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Scaling the Schrödinger Equation

C. QUIGG^{*}

Fermi National Accelerator Laboratory[†]
Batavia, Illinois 60510

and

JONATHAN L. ROSNER[‡]

School of Physics and Astronomy
University of Minnesota
Minneapolis, Minnesota 55455

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^{*} Alfred P. Sloan Foundation Fellow; also at Enrico Fermi
Institute, University of Chicago, Chicago, Illinois 60637.

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The notion of asymptotic freedom¹ in gauge theories of the strong interactions has lent respectability to the hope that hadrons which are composed of heavy quarks may be described by the nonrelativistic Schrödinger equation.² An impressive phenomenology of the psion family has been constructed³ following the analogy between (e^+e^-) positronium and ($c\bar{c}$) charmonium. In most versions of the model, the interquark potential (which is thought to be mediated by the exchange of massless gluons) has been assumed to be a superposition of a Coulomb term and a linear confining potential.⁴ However, no compelling derivation of this form from the underlying field theory has been given.⁵ Since the nature of the potential is unknown, it is of some interest to obtain general results which permit the properties of the potential to be inferred from experiment. In this vein, an introductory discussion has been given by Jackson⁶ and several important theorems on the order of levels and on leptonic widths of vector mesons have been proved by Martin⁷ and by Grosse.⁸ With the accumulation of many precise experimental results, one may contemplate the approximate solution of the inverse scattering problem, whereby bound state properties determine the potential.⁹

In this short article we compile several scaling formulae which exhibit the dependence of level spacings and other dimensionful quantities upon the reduced quark mass μ and upon the principal quantum number n , for potentials of the form

$$V(r) = ar^\epsilon \quad . \quad (1)$$

We encountered these formulae in the course of answering¹⁰ the question: for what form of $V(r)$ are the intervals between eigenvalues of the Schrödinger equation independent of reduced mass? Special cases of the relations are widely known, and we have found some of the general cases as textbook problems. But in spite of the elementary nature of these results, they are rather unfamiliar to high-energy physicists. Therefore, because they promise considerable utility for the study of the quarkonium potential, it appears worthwhile to assemble them in one place, together with derivations.

We first show that for potentials of the form (1) the scale of level spacings is given by

$$\Delta E \propto \mu^{-\epsilon/(2+\epsilon)} \quad . \quad (2)$$

In the reduced radial Schrödinger equation (for $u(r) = rR(r)$, where the Schrödinger wavefunction is $\Psi(\underline{r}) = R(r)Y_{\ell m}(\theta, \phi)$)

$$\frac{-u''(r)}{2\mu} + \left[\frac{\ell(\ell+1)}{2\mu r^2} + V(r) - E \right] u(r) = 0 \quad (3)$$

with $\hbar = c = 1$, define the dimensionless parameter

$$\rho = \mu^p m_0^{1-p} r \quad . \quad (4)$$

Here m_0 is a constant with dimensions of mass and the power p will be specified below. With the replacement

$$u(r) \equiv w(\rho) = w(\mu^p m_0^{1-p} r) \quad , \quad (5)$$

we have

$$u''(r) = \mu^{2p} m_0^{2(1-p)} w''(\rho) \quad , \quad (6)$$

so that

$$-\frac{1}{2} \mu^{2p-1} m_0^{2(1-p)} w''(\rho) + \frac{1}{2} \mu^{2p-1} m_0^{2(1-p)} \ell(\ell+1) w(\rho) / \rho^2 + \left[a \rho^\epsilon \mu^{-\epsilon p} m_0^{\epsilon(p-1)} - E \right] w(\rho) = 0 \quad . \quad (7)$$

We now set

$$2p-1 = -\epsilon p \quad (8)$$

and divide (7) by $\mu^{2p-1} m_0^{2(1-p)}$, obtaining

$$-\frac{1}{2} w''(\rho) + \ell(\ell+1) w(\rho) / 2\rho^2 + \left[\frac{a \rho^\epsilon}{m_0^{1+\epsilon}} - \frac{E}{\mu^{2p-1} m_0^{2-2p}} \right] w(\rho) = 0 \quad . \quad (9)$$

We have now isolated the μ -dependence in the term $E/\mu^{2p-1} m_0^{2-2p}$; thus the scale of energy level spacings is given by $\Delta E \sim \mu^{2p-1}$.

Solving (8) for $p = 1/(2+\epsilon)$, we obtain eq. (2). Note that the quantity $a/m_0^{1+\epsilon}$ must be dimensionless; it is therefore convenient to define

$$a = \lambda m_0^{1+\epsilon} \quad , \quad (10)$$

where λ is a dimensionless strength. Thus the strength of the Coulomb potential is dimensionless, so that μ must set the scale of energy levels.

According to eq. (4) and (10), quantities with dimensions of length L scale as

$$L \sim \mu^{-1/(2+\epsilon)} \quad (11)$$

The size of a bound state with given quantum numbers is such a quantity. The matrix elements of electric and magnetic multipole operators scale as

$$\langle n' | E_j | n \rangle \sim L^j \quad (12)$$

and¹¹

$$\langle n' | M_j | n \rangle \sim L^{j-1}/\mu \quad (13)$$

Since the radiative widths are given by

$$\Gamma(E_j \text{ or } M_j) \sim p_\gamma^{2j+1} |\langle n' | E_j \text{ or } M_j | n \rangle|^2 \quad (14)$$

and

$$p_\gamma \sim \Delta E \sim \mu^{-\epsilon/(2+\epsilon)} \quad (15)$$

we find

$$\Gamma(E_j) \sim \mu^{-[2j(1+\epsilon)+\epsilon]/(2+\epsilon)} \quad (16)$$

and

$$\Gamma(M_j) \sim \mu^{-[2j(1+\epsilon)+(3\epsilon+2)]/(2+\epsilon)} \quad (17)$$

For potentials weaker near $r=0$ than a Coulomb potential, i.e. for $\epsilon > -1$, the relative importance of higher multipoles decreases as μ increases. It is amusing that for $-2 < \epsilon < -1$ the limit of very large μ can lead to a growing importance of high multipoles in radiative decays. For the Coulomb

potential, $\epsilon=-1$, the j -dependence drops out, and all decay rates scale as $\Gamma \sim \mu$. Again this follows from the dimensionless nature of the Coulomb coupling constant.

Probability densities $|\Psi(\underline{r})|^2$ have dimensions of inverse volume L^{-3} so scale as

$$|\Psi(\underline{r})|^2 \sim \mu^{3/(2+\epsilon)} \quad . \quad (18)$$

Such quantities are of interest, for example, in the decays of massive vector mesons V which are 3S_1 bound states of a quark and antiquark, for which¹²

$$\Gamma(V \rightarrow \ell^+ \ell^-) = 16\pi\alpha^2 e_Q^2 |\Psi(0)|^2 / M(V)^2 \quad , \quad (19)$$

where e_Q is the quark charge and $M(V)$ is the vector meson mass. For $\epsilon > -1$, the scale of $M(V)$ will itself be set by μ for the low-lying levels.¹³ Consequently we find

$$\Gamma(V \rightarrow \ell^+ \ell^-) \sim \mu^{-(1+2\epsilon)/(2+\epsilon)} \quad , \quad \epsilon \geq -1 \quad . \quad (20)$$

Again for the Coulomb potential the width is proportional to μ .

The ratios of radiative to leptonic widths are of concern for massive states:

$$\Gamma(Ej)/\Gamma(V \rightarrow \ell^+ \ell^-) \sim \mu^{(1-2j)(1+\epsilon)/(2+\epsilon)} \quad , \quad \epsilon \geq -1 \quad ; \quad (21)$$

$$\Gamma(Mj)/\Gamma(V \rightarrow \ell^+ \ell^-) \sim \mu^{-(1+2j)(1+\epsilon)/(2+\epsilon)} \quad , \quad \epsilon \geq -1. \quad (22)$$

Since $j \geq 1$, the exponents in (21) and (22) are both negative for $\epsilon > -1$. Hence leptonic decays will dominate over radiative

transitions as μ increases.

For $\epsilon < -1$, relation (20) is only expected to hold for the ground state. For the excited states $M(V^*) \sim \Delta E \nu \mu^{-\epsilon/(2+\epsilon)}$, and

$$\Gamma(V^* \rightarrow \ell^+ \ell^-) \sim \mu^{(3+2\epsilon)/(2+\epsilon)}, \quad \epsilon < -1 \quad (23)$$

In this case the radiative transitions take on an increasing importance as μ increases, because

$$\Gamma(Ej)/\Gamma(V^* \rightarrow \ell^+ \ell^-) \sim \mu^{-(2j+3)(1+\epsilon)/(2+\epsilon)}, \quad (24)$$

and

$$\Gamma(Mj)/\Gamma(V^* \rightarrow \ell^+ \ell^-) \sim \mu^{-(2j+5)(1+\epsilon)/(2+\epsilon)}. \quad (25)$$

For potentials which vary more slowly than any power of r , eq. (2) indicates that ΔE varies more slowly than any power of μ . The potential

$$V(r) = C \ln(r/r_0) \quad (26)$$

gives rise to level spacings which are¹⁰ strictly independent of μ . The length-scaling arguments following from eq. (11) are valid with $\epsilon = 0$ for the logarithmic potential (26). The limit $\epsilon = \infty$ corresponds to a square well if we take m_0^{-1} to be its size a_0 :

$$V(r) = \lim_{\epsilon \rightarrow \infty} \frac{\lambda}{a_0} \left(\frac{r}{a_0} \right)^\epsilon \longrightarrow \begin{cases} \infty, & r > a_0 \\ 0, & r < a_0 \end{cases} \quad (27)$$

The scaling properties of some of the quantities we have discussed are summarized in Table I for several commonplace potentials. We show in Fig. 1 the dependence of the leptonic widths of ground-state vector mesons upon the vector meson mass. If nonrelativistic considerations were valid (a hypothesis we do not believe for ρ, ω, ϕ), we would conclude from eq. (20) that the effective power of the potential is

$$\varepsilon(\text{leptonic widths}) = -0.40 \pm 0.10 \quad . \quad (28)$$

Application of this kind of result to more massive quarkonium states is an attractive future possibility.

We now turn to an exploration of the dependence of physically interesting quantities upon the principal quantum number n . This discussion is much less extensive than the preceding one. It is motivated by the observation¹⁰ that for a logarithmic potential of the form (26) the s-wave wavefunctions obey $|\psi_n(0)|^2 \sim n^{-1}$. Relations of this kind are of interest, for example, in the interpretation of leptonic widths of vector mesons. We will show by means of the semiclassical (WKB) approximation¹⁴ that for power-law potentials of the form (1)

$$|\psi_n(0)|^2 \sim n^{2(\varepsilon-1)/(2+\varepsilon)}, \quad \varepsilon > 0 \quad , \quad (29)$$

and

$$|\psi_n(0)|^2 \sim n^{(\varepsilon-2)/(2+\varepsilon)}, \quad -2 < \varepsilon < 0 \quad . \quad (30)$$

for large n .

The s-wave WKB bound-state wavefunction is

$$u(r) = \frac{N}{[2\mu(E-V(r))]^{1/4}} \sin \left\{ \int_0^r dr' [2\mu(E-V(r'))]^{1/2} \right\} , \quad (31)$$

where N is a normalization constant. The quantization condition is that:

$$\int_0^{r_0} dr' [2\mu(E-V(r'))]^{1/2} = (n - \frac{1}{4})\pi , \quad n=1,2,\dots \quad (32)$$

For a power-law potential (1), the integral in (32) can be performed exactly,¹⁵ yielding

$$|E_n| = a^{\frac{2}{2+\epsilon}} (2\mu)^{\frac{-\epsilon}{2+\epsilon}} [E(\epsilon)(n-\frac{1}{4})]^{\frac{2\epsilon}{2+\epsilon}} , \quad (33)$$

where

$$E(\epsilon) = \frac{2\epsilon\sqrt{\pi} \Gamma(\frac{3}{2} + \frac{1}{\epsilon})}{\Gamma(\frac{1}{\epsilon})} \quad \epsilon > 0 , \quad (34)$$

and

$$E(\epsilon) = \frac{2|\epsilon|\sqrt{\pi} \Gamma(1-\frac{1}{\epsilon})}{\Gamma(-\frac{1}{2}-\frac{1}{\epsilon})} \quad -2 < \epsilon < 0 . \quad (35)$$

The behavior of $|\Psi_n(0)|^2$ for $\epsilon > 0$ may be obtained by noting¹⁶ that

$$|\Psi(0)|^2 = \frac{\mu}{2\pi} \left\langle \frac{dV}{dr} \right\rangle . \quad (36)$$

Consequently

$$|\Psi(0)|^2 = \frac{\mu}{2\pi} \cdot 4\pi \int_0^{r_0} dr [u(r)]^2 a\epsilon r^{\epsilon-1} , \quad (37)$$

where $u(r)$, given by Eq. (31), satisfies the normalization condition

$$1 = 4\pi \int_0^{r_0} dr [u(r)]^2 \quad . \quad (38)$$

The integral in (37) is elementary if we make the approximation that the average value of the \sin^2 term is $\frac{1}{2}$; one then obtains

$$|\Psi(0)|^2 = (2\mu E)^{\frac{1}{2}} N^2 \quad (\epsilon > 0) \quad . \quad (39)$$

A similar trick applies to the evaluation of N^2 , and gives

$$N^2 = (2\mu)^{\frac{1}{2}} E^{\frac{\epsilon-2}{2\epsilon}} a^{1/\epsilon} \frac{|\epsilon|}{2\pi^{3/2}} N(\epsilon) \quad , \quad (40)$$

where

$$N(\epsilon) = \frac{\Gamma(\frac{1}{2} + \frac{1}{\epsilon})}{\Gamma(\frac{1}{\epsilon})} \quad \epsilon > 0 \quad , \quad (41)$$

and

$$N(\epsilon) = \frac{\Gamma(1 - \frac{1}{\epsilon})}{\Gamma(\frac{1}{2} - \frac{1}{\epsilon})} \quad -2 < \epsilon < 0 \quad . \quad (42)$$

Combining (39) and (40), and using Eq. (33) to express E in terms of n , we find

$$|\Psi(0)|^2 = (2\mu a)^{\frac{3}{2+\epsilon}} (n - \frac{1}{4})^{\frac{2(\epsilon-1)}{2+\epsilon}} G(\epsilon) \quad , \quad \epsilon > 0 \quad , \quad (43)$$

where

$$G(\epsilon) = (2\pi^2)^{-1} \left(1 + \frac{2}{\epsilon}\right)^{\frac{2(\epsilon-1)}{2+\epsilon}} \left[\frac{\epsilon \sqrt{\pi} \Gamma(\frac{1}{2} + \frac{1}{\epsilon})}{\Gamma(\frac{1}{\epsilon})} \right]^{\frac{3\epsilon}{2+\epsilon}} \quad , \quad \epsilon > 0. \quad (44)$$

For $\epsilon < 0$, it was remarked some time ago¹⁷ that the WKB approximation is improved by replacing $\ell(\ell+1) \rightarrow (\ell+\frac{1}{2})^2$. This adds a repulsive S-wave potential $1/(8\mu r^2)$ to $V(r)$ and thus imposes a lower cutoff on the r -integral in (37). This cutoff can be shown unimportant in the evaluation of N^2 , but for (37) we find a result parallel to (39):

$$|\psi(0)|^2 = \frac{1}{4} \frac{|\epsilon|}{2+\epsilon} N^2 (8\mu|a|)^{\frac{1}{2+\epsilon}} \frac{\Gamma\left(\frac{-\epsilon}{2(2+\epsilon)}\right) \sqrt{\pi}}{\Gamma\left(\frac{1}{2+\epsilon}\right)}. \quad (45)$$

Combining (45) with (40), (42), (33), and (35), one can then arrive at a result parallel to (43), which yields (30).

For $\epsilon \rightarrow 0$ both (29) and (30) imply that $|\psi(0)|^2$ should behave as n^{-1} for large n . This is precisely the behavior found in Ref. (10) for the potential $V(r) \sim \ln r$.

We plot in Fig. 2 the values of $|\psi_n(0)|^2$ for $\psi(3095)$, $\psi(3684)$, and $\psi(4414)$, which we regard as 1S, 2S, and 4S or 5S levels of the charmonium system. Blithely applying our semiclassical result (29) and (30) to the power-law fits shown in Fig. 2, we find

$$\begin{aligned} \epsilon &= 0.01 \pm 0.14 && (4S \text{ assignment}); \\ \epsilon &= 0.05 \pm 0.13 && (5S \text{ assignment}). \end{aligned} \quad (47)$$

As was the case for our discussion of the experimental mass dependence of $|\psi(0)|^2$, this is purely an illustrative application. We look forward to data which will permit the use of (29) and (30) within their justified range.

Within the semiclassical context, it is possible in principle to find the shape of the potential given the dependence of the bound state energies on the principal quantum number n . The result applicable to our three-dimensional problem (cf. Ref. 18) is

$$r(V) = \frac{2}{\sqrt{2\mu}} \int_{E_0}^V \frac{dn}{dE}(E) \frac{dE}{\sqrt{V-E}}, \quad (48)$$

where E_0 is the zero-point energy. Eq. (48) is obtained by differentiating (32) with respect to E , multiplying by $(V-E)^{-\frac{1}{2}}$, and integrating with respect to E .

If the known energy levels satisfied the relation

$$E - E_0 = C n^q, \quad (49)$$

we would then find

$$V - E_0 \sim r^{2q/(2-q)}. \quad (50)$$

Eqs. (49) and (50) are equivalent to Eqs. (1) and (33), but with the present sketchy knowledge of the energy levels for charmonium, it is very hard to determine q without independent information on E_0 ! The use of information on the behavior of $|\Psi(0)|^2$ is clearly superior.

To summarize: the dependence of level spacings, transition rates, bound state sizes, and other quantities on quark masses and on principal quantum numbers can be exploited very simply to gain an idea of the structure of the quark-antiquark force.

We have collected some of these simple results in the hope that they will be of help in interpreting data on the new heavy particles, and in refreshing the reader (as these results refreshed us!) on some elementary properties of the quantum mechanics of bound states.

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FOOTNOTES AND REFERENCES

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- ¹¹ For simplicity, in discussing the magnetic transitions we assume that only one mass scale is important, whether because the two particles bound to one another have equal masses ($=2\mu$), or because one (with mass $\approx \mu$) is much lighter than the other.
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- ¹³ This is because $\Delta E/\mu$ does not grow as μ increases for $\epsilon > -1$. We assume $M(V) \approx 2(2\mu) +$ small binding corrections, when the mass of each constituent of V is 2μ .
- ¹⁴ See, e.g., L.D. Landau and E.M. Lifshitz, Quantum Mechanics, translated by J.B. Sykes and J.S. Bell (Addison-Wesley, Reading, Mass., 1958), c.VII; and I.I. Gol'dman and V.D. Krivchenkov, Problems in Quantum Mechanics, translated by E. Marquit and E. Lepa (Addison-Wesley, Reading, Mass., 1961), §1.
- ¹⁵ Gol'dman and Krivchenkov, Ref. 14, present a slightly different argument leading to $(n-\frac{1}{2})$ on the right-hand side of (33) for one-dimensional problems.
- ¹⁶ This is proved by solving (3) (with $\ell=0$) for $dV(r)/dr$, taking the expectation value, and integrating by parts.
- ¹⁷ See R.E. Langer, Phys. Rev. 51, 669 (1937); apparently the original observation is due to H.A. Kramers, Zeits. f. Physik 39, 836 (1926).
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TABLE I. Scaling properties of some physical quantities in various potentials.

Potential	ϵ	ΔE	length scale	$\Gamma(E1)$	$\Gamma(M1)$	$\Gamma(\nu \rightarrow e^+e^-)$
Coulomb	-1	μ	μ	μ	μ	μ
Logarithmic	0	μ^0	$\mu^{-1/2}$	μ^{-1}	μ^{-2}	$\mu^{-1/2}$
Linear	1	$\mu^{-1/3}$	$\mu^{-1/3}$	$\mu^{-5/3}$	μ^{-3}	μ^{-1}
Harmonic Oscillator	2	$\mu^{-1/2}$	$\mu^{-1/4}$	μ^{-2}	$\mu^{-7/2}$	$\mu^{-5/4}$
Square Well	∞	μ^{-1}	μ^0	μ^{-3}	μ^{-5}	μ^{-2}

FIGURE CAPTIONS

- Fig. 1 Leptonic widths of the vector mesons divided by the square of the effective quark charge, $e_Q^2 = 1/2, 1/18, 1/9, 4/9$ for ρ, ω, ϕ, ψ . The solid line is a "best fit" proportional to M^p with $p = -0.12 \pm 0.11$. For a closely related plot see Ref. 6.
- Fig. 2 Square of the wave function at the origin deduced from leptonic widths of the psions. Possible mixing between the 2^3S_1 (3684) and 3^3D_1 (3772) levels has been neglected. The solid line is a best fit proportional to n^p , with $p = -0.98 \pm 0.20$, assuming the conventional 4S assignment for $\psi(4414)$. The dashed line, which refers to an alternative 5S assignment for $\psi(4414)$, corresponds to $p = -0.92 \pm 0.18$.

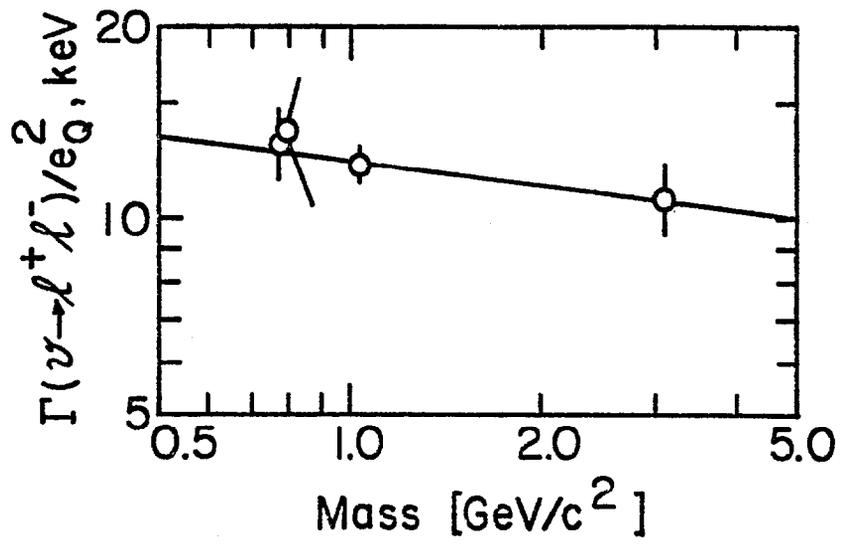


Fig. 1

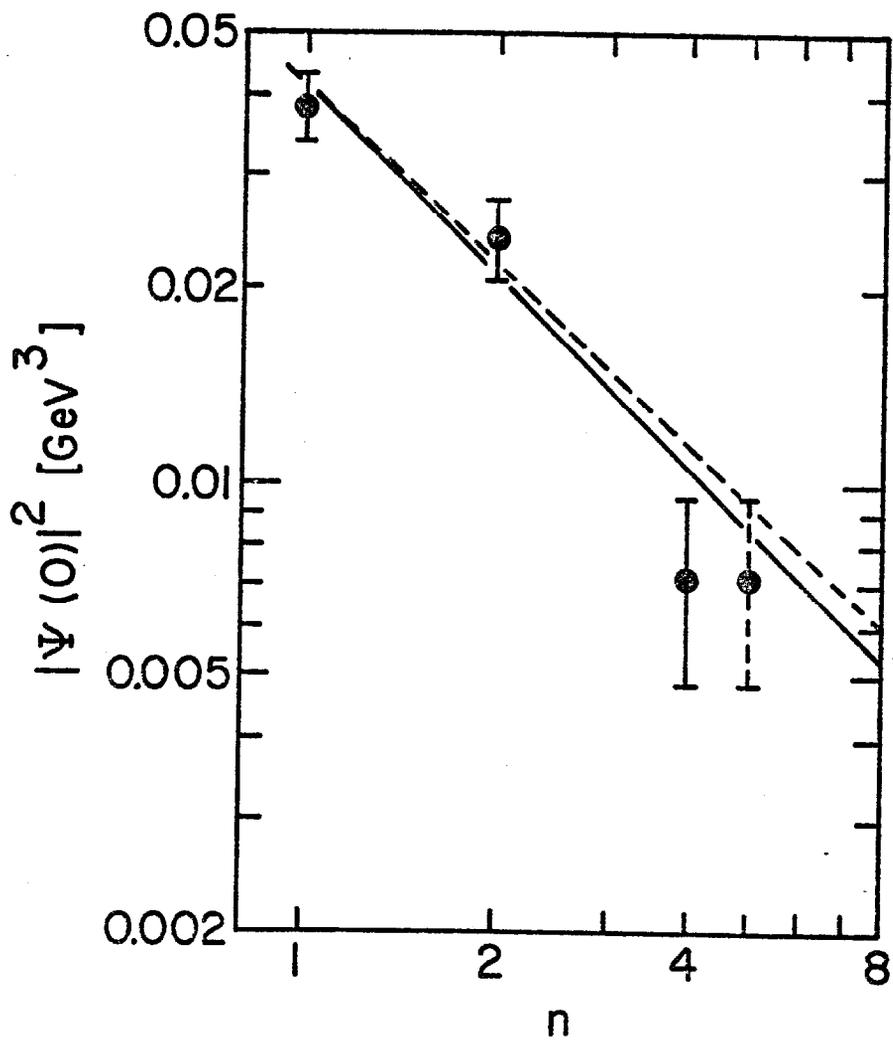


Fig. 2