



Quarkonium Level Spacings

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Motivated by the apparent equal spacing $M(\Upsilon') - M(\Upsilon) = M(\psi') - M(\psi)$, we show that the potential for which quarkonium level spacings are independent of quark mass, in the nonrelativistic limit, is $V(r) = C \ln(r/r_0)$. We enumerate consequences of the logarithmic potential and present an alternative interpretation of the data.



ERRATA

Please note the following corrections:

1. On page 3, last paragraph, first line, after "(1 GeV)]", insert "in (3) with $m = 1 \text{ GeV}/c^2$ ".
2. In eq. (6) and on page 4, line 2, replace " m_0 " by " m_1 ".

The $\Upsilon(9.5 \text{ GeV}/c^2)$ recently discovered at Fermilab [4] appears to have a level structure [2] qualitatively similar to that of the psion family [3]. It is therefore appealing to regard $\Upsilon, \Upsilon', \dots$ as $Q\bar{Q}$ bound states of a new heavy quark Q [4], just as the psions have been interpreted with impressive success as levels of the charmonium system. Remarkably, it appears that [2]

$$M(\Upsilon') - M(\Upsilon) \simeq M(\psi') - M(\psi) \simeq 0.59 \text{ GeV}/c^2 \quad . \quad (1)$$

What would it mean if this equality were not coincidental, but were strictly independent of the quark mass? We shall show that, in the nonrelativistic limit, the potential between quark and antiquark must be

$$V(r) = C \ln(r/r_0) \quad . \quad (2)$$

With a strength $C \simeq 3/4 \text{ GeV}$ the potential (2) not only reproduces many features of the psion family⁴, but also allows nearly instantaneous calculation of masses and leptonic widths of the entire Υ family or indeed of any bound states of more massive quarks. In this note we describe some features of the logarithmic potential as applied to quarkonium families, comparing briefly with the more conventional Coulomb + linear potentials considered by many authors [4, 6-11].

Bound states of a quark (of mass m) and its antiquark in a potential $V(r)$ are the eigenstates of the radial Schrödinger equation

$$\left\{ -\frac{1}{mr^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) + \frac{\ell(\ell+1)}{mr^2} + V(r) - E \right\} \Psi(r) = 0 \quad . \quad (3)$$

Define the dimensionless parameter $\rho \equiv r\sqrt{mm_0}$, where m_0 is an arbitrary scale, and let $\Psi(r) \equiv \mathcal{N}v(\rho)$, where the normalization \mathcal{N} will be discussed below. Then the radial equation can be recast as

$$\left\{ -\frac{1}{\rho^2} \frac{d}{d\rho} \left(\rho^2 \frac{d}{d\rho} \right) + \frac{\ell(\ell+1)}{\rho^2} + \frac{V(\rho/\sqrt{mm_0})}{m_0} - \frac{E}{m_0} \right\} v(\rho) = 0 \quad . \quad (4)$$

The differences of eigenvalues are independent of m if and only if under a scale transformation $m \rightarrow \lambda m$,

$$V(\rho/\sqrt{\lambda mm_0}) = V(\rho/\sqrt{mm_0}) + f(\lambda) \quad . \quad (5)$$

The solution to Eq. (5) may be obtained by differentiating with respect to λ and ρ and separating variables. It is just the potential (2), with $f(\lambda) = -\frac{1}{2} C \ln \lambda$.

The potential $V(r) = (1 \text{ GeV}) \ln [r \cdot (1 \text{ GeV})]$ in (3) with $m \equiv 1 \text{ GeV}/c^2$ gives rise to the levels indicated in Fig. 1. With the choice $C = 3/4 \text{ GeV}$, the splittings (1) are reproduced and a rescaled Fig. 1 depicts the levels of the ψ , Υ , and higher-lying families.⁵ Eigenvalues E of Fig. 1 are related to the masses of physical states by

$$M = (C/1 \text{ GeV})E + m_1 \quad . \quad (6)$$

By fitting the masses of $\psi(3.095)$, $\psi'(3.684)$, and $\Upsilon(9.40)$, we find

$C = 0.733$ GeV, $m_1(\psi \text{ family}) = 2.329$ GeV, and $m_0(\Upsilon \text{ family}) = 8.634$ GeV.

The resulting level schemes for the ψ and Υ families and for a hypothetical ζ family⁶ are shown in Fig. 2, together with the experimental information.⁷

Before discussing these results let us remark that although $V(r) = C \ln(r/r_0)$ is unique in giving level spacing independent of the quark mass it is by no means alone in reproducing the equal spacing rule (1) for ψ and Υ . The potential of Ref. 4,

$$V(r) = -\frac{4}{3} \alpha_s / r + r/a^2, \quad (7)$$

gives $M(\Upsilon') - M(\Upsilon) \approx \frac{2}{3} [M(\psi') - M(\psi)]$ if, independent of the quark mass, $\alpha_s = 0.19$ and $a = 2.22 \text{ GeV}^{-1}$. The Υ splitting is even smaller if $\alpha_s(m_c) = 0.19 > \alpha_s(m_Q)$. Now, for a potential $V \sim r^\epsilon$, the level spacing behaves as $m^{-\epsilon/(2+\epsilon)}$. Thus it is possible to choose a certain combination of a Coulomb potential ($\Delta E \sim m$) and a linear potential ($\Delta E \sim m^{-1/3}$) for which the spacing between the 4S and 2S levels is 0.59 GeV for precisely two values of the quark mass. For example, with $\alpha_s \approx 0.42$ and $a = 2.48 \text{ GeV}^{-1}$ in (7), we have no difficulty in reproducing (1), as shown in Fig. 2(c). For quarkonia heavier than Υ this "modified Coulomb" potential predicts a 2S-1S splitting which increases with mass.

For the charmonium system the logarithmic potential has a denser spectrum than does the modified Coulomb potential. Both level schemes are in respectable agreement with the data as now known. They

differ in their assignments of $\psi(4.414)$: for the logarithmic potential it is a 5S level, whereas it is a 4S level for the modified Coulomb potential [6-10]. The logarithmic potential predicts a state $\psi(4.25)$ which must be found if the potential is to be taken seriously.⁸

For the Υ family the parallel between the level schemes given by the two potentials is very striking. In particular, both yield $\Upsilon'' - \Upsilon' \simeq 0.32$ GeV, to be compared with the experimental suggestion [2] of 0.39 ± 0.13 GeV. One should not overlook the possibility that the putative third level may in fact be the unresolved 3S and 4S states.

For the conjectural ζ family the 2S-1S spacing has begun to increase for the modified Coulomb potential (although the 3S-2S spacing is unchanged). Together with the near-degeneracy of the 2S and 2P states, this reflects the increasing importance of the Coulomb component for low-lying levels.

The leptonic widths of the massive vector mesons \mathcal{V} provide another test for the logarithmic potential, because they are sensitive to the magnitude of the wave function at the origin:

$$\Gamma(\mathcal{V} \rightarrow l^+ l^-) \propto |\Psi(0)|^2 e_Q^2 / M(\mathcal{V})^2, \quad (8)$$

where e_Q is the charge of the heavy quark. Having solved the s-wave Eq. (4) subject to the boundary conditions $(\rho v(\rho)) = 0$, $(d/d\rho)(\rho v(\rho)) = 1$ at $\rho = 0$, we obtain $|v(0)|^2$ from the normalization condition $\int_0^\infty \rho^2 d\rho |v(\rho)|^2 = 1$. The requirement $\int_0^\infty r^2 dr |\Psi(r)|^2 = 1/4\pi$ then fixes

$$|\Psi(0)|^2 = (mC)^{3/2} |\psi(0)|^2 / 4\pi \quad (9)$$

If we make the crude approximation that $m \propto M(1^3S_1)$, i. e. that the quark mass is proportional to the mass of the ground state, we find that with the logarithmic potential $\Gamma(1^3S_1 \rightarrow \ell^+ \ell^-) \propto M(\mathcal{V})^{-\frac{1}{2}}$. A weak dependence upon $M(\mathcal{V})$ has been noted in the comparison of ρ, ω, ϕ , and ψ leptonic widths [13-15]. The result (9) is a special case of the relation $|\Psi(0)|^2 \sim m^{3/(2+\epsilon)}$ for a potential $V \sim r^\epsilon$, which can be proved by elementary dimensional arguments. Some illustrations are given in Ref. [15]. Within a family of vector mesons, $n |\Psi(0)|^2$, where n is the principal quantum number, is approximately constant for the logarithmic potential. (For $V \sim r^\epsilon$ with $\epsilon > 0$, $|\Psi(0)|^2 \sim n^{2(\epsilon-1)/(2+\epsilon)}$.)

Predicted leptonic widths are compared with experiment in Table I, where the expectations of the modified Coulomb potential are also shown. The logarithmic potential fares rather well. The width predicted by the modified Coulomb potential for $\psi(4.414)$ is uncomfortably large. This could be decreased by a further increase in α_s , but it is not our purpose here to engage in fine-tuning of the modified Coulomb potential. At the Υ mass the modified Coulomb potential gives rise to much larger leptonic widths than does the logarithmic potential. This is because $|\Psi(0)|^2 \sim m^3$ (rather than $m^{3/2}$) for the Coulomb potential.

The numerical coefficient $16\pi\alpha^2$ (see, e.g., [15]) in (8) would imply $\Gamma(\psi \rightarrow \ell^+ \ell^-) = 8.82 \text{ keV}$ for the modified Coulomb potential and

$\Gamma(\psi \rightarrow \ell^+ \ell^-) = 3.83 \text{ keV} \times (m/1 \text{ GeV})^{3/2}$ for the logarithmic potential.

Thus $\Gamma(\psi \rightarrow \ell^+ \ell^-)$ can be fitted with the logarithmic potential, albeit at the expense of a fairly small charmed quark mass. The large value predicted by the modified Coulomb potential is one reason why smaller values of α_s have generally been taken [4, 6], though not in all models [8-10].

The virial theorem⁹ for the expectation value of the kinetic energy T,

$$\langle T \rangle = \left\langle \frac{r}{2} \frac{dV(r)}{dr} \right\rangle, \quad (10)$$

takes an especially simple form for the potential (2):

$$\frac{\langle T \rangle}{mc^2} = \langle \beta^2 \rangle = C/2mc^2. \quad (11)$$

Thus, for $C \approx 3/4 \text{ GeV}$ and for $m \approx 1.5 \text{ GeV}/c^2$ (charmed quarks), $\langle \beta^2 \rangle^{1/2} \approx \frac{1}{2}$ and the nonrelativistic approximation is rather crude.

This undercuts, to some extent, our experimental motivation for (2). However, in contrast to the situation for any potential of the form r^ϵ , $\epsilon > 0$, the nonrelativistic approximation does not deteriorate further with increasing excitation energy. Moreover, for $m \approx 5 \text{ GeV}/c^2$ (the constituents of Υ) the nonrelativistic approximation is much better. The virial theorem (11) tells us that the potential (2) is far too crude to describe bound states of the light quarks u, d, s. As a result, we do not expect the $\rho' - \rho$, $\omega' - \omega$, and $\phi' - \phi$ spacings to be the same as $\psi' - \psi$. We are also reluctant to apply potential arguments to light

quark-heavy quark bound states. Our diffidence leaves us with no prediction for the threshold for Zweig-rule-allowed strong decays of quarkonium.

Fine structure effects in a logarithmic potential involve terms all proportional to $C/m\rho^2$ when suitably rescaled. Since such terms can be absorbed into the centrifugal barrier, their effects can be evaluated directly from Fig. 1 without recourse to perturbation theory. However, as in Ref. [7], if the entire potential is ascribed to a $\gamma_\mu \otimes \gamma^\mu$ interaction, the large ψ - η_c splitting (regarding for the moment X(2.83) as η_c) is reproduced only at the expense of $2^3 P_J$ splittings which are about $2\frac{1}{2}$ times too large.

The rates for E1 radiative transitions scale as $1/m$ in the logarithmic potential. (Distances scale as $m^{-1/(2+\epsilon)}$ in a potential $V \sim r^\epsilon$.) The calculated rates $\Gamma(\psi' \rightarrow \gamma\chi(^3P_J))$ are about three times too large, again indicating the crudeness of the logarithmic potential for psion phenomenology.

To summarize, we have shown that a quarkonium level spacing independent of the quark mass entails a logarithmic potential in the nonrelativistic limit. This amusing result in elementary quantum mechanics may also provide a very useful computational tool for quarkonium spectroscopy in the Υ regime. For the logarithmic potential to be taken seriously as a candidate for the quark-quark interaction, a necessary condition is the existence of a 4S charmonium level near 4.25 GeV.

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FOOTNOTES

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- ³Operated by Universities Research Association Inc. under contract with the Energy Research and Development Administration.
- ⁴A potential of the form (2) is one of several discussed for the pion family [5] .
- ⁵The feature of exponentially rising Regge trajectories is unusual but perhaps tolerable. Open decay channels may modify (2) at large distances and for large excitation energies. The Regge trajectories have been obtained numerically. We thank T. Yamanouchi for the observation that the WKB approximation for the s-wave levels, $E_n = \ln [\sqrt{\pi}(2n - \frac{1}{2})]$, is excellent.
- ⁶We have no theoretical reason to anticipate that the next quarkonium should appear at $17 \text{ GeV}/c^2$. This is the highest lepton pair mass yet observed [1, 2] . The name ζ is derived from the Greek $\zeta \eta \tau \epsilon \iota \nu$ (to seek or inquire) and from the Yiddish (grandfather).

⁷We have included a large number of d-states to raise the possibility of $^3S_1 - ^3D_1$ mixing, which appears [12] to be substantial for the 2^3S_1 and 1^3D_1 charmonium states.

⁸The logarithmic potential is not the only one to give a level $\psi(4.25)$; see Ref. [8].

⁹For a recent exposition, see [16]. We thank H. Lipkin for a discussion of this theorem.

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Table I. Masses and leptonic widths in the ψ and Υ families. $\Gamma(\Upsilon \rightarrow \ell^+ \ell^-)$ is tabulated for $e_Q = -1/3$. Input values are underlined.

Logarithmic Potential		Experiment	Modified Coulomb Potential	
Mass (GeV/c ²)	$\Gamma(\mathcal{V} \rightarrow \ell^+ \ell^-)$ (keV)	$\Gamma(\mathcal{V} \rightarrow \ell^+ \ell^-)$ (keV)	Mass (GeV/c ²)	$\Gamma(\mathcal{V} \rightarrow \ell^+ \ell^-)$ (keV)
$\psi(\underline{3.095})$	<u>4.80</u>	4.8 ± 0.6^a	$\psi(3.101)$	<u>4.80</u>
$\psi(\underline{3.684})$	1.73	2.1 ± 0.3^a	$\psi(3.685)$	2.24
$\psi(4.008)$	1.00	b	$\psi(4.108)$	1.71
$\psi(4.233)$	0.68	level not observed at present		
$\psi(4.405)$	0.51	0.44 ± 0.14^c	$\psi(4.469)$	1.21
$\psi(4.544)$	0.41			
$\Upsilon(\underline{9.40})$	0.69^d		$\Upsilon(\underline{9.40})$	1.88
$\Upsilon(9.99)$	0.31		$\Upsilon(\underline{9.99})$	0.64
$\Upsilon(10.31)$	0.20		$\Upsilon(10.32)$	0.45
$\Upsilon(10.54)$	0.15		$\Upsilon(10.58)$	0.37
$\Upsilon(10.71)$	0.12			
$\Upsilon(10.85)$	0.09			

a) See Ref. [3].

b) Level is to be identified with $\psi(4.028)$; $\Gamma(\ell^+ \ell^-)$ is uncertain.

c) Level is to be identified with $\psi(4.414)$; see Ref. [3].

d) Assuming $\Gamma(1^3S_1 \rightarrow \ell^+ \ell^-) \propto M(\mathcal{V})^{-\frac{1}{2}}$.

FIGURE CAPTIONS

Fig. 1: Regge trajectories of the logarithmic potential.

Fig. 2: Level schemes of the ψ , Υ , and ζ families in (a) nature, (b) the logarithmic potential, and (c) the "modified Coulomb" potential described in the text. The data are from Ref. [4] for the ψ states and from Refs. [1, 2] for the Υ states.

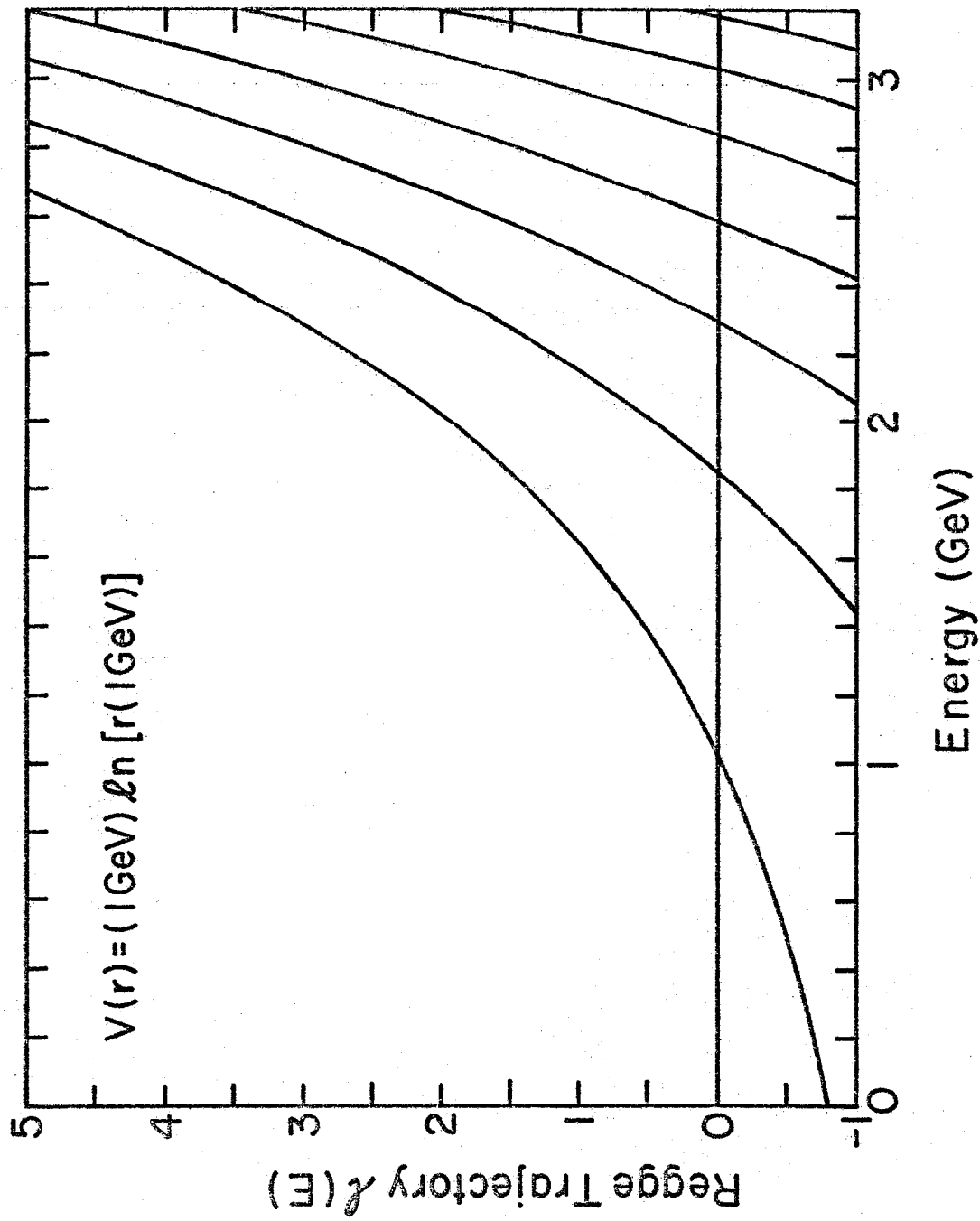


Fig. 1

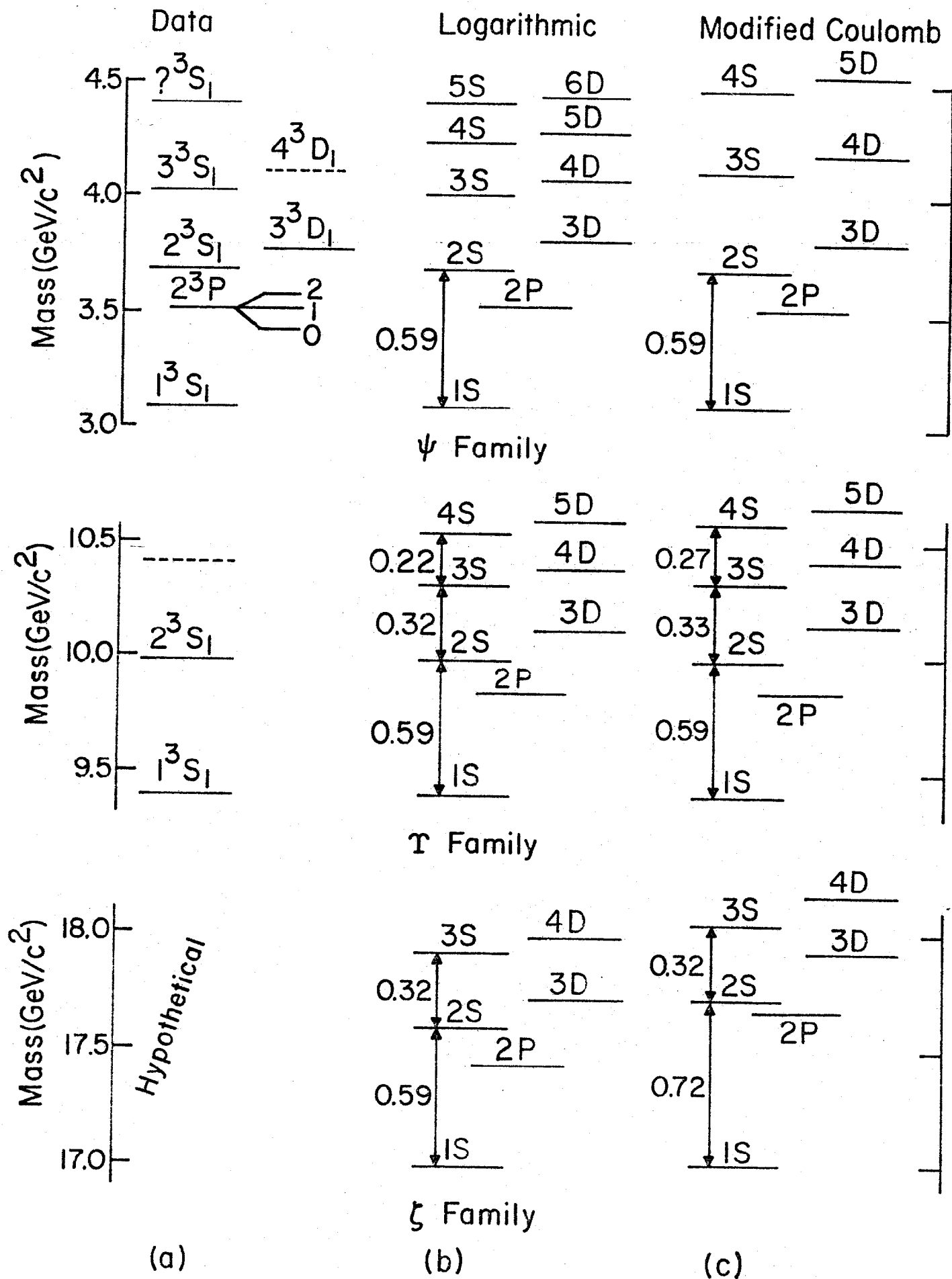


Fig. 2

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