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Instantons in Gauge Groups Larger Than SU(2)

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ABSTRACT

We discuss the construction of instanton solutions to gauge theory for gauge groups larger than SU(2) and show that this may be done either by embedding SU(2) in the larger group or by embedding the full O(4) group. In the latter case Field configurations corresponding to instantons of positive and negative winding numbers are obtained.

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I. INTRODUCTION

Instanton solutions in gauge groups larger than SU(2) may be obtained from those in SU(2)[1] by considering all inequivalent embeddings of SU(2) in the desired larger group.[2] Since the solutions in SU(2) are obtained by mapping the space time algebra of O(4) onto the internal symmetry algebra of SU(2) the solutions obtained by the various embeddings are then those of mapping O(4) onto various SU(2) subalgebras of the larger algebra. In all of these cases the field tensors obtained are either self-dual or antiself-dual corresponding thus to field configurations with positive or negative winding number only; one does not obtain a mixture of these. The action integral is then given by the absolute value of the winding number multiplied by a constant. This procedure of embedding SU(2) into larger algebras is discussed thoroughly in Ref. 2 to which we refer.

We wish to point out in this paper that a new class of solutions may be obtained in higher groups by mapping the space time O(4) algebra onto full O(4) = SU(2) $\times$ SU(2) subalgebras of the larger algebra. These solutions do not lead to self-dual (or antiself-dual) field strength tensors and hence can accommodate mixtures of positive and negative winding number field configurations.

In section II we discuss a general form of the solution to the gauge field equations. The case of the gauge group SU(2) is then briefly discussed in section III. In section IV we discuss in general the case of gauge groups larger than SU(2) and in section V we specialize, as an example, to the group SU(4). We then discuss the general case further in section VI and end with some general remarks in section VII.

II. A GENERAL FORM OF THE SOLUTION

Let G be the gauge group with generators represented by the matrices λ^a, $a = 1 \ldots \eta$ such that

$$\left[\lambda^a, \lambda^b\right] = i f_{abc} \lambda^c \quad . \tag{2.1}$$

Let A_μ^a be the gauge field potentials and define the matrix

$$A_\mu = A_\mu^a \lambda^a \quad . \tag{2.2}$$

The field strength tensor then reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] \tag{2.3}$$

and the equation of motion becomes

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} - i\left[A_\mu, F_{\mu\nu}\right] = 0 \quad . \tag{2.4}$$

As an ansatz we consider the following general form for the solution to Eq. (2.4)[4]

$$A_\mu = - \eta_{\mu\nu} \partial_\nu \ln \phi(x) \quad . \tag{2.5}$$

Here ϕ is a scalar function of x_μ, and $\eta_{\mu\nu}$ are a set of six matrices arranged in the form of a second rank antisymmetric tensor. This is possible if these six matrices satisfy the algebra of the group O(4), namely

$$\left[\eta_{\mu\nu}, \eta_{\kappa\lambda}\right] = i\left[\delta_{\mu\kappa}\eta_{\nu\lambda} - \delta_{\mu\lambda}\eta_{\nu\kappa} - \delta_{\nu\kappa}\eta_{\mu\lambda} + \delta_{\nu\lambda}\eta_{\mu\kappa}\right] \quad . \tag{2.6}$$

For in this case their algebra is not changed under space transformations over the indices μ, ν.

Furthermore, if $\eta_{\mu\nu}$ satisfy the algebra of O(4) the equation of motion Eq. (2.4) is satisfied provided that:

either
$$\partial^2\phi = 0 \tag{2.7a}$$

or
$$\partial^2\phi = \lambda\,\phi^3 \quad . \tag{2.7b}$$

For, using Eq. (2.5) in Eq. (2.3) and making use of Eq. (2.6) we obtain

$$\begin{aligned} F_{\mu\nu} &= \eta_{\mu\alpha}\partial_\nu\partial_\alpha \ln\phi - \eta_{\nu\alpha}\partial_n\partial_\alpha \ln\phi \\ &\quad + \left[\delta_{\mu\nu}\eta_{\alpha\beta} - \delta_{\mu\beta}\eta_{\alpha\nu} - \delta_{\alpha\nu}\eta_{\mu\beta} + \delta_{\alpha\beta}\eta_{\mu\nu}\right] \\ &\quad \times (\partial_\alpha \ln\phi)(\partial_\beta \ln\phi) \end{aligned} \tag{2.8}$$

and the equation of motion becomes after using Eq. (2.6) again:

$$D_\mu F_{\mu\nu} = -\eta_{\nu\alpha}\left[\phi^2\partial_\alpha\left(\frac{1}{\phi^3}\,\partial^2\phi\right)\right] = 0 \quad . \tag{2.9}$$

This is clearly satisfied if either Eq. (2.7a) or (2.7b) is satisfied. Thus, the problem of finding a set of exact solutions to the field equations of a non-abelian gauge theory is reduced to constructing the matrices $\eta_{\mu\nu}$ satisfying Eq. (2.6) and solving Eqs. (7a) or (7b). A general solution to Eq. (7a) is 4

$$\phi = 1 + \sum_{i}^{N} \frac{\beta_i^{\,2}}{|x - R_i|^2} \quad , \quad x \neq R_i \ldots \tag{2.10}$$

where x, R_i are four vectors in Euclidean space; and β_i are arbitrary parameters. A solution to Eq. (7b) is also known:[3]

$$\phi = \sqrt{\lambda}\, \gamma \sqrt{8} \left(1 + \gamma^2 (x - a)^2\right)^{-1} \tag{2.11}$$

where γ and a are arbitrary parameters.

We turn now to the problem of constructing the matrices $\eta_{\mu\nu}$ in G. Let us recall that if we define:

$$\begin{aligned} \eta_{12} &= J_3 & \eta_{01} &= K_1 \\ \eta_{31} &= J_2 & \eta_{02} &= K_2 \\ \eta_{23} &= J_1 & \eta_{03} &= K_3 \end{aligned} \tag{2.12}$$

then Eq. (2.6) becomes

$$\begin{aligned} \left[J_i, Jj\right] &= i\,\epsilon_{ijk}\, J_k \\ \left[J_i, K_j\right] &= i\,\epsilon_{ijk}\, K_k \\ \left[K_i, K_j\right] &= i\,\underline{\epsilon_{ijk}}\, J_k \end{aligned}$$

Furthermore if we define

$$\underset{\sim}{S} = \tfrac{1}{2}(\underset{\sim}{J} + \underset{\sim}{K}) \qquad \underset{\sim}{U} = \tfrac{1}{2}(\underset{\sim}{J} - \underset{\sim}{K}) \quad . \tag{2.13}$$

Then

$$\begin{aligned} \left[S_i, S_j\right] &= i\, \epsilon_{ijk}\, S_k \\ \left[U_i, U_j\right] &= i\, \epsilon_{ijk}\, U_k \\ \left[S_i, U_j\right] &= 0 \end{aligned} \tag{2.14}$$

showing the well-known fact that $O(4) \equiv SU(2) \otimes SU(2)$.

III. THE CASE OF THE GROUP G = SU(2)

The problem of constructing $\eta_{\mu\nu}$ when G = SU(2) is easily solved. In this case the generators of SU(2) may be identified either with the matrices $\underset{\sim}{S}$ above and the $\underset{\sim}{U}$ matrices set to zero, or are identified with the $\underset{\sim}{U}$ matrices and $\underset{\sim}{S}$ set to zero. In other words we may have either one of two possibilities:

$$\text{i)}\ \underset{\sim}{S} = \underset{\sim}{\sigma}/2 \qquad \underset{\sim}{U} = 0 \tag{3.1a}$$

$$\text{ii)}\ \underset{\sim}{U} = \underset{\sim}{\sigma}/2 \qquad \underset{\sim}{S} = 0 \tag{3.1b}$$

where $\underset{\sim}{\sigma}$ are the three pauli matrices.

The first choice implies the identification

$$\underset{\sim}{J} = \underset{\sim}{K} = \underset{\sim}{\sigma}/2 \tag{3.2a}$$

whereas the second choice implies the identification

$$\underset{\sim}{J} = -\underset{\sim}{K} = \underset{\sim}{\sigma}/2 \quad . \tag{3.2b}$$

These two mappings of O(4) onto SU(2) lead to the t'Hooft instanton solutions.[4] For if we denote by $\eta_{\mu\nu}$ and $\overline{\eta}_{\mu\nu}$ the matrices of the mappings (3.2a) and (3.2b) respectively then we have

$$A_\mu = -\eta_{\mu\nu}\partial_\nu \ln \phi \tag{3.3a}$$

$$\overline{A}_\mu = -\overline{\eta}_{\mu\nu}\partial_\nu \ln \phi \tag{3.3b}$$

and if we write these in component form we have

$$A_\mu^a = -\eta^a_{\mu\nu}\ \partial_\nu \ \ln \phi \tag{3.4a}$$

$$\overline{A}_\mu^a = -\overline{\eta}^a_{\mu\nu}\ \partial_\nu \ \ln \phi \tag{3.4b}$$

where $\eta^a_{\mu\nu}$, $\overline{\eta}^a_{\mu\nu}$ are the self (anti-self) adjoint tensors given by t' Hooft[4] and ϕ is given by Eq. (2.10).

Going back to the forms (3.3a) and (3.3b) we find that because of Eqs. (3.2a, b) we have

$$\epsilon_{\mu\nu\lambda\sigma}\eta_{\sigma\alpha} = -\left[\delta_{\alpha\mu}\eta_{\nu\lambda} + \delta_{\alpha\lambda}\eta_{\mu\nu} + \delta_{\alpha\nu}\eta_{\lambda\mu}\right] \tag{3.5}$$

and

$$\epsilon_{\mu\nu\lambda\sigma}\overline{\eta}_{\sigma\alpha} = +\left[\delta_{\alpha\mu}\overline{\eta}_{\nu\lambda} + \delta_{\alpha\lambda}\overline{\eta}_{\mu\nu} + \delta_{\alpha\nu}\overline{\eta}_{\lambda\mu}\right] \quad . \tag{3.6}$$

These in turn lead to the fact that the field strength tensors $F_{\mu\nu}$ and $\overline{F}_{\mu\nu}$ are antiself-dual and self-dual respectively for any ϕ that solves Eq. (2.7a). For the solution of Eq. (2.7b) the reverse is true. Thus in these cases the winding number is either a negative integer or a positive integer with the action integral given by a constant multiple of the absolute value of this number. In the case of Eq. (2.7a) the solution (3.4a) has winding number -N and that of Eq. (3.4b) winding number +N (N being the number of distinct terms in the sum defining ϕ).

IV. GAUGE GROUPS LARGER THAN SU(2)

It must be clear from the above discussion that, in a gauge group G larger than SU(2), the construction of the matrices $\eta_{\mu\nu}$ may proceed by identifying either $\underset{\sim}{S}$ or $\underset{\sim}{U}$ with the matrices satisfying any of the SU(2) subalgebras of the larger algebra G and taking the others to be zero. Thus if λ_i, i = 1, 2, 3, are three matrices among the generators of G which satisfy an SU(2) algebra we may take either of the following two possibilities:

$$\underset{\sim}{S} = \underset{\sim}{\lambda} \qquad \underset{\sim}{U} = 0 \tag{4.1a}$$

$$\underset{\sim}{U} = \underset{\sim}{\lambda} \qquad \underset{\sim}{S} = 0 \quad . \tag{4.1b}$$

The matrices $\eta_{\mu\nu}$ $(\overline{\eta}_{\mu\nu})$ thus obtained would lead to antiself (self) dual field tensors as before. The set of fields A_μ^a have only three nonzero members for values of a corresponding to the three λ_i matrices chosen.

The above procedure is precisely what is followed in obtaining instanton solutions via all inequivalent embeddings of SU(2) in G. The classification of all these possible solutions has been done in Ref. 2 to which we refer for more details.

It must be apparent from our presentation, however, that the above class of solutions in G is not the only one if the algebra of G contains an O(4) subalgebra. For one may construct a new set of $\eta_{\mu\nu}$ matrices in this case by identifying $\underset{\sim}{S}$ and $\underset{\sim}{U}$ with this O(4) subalgebra. Thus if the six matrices σ_i, i = 1, 2, 3, and γ_i, i = 1, 2, 3, among the generators of G satisfy the algebra of $O(4) \equiv SU(2) \otimes SU(2)$:

$$
\begin{aligned}
\left[\sigma_i, \sigma_j\right] &= i\,\epsilon_{ijk}\,\sigma_k \\
\left[\gamma_i, \gamma_j\right] &= i\,\epsilon_{ijk}\,\gamma_k \qquad (4.2) \\
\left[\sigma_i, \sigma_j\right] &= 0
\end{aligned}
$$

then two identifications are possible:

$$\underset{\sim}{S} = \underset{\sim}{\sigma} \qquad \underset{\sim}{U} = \underset{\sim}{\gamma} \qquad (4.3a)$$

$$\underset{\sim}{S} = \underset{\sim}{\gamma} \qquad \underset{\sim}{U} = \underset{\sim}{\sigma} \qquad (4.3b)$$

leading to a new set of $\eta_{\mu\nu}$ matrices and hence a new class of solutions.

In these two cases we have

either $\qquad \underset{\sim}{J} = \underset{\sim}{\sigma} + \underset{\sim}{\gamma} \qquad \underset{\sim}{K} = \underset{\sim}{\sigma} - \underset{\sim}{\gamma}$ (4.4a)

or $\qquad \underset{\sim}{J} = \underset{\sim}{\sigma} + \underset{\sim}{\gamma} \qquad K = \underset{\sim}{\gamma} - \underset{\sim}{\sigma}$ (4.4b)

leading to field potentials $A_\mu^{\ a}$ with six nonzero members for values of a corresponding to the six generators σ_i and γ_i of G. In these cases since $\underset{\sim}{J}$ and $\underset{\sim}{K}$ are identified with different matrices the relationship of Eqs. (3.5) and (3.6) are not satisfied and hence the field strength tensors are not necessarily antiself (or self) dual.

These solutions may therefore correspond to a mixture of positive and negative winding number field configurations.

As an example of a group, aside from O(4),[*,5] that contains an O(4) subgroup we take SU(4). Our discussion will then center on it as an example for the procedure outlined above. The group SU(3) does not contain an O(4) subgroup and hence the only solutions we know in it are those obtained by all inequivalent embeddings of SU(2) in SU(3) as discussed in Ref. 2.

V. THE CASE G = SU(4)

The generators of SU(4) may be represented by the fifteen 4×4 matrices given in Appendix A. The matrices $\sigma_i = \frac{1}{2}\lambda_i$, i = 1, 2, 3, generate an SU(2) subalgebra and commute with another SU(2) subalgebra generated by $\frac{1}{2}\lambda_{13}$, $\frac{1}{2}\lambda_{14}$ and $1/\sqrt{6}\,\lambda_{15} - 1/2\sqrt{3}\,\lambda_8$ which we denote by γ_1, γ_2 and γ_3 respectively. Thus a construction of $\eta_{\mu\nu}$ may proceed now by identifying either $\underset{\sim}{S}$ or $\underset{\sim}{U}$ with either the $\underset{\sim}{\sigma}$ or $\underset{\sim}{\gamma}$ matrices above. Thus the two possibilities

$$\underset{\sim}{J} = \underset{\sim}{\sigma} + \underset{\sim}{\gamma} \qquad \underset{\sim}{K} = \underset{\sim}{\sigma} - \underset{\sim}{\gamma} \tag{5.1a}$$

$$\underset{\sim}{J} = \underset{\sim}{\sigma} + \underset{\sim}{\gamma} \qquad \underset{\sim}{K} = \underset{\sim}{\gamma} - \underset{\sim}{\sigma} \tag{5.1b}$$

lead us to two different field configurations. These are:

$$\begin{aligned} A_i^{\pm} &= \left[\underset{\sim}{\nabla} \ln \phi \times (\underset{\sim}{\sigma} + \underset{\sim}{\gamma})\right]_i \pm \partial_0 \ln \phi \, (\underset{\sim}{\sigma} - \underset{\sim}{\gamma})_i \\ A_0^{\pm} &= \mp (\underset{\sim}{\sigma} - \underset{\sim}{\gamma}) \cdot \underset{\sim}{\nabla} \ln \phi \end{aligned} \tag{5.2}$$

where the + and - signs correspond to the two possibilities (5.1a) and (5.1b) respectively.

If we now take the simplest solution to Eq. (2.7a) namely

$$\phi = 1 + \frac{\beta^2}{R^2} \; ; \; R^2 = x_1^2 + x_2^2 + x_3^2 + x_0^2 \neq 0 \;, \tag{5.3}$$

we find for the potentials of Eq. (5.2):

$$A_i^{\pm} = \frac{-2\beta^2}{R^2(R^2+\beta^2)}\left[\epsilon_{ijk}\, x_j\, (\sigma+\gamma)_k \pm x_0(\sigma-\gamma)_i\right]$$

$$A_0^{\pm} = \mp \frac{(-2\beta^2)}{R^2(R^2+\beta^2)}\ \underset{\sim}{x}\cdot(\underset{\sim}{\sigma}-\underset{\sim}{\gamma}) \quad . \tag{5.4}$$

Furthermore

$$A_i^{+} = \frac{-2\beta^2}{R^2(R^2+\beta^2)}\left[\epsilon_{ijk}x_j\sigma_k + x\sigma_i + \epsilon_{ijk}x_j\gamma_k - x_0\gamma_i\right] = A_i^{(\sigma)} + \overline{A}_i^{(\gamma)} \tag{5.5}$$

$$A_0^{+} = \frac{2\beta^2}{R^2(R^2+\beta^2)}(\underset{\sim}{x}\cdot\underset{\sim}{\sigma} - \underset{\sim}{x}\cdot\underset{\sim}{\gamma}) = A_0^{(\sigma)} + \overline{A}_0^{(\gamma)}$$

where $A_i^{(\sigma)}\left(\overline{A}_i^{(\gamma)}\right)$ and $A_0^{(\sigma)}\left(\overline{A}_0^{(\gamma)}\right)$ are the field potentials one would obtain in an ordinary embedding of SU(2) in G. $A^{(\sigma)}$ is the one obtained by the identification $\underset{\sim}{S} = \underset{\sim}{\sigma}, \underset{\sim}{U} = 0$ where as $\overline{A}^{(\gamma)}$ is obtained by the identification $\underset{\sim}{S} = 0 \quad \underset{\sim}{U} = \underset{\sim}{\gamma}$. This $A^{(\sigma)}$ has winding number minus one and $\overline{A}^{(\gamma)}$ winding number plus one. Since, however, the $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ matrices commute this implies that the total winding number for A_μ^{+} is zero. For then we have:

$$F_{\mu\nu}^{+} = F_{\mu\nu}^{(\sigma)} + \overline{F}_{\mu\nu}^{(\gamma)} \tag{5.6}$$

where
$$F^{*\,(\sigma)}_{\mu\nu} = -F_{\mu\nu}^{(\sigma)}$$

and
$$\overline{F}^{*\,(\gamma)}_{\mu\nu} = +\overline{F}_{\mu\nu}^{(\gamma)} \quad . \tag{5.7}$$

where $F^{*}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Thus as expected

$$F^{*\,+}_{\mu\nu} = -F_{\mu\nu}^{(\sigma)} + \overline{F}_{\mu\nu}^{(\gamma)} \neq F_{\mu\nu}^{+} \quad . \tag{5.8}$$

The winding number is clearly however:

$$q = \frac{1}{8\pi^2} \mathrm{Tr} \int F^{+}_{\mu\nu} {}^{*}F^{+}_{\mu\nu} d^4x = 0 \tag{5.9}$$

since

$$\left[F_{\mu\nu}^{(\sigma)}, \overline{F}_{\mu\nu}^{(\gamma)} \right] = 0 \quad . \tag{5.10}$$

Moreover, since

$$\mathrm{Tr} \left\{ F_{\mu\nu}^{(\sigma)}, \overline{F}_{\mu\nu}^{(\gamma)} \right\} = 0 \tag{5.11}$$

which follows from the λ matrices generating SU(4) we obtain for the action integral

$$S = \frac{1}{4g^2} \mathrm{Tr} \int F_{\mu\nu}^{+} F_{\mu\nu}^{+} d^4x = \frac{8\pi^2}{g^2} \times 2 \quad . \tag{5.12}$$

The potentials A_{μ}^{-} have the same properties since the roles of the $\vec{\underset{\sim}{\sigma}}$ and $\vec{\underset{\sim}{\gamma}}$ matrices are only interchanged.

Thus in both cases we have a superposition of two instantons of winding number plus one and minus one.

We note that had we made the identification $\underset{\sim}{S} = \underset{\sim}{\sigma} + \underset{\sim}{\gamma}$ $\underset{\sim}{U} = 0$ (or $\underset{\sim}{S} = 0$, $\underset{\sim}{U} = \underset{\sim}{\sigma} + \underset{\sim}{\gamma}$) which are possible we would have obtained a winding number minus two (two) instanton with the same action integral as above and whose field strength tensor is antiself (self) dual. This would then be the sum of two winding number minus one (one) instantons (pointed out in Ref. 2) and would be another one of the possible embeddings of SU(2) in G. The embeddings of the full O(4) algebra of Eqs. (5.1a), (5.1b) lead then to a sum of instantons of positive and negative winding number.

For the most general form of ϕ, namely

$$\phi = 1 + \sum_{i=1}^{N} \frac{\beta_i^2}{|x - R_i|^2} \tag{5.13}$$

the potentials $A_\mu^{(\sigma)}$ and $\overline{A}_\mu^{(\gamma)}$ correspond to instantons with winding number -N and +N respectively. However A_μ^+ has winding number zero and the action integral for A_μ^+ is $8\pi^2/g^2 \cdot 2N$. A_μ^- has the same properties although the roles of the $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ matrices are interchanged.

VI. CASE OF LARGER GROUPS

The same procedure as above clearly applies for any large group G containing an O(4) subgroup. In the general case the $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ matrices may correspond to embeddings of SU(2) in G of unequal indices m and n. (For SU(4) above both $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ correspond to embeddings of SU(2) of index one). In such a case and for the choice of $\phi(x)$ as in Eq. (2.7a) the field potentials $A_\mu^{(\sigma)}$ and $\bar{A}_\mu^{(\gamma)}$ which are obtained as in Eq. (5.4) would carry winding numbers $-mN$ and nN respectively. In this case the full field potentials $A_\mu^+ = A_\mu^{(\sigma)} + \bar{A}_\mu^{(\gamma)}$ and $A_\mu^- = \bar{A}_\mu^{(\sigma)} + A_\mu^{(\gamma)}$ have winding number $(n - m)N$ and $(m - n)N$ respectively whereas their action integrals are the same and equal to $2\pi^2/g^2(m + n)N$. Thus in general A_μ^+ and A_μ^- will have opposite winding numbers whereas they have the same action integrals.

Most inequivalent embeddings of O(4) into any larger group G may be inferred by considering all inequivalent embeddings of commuting SU(2) subgroups into G. The values of n and m above are then the indices of these SU(2) embeddings and one may use their (ordered) Dynkin characteristics to classify the various embeddings of O(4). Thus the pair $(\sigma, \gamma) = (m, n)$ will denote an embedding of O(4) with winding number $(n - m)N$ (where N is the number of distinct terms defining $\phi(x)$) whereas $(\sigma, \gamma) = (n, m)$ will have winding number $(m - n)N$. For both field configurations however the action integral is $8\pi^2/g^2\ (m + n)N$.

The general classification of all inequivalent embeddings of O(4) in larger groups is discussed in Ref. 6 to which we refer. We give in Table I the values of m and n for some groups.

VII. CONCLUSION

It must be clear from the forms of $A_\mu^{\pm}$ in Eq. (5.4) that the constituent fields $A_\mu^{(\sigma)}$ and $\bar{A}_\mu^{(\gamma)}$ need not have the same space time dependent functions ϕ. For what is important here is that the $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$ matrices commute. Thus we may write general solutions of the form

$$A_\mu^{+} = A_\mu^{(\sigma)}(\phi_1) + \bar{A}_\mu^{(\gamma)}(\phi_2)$$

and

$$A_\mu^{-} = \bar{A}_\mu^{(\sigma)}(\phi_1) + A_\mu^{(\gamma)}(\phi_2)$$

where ϕ_1 and ϕ_2 may be different sums of the form of Eq. (2.7a). In this case if $\underset{\sim}{\sigma}$, $\underset{\sim}{\gamma}$ correspond to embeddings of indices m and n respectively then the winding number for A_μ^{+} is $nN_2 - mN_1$ and for A_μ^{-}, $mN_1 - nN_2$ whereas their action integrals are the same and equal to $8\pi^2/g^2(nN_2 + mN_1)$.

Thus in a gauge group G that contains an O(4) subgroup our construction allows for any arbitrary mixture of instantons with positive and negative winding number. The action integral is always $8\pi^2/g^2$ times the total number of such instantons whereas the total winding number is the algebraic sum of the winding number of these instantons.

These solutions correspond to local minima of the action and have vanishing Euclidean energy momentum tensor just like the usual solutions obtained by embeddings of SU(2). For, consider a field of the form

$$A_\mu^+ = A_\mu^{(\sigma)}(\phi_1) + \overline{A}_\mu^{(\gamma)}(\phi_2) \quad .$$

Since $\vec{\sigma}$ and $\vec{\gamma}$ commute we have

$$F_{\mu\nu}^+ = F_{\mu\nu}^{(\sigma)}(\phi_1) + \overline{F}_{\mu\nu}^{(\gamma)}(\phi_2) \quad .$$

Furthermore by virtue of Eq. (5.11) we have

$$S^+ = \frac{1}{g^2} \int \mathrm{Tr}\left\{F_{\mu\nu}^{(\sigma)}(\phi_1)F_{\mu\nu}^{(\sigma)}(\phi_1) + \overline{F}_{\mu\nu}^{(\gamma)}(\phi_2)\overline{F}_{\mu\nu}^{(\gamma)}(\phi_2)\right\} d^4x \quad .$$

Now it is well-known that by using the inequality

$$\int (F_{\mu\nu} \pm {}^*F_{\mu\nu}) d^4x \geq 0$$

we can show that

$$\int F_{\mu\nu}^{(\sigma)}(\phi_1)F_{\mu\nu}^{(\sigma)}(\phi_1) \geq \frac{8\pi^2}{g^2} |m| N_1$$

and that

$$\int \overline{F}_{\mu\nu}^{(\gamma)}(\phi_2)F_{\mu\nu}^{(\sigma)}(\phi_2) \geq \frac{8\pi^2}{g^2} |n| N_2$$

where if ϕ_1 and ϕ_2 are of the form of Eq. (5.13) with N_1 and N_2 terms respectively, the bound is saturated. Thus it is clear that in general

$$S^+ \geq \frac{8\pi^2}{g^2}(|m| N_1 + |n| N_2)$$

and our solutions saturate the bound. Hence a solution characterized by the numbers m, n, N_1 and N_2 which would have winding number $q = nN_2 - mN_1$ is topologically inequivalent to another solution with the same net winding number q but different indices n and m as they are both local minima around two different values of the total action. For example the field configuration of one instanton and one anti instanton gives a local minimum of the action at value $2 \cdot 8\pi^2/g^2$ although its net winding number is zero. In fact one can see easily that for any value of the action $8\pi^2/g^2 \times K$ there are K + 1 topologically inequivalent local minima corresponding to all possible O(4) embeddings that we exhibit.

Furthermore it is useful to point out that the euclidean energy momentum tensor $\theta_{\mu\nu} = \frac{1}{4} \mathrm{Tr}\, (F_{\mu\alpha} - {}^*F_{\mu\alpha})(F_{\alpha\nu} + {}^*F_{\alpha\nu})$ also vanishes for our solutions by virtue of Eq. (5.11) and that this fact as well as the value of the action integral are independent of the positions of the poles in the functions ϕ_1 and ϕ_2 of Eq. (5.13).

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APPENDIX

The matrix representation for the generators of SU(4) are taken in the text as:

$$\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \lambda_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

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†Footnote:

In general O(4) has two topological invariants, namely

$$q_1 \approx \int d^4x \, F^{\gamma\delta}_{\mu\nu} \, F^{\gamma\delta}_{\alpha\beta} \, \epsilon_{\mu\nu\alpha\beta}$$

and

$$q_2 \approx \int d^4x \, \epsilon_{\mu\nu\alpha\beta} \epsilon_{\gamma\delta\sigma\xi} F^{\gamma\delta}_{\mu\nu} \, F^{\sigma\xi}_{\alpha\beta} \quad .$$

However, q_2 is not an invariant of a larger group in which O(4) is embedded and hence may not be used to classify pseudoparticle solutions in these larger groups.

SOME
TABLE I. Embedding of O(4) into Some Groups G. [m, n are indices for the possible embeddings of two commuting SU(2) groups]

G:	m	n	
SU(4)	1	1	
SU(5)	1	1	
	1	4	
	4	1	
SU(6)	1	1	three ways
	1	4	
	4	1	
	4	4	
	10	1	
	1	10	
SU(7)	1	1	three ways
	4	1	
	1	4	
	4	2	
	2	4	
	4	4	
	10	1	
	1	10	

SU(7) cont'd	10	4	
	4	10	
	20	1	
	1	20	
SU(8)	1	1	six ways
	2	2	
	1	3	
	3	1	
	4	1	
	1	4	
	2	4	
	4	2	
	4	4	
	8	1	
	1	8	
	1	10	
	10	1	
	10	2	
	2	10	
	4	10	
	10	4	

SU(8) cont'd	10	10	
	20	1	
	1	20	
	20	4	
	4	20	
	35	1	
	1	35	
E(7)	1	1	many ways
	4	1	
	1	4	

and many others