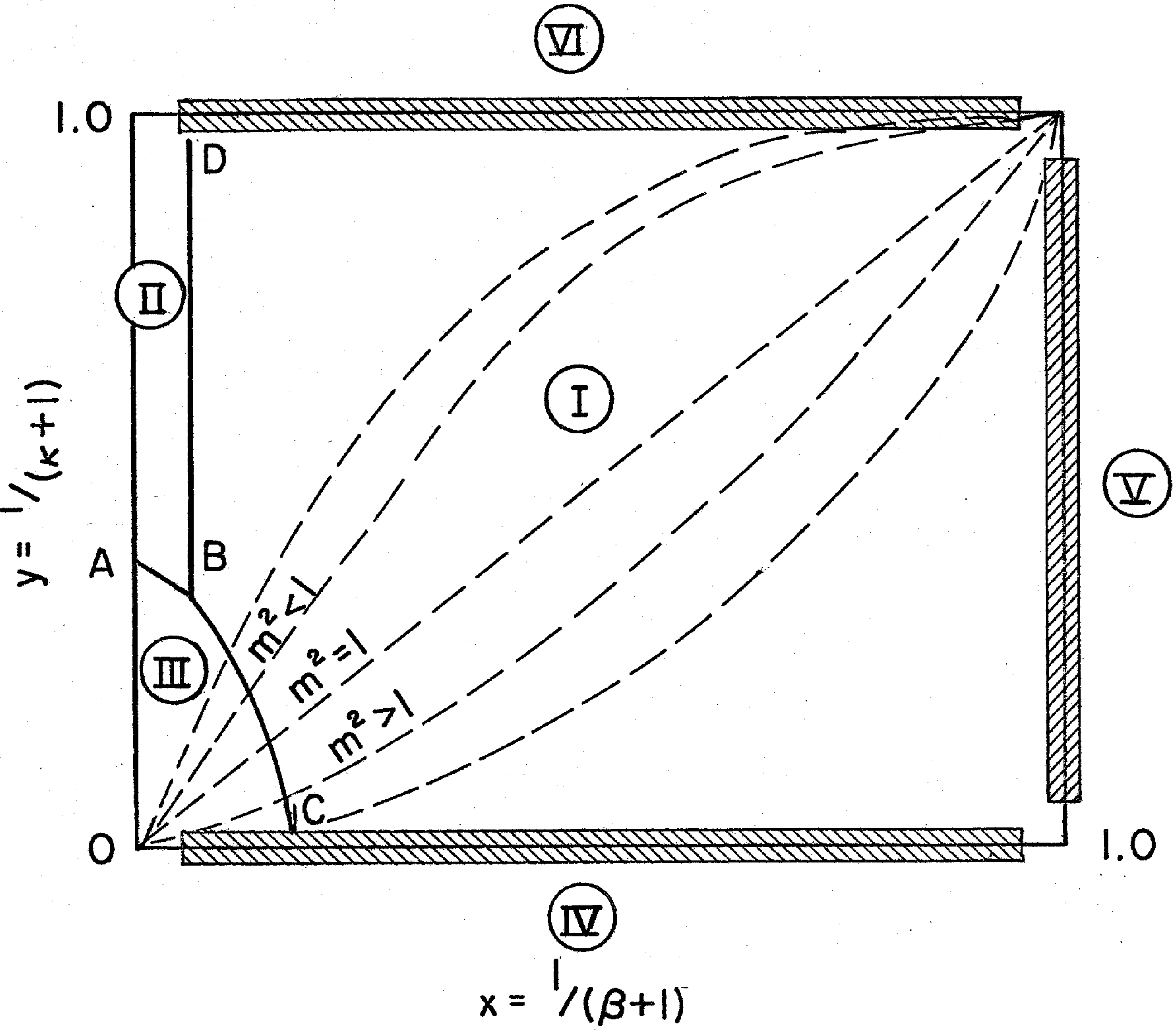
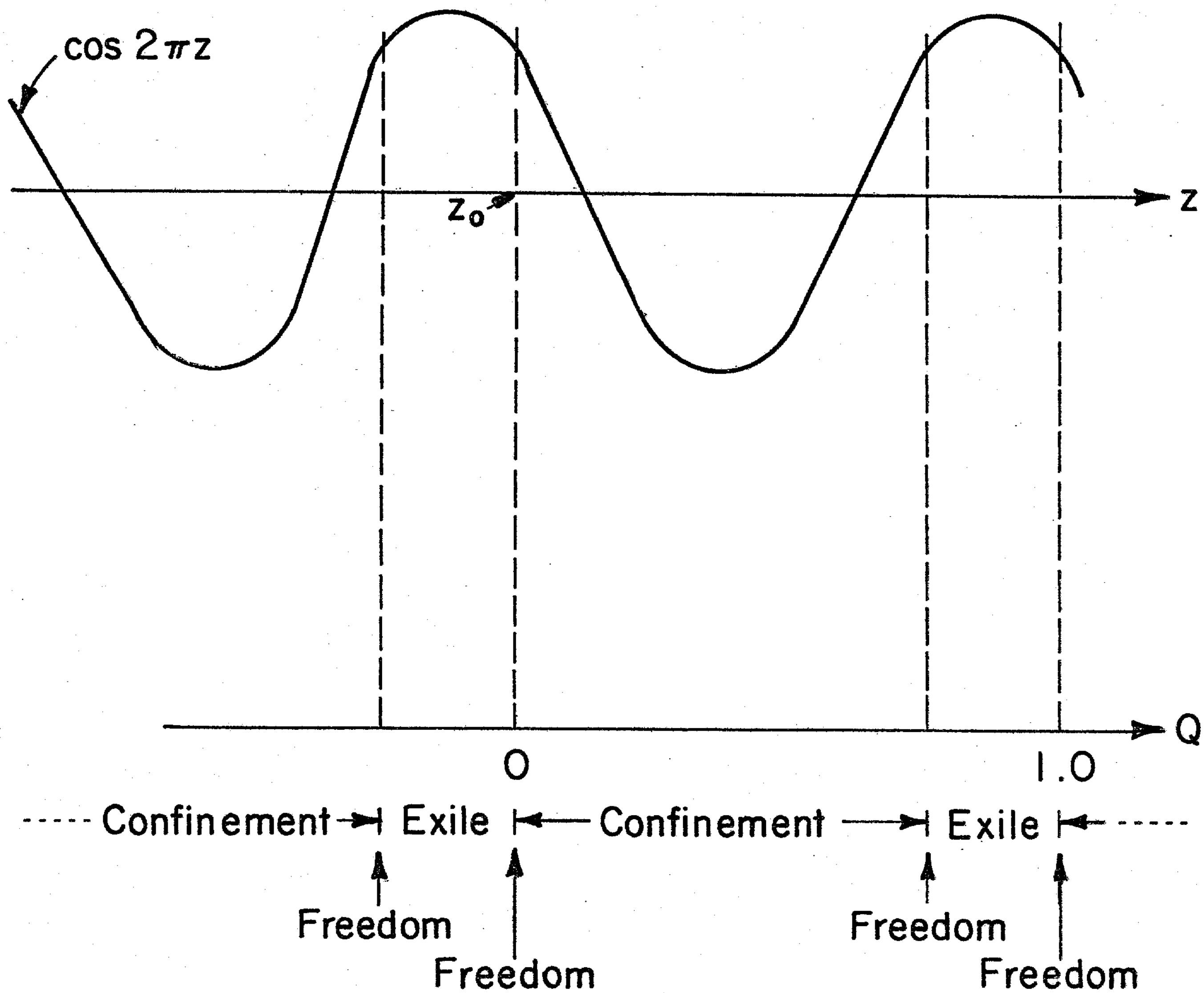


Fig. 1


cos 2πz
z
z_0
Q
0
1.0
Confinement
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Confinement
Exile
Freedom
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Freedom
Freedom

Fig. 2

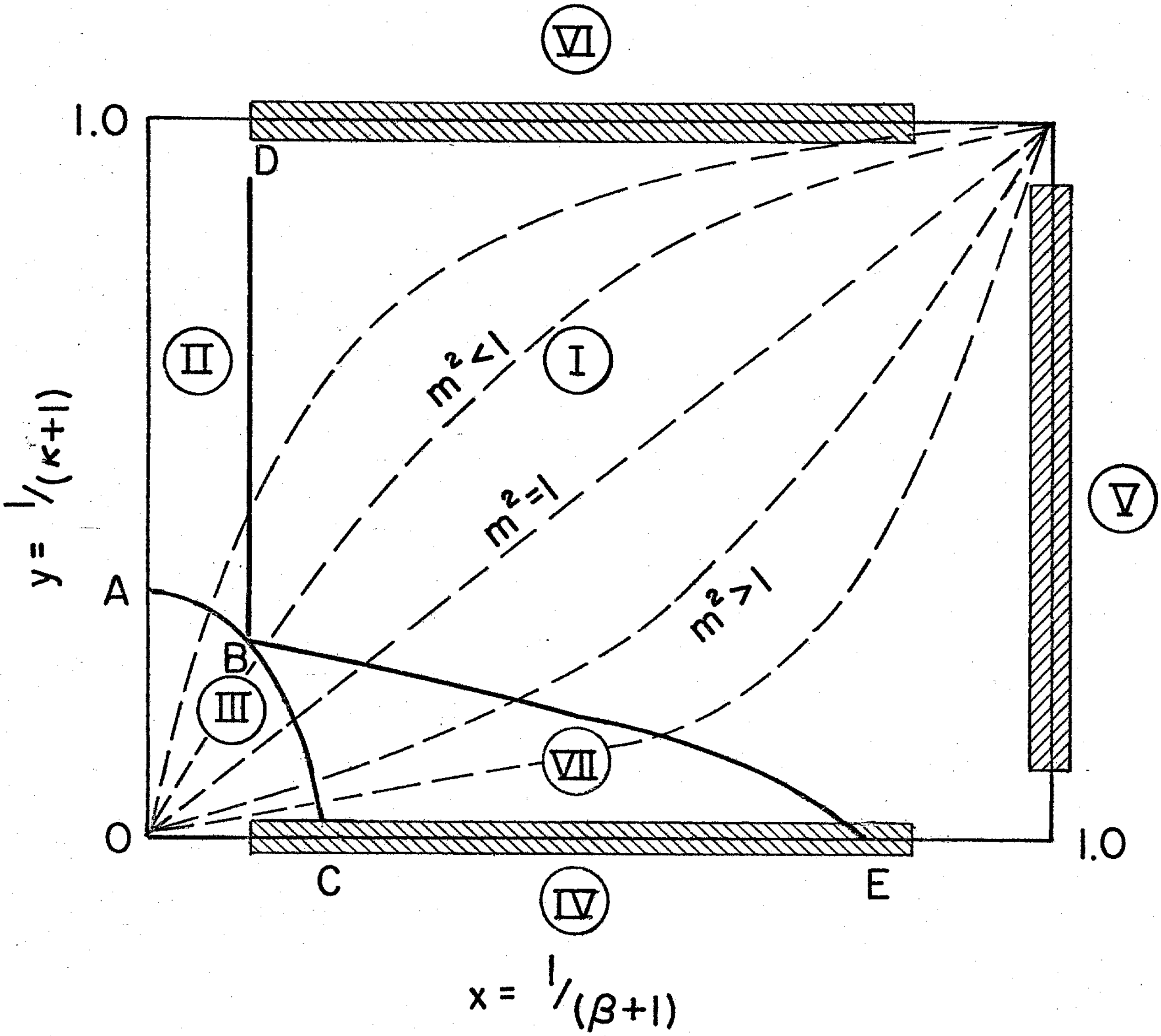

Fig. 3

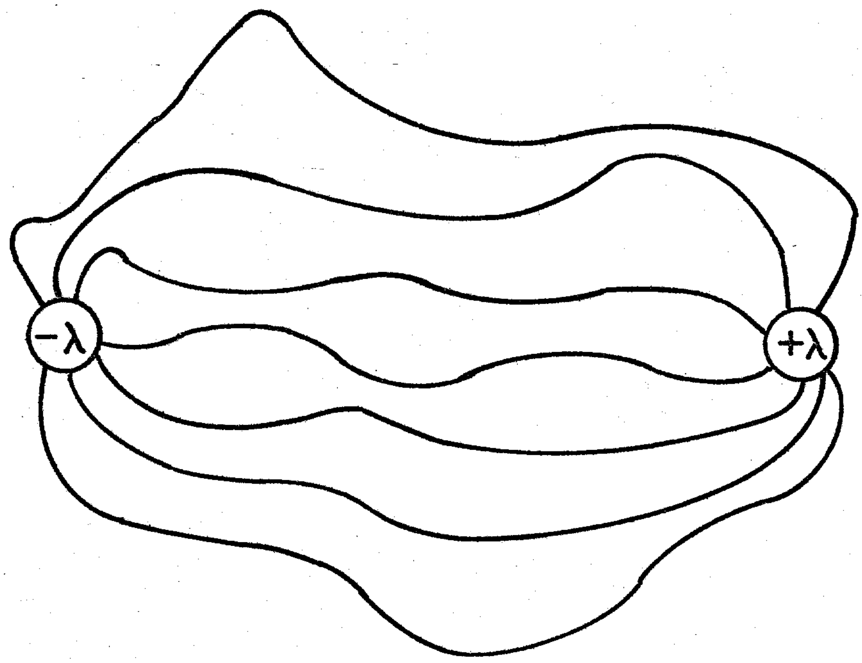

Fig. 4