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Classification of Pseudo Particle Solutions in Gauge Theories

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ABSTRACT

We discuss the classification of pseudo particle solutions obtained using all inequivalent embeddings of SU(2) in a semi-simple algebra G. We also discuss some simple consequences of this classification on the interaction of Belavin et al. pseudo particles.

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I. INTRODUCTION

It has been suggested by Belavin et al.[1] and more recently by Wilczek[2] that one may construct pseudo-particle --or instanton[3]-- solutions of the gauge field equations in the group G by embedding the group SU(2) in G. Due to the fact that there exist inequivalent imbeddings of SU(2) in SU(N) ($N > 2$), Wilczek was able to draw some conclusions about the 4-particle interactions from pseudo-particle solutions in SU(3), and to construct solutions of winding number $q > 1$ in SU(N).

An embedding of SU(2) in SU(N), for example, is completely defined by a set of N numbers known as the defining vector of the embedding, and can be graphically represented by a diagram known as the Dynkin characteristic[4]. Half the square of the length of the defining vector is called the index of the embedding. In most cases, the knowledge of the index is sufficient to specify completely the embedding. However, it may happen that inequivalent embeddings admit the same index: in this case reference should be made to the corresponding Dynkin characteristics.

We show that the charge -- or Pontryagin index -- is a multiple of the index of a particular embedding of SU(2) in G. We illustrate our classification by performing explicitly the classification of the embeddings for some simple algebras G (SU(N), for N = 3,4,5,6,7,8 and the exceptional algebra E_7).

The gauge transformation properties of the solutions are completely determined by the Dynkin characteristic: this allows us to assign relevant quantum numbers for the various solutions obtained in this manner. We do this in the cases $G = SU(3)$ and $SU(4)$ as examples. Moreover a mathematical property of the embeddings is used to discuss the interaction of instantons.

This paper is organized as follows. In Section II, we recall the pseudo particle solution of Belavin et al. for $G = SU(2)$, $q = 1$, and their extension to $G = SU(N) \supset SU(2)$, $q > 1$ by Wilczek. Section III deals with the mathematical apparatus needed for our classification. This classification of the instanton solutions is achieved in Section IV. Finally Section V is devoted to a discussion of some simple physical properties (interaction) of these solutions.

II. PSEUDO PARTICLE SOLUTIONS

We consider the gauge group SU(2) in the four dimensional Euclidean space, and define the gauge fields by:

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + i\ \epsilon_{ijk}\ A^j_\mu A^k_\nu \quad i = 1,2,3 \text{ and } \mu, \nu = 1,2,3,4. \tag{II. 1}$$

If we use matrix notation and write

$$A_\mu = A^i_\mu \frac{\sigma_i}{2}\ , \tag{II. 2}$$

σ_i being the usual Pauli matrices, then:

$$F_{\mu\nu} = F^i_{\mu\nu} \frac{\sigma_i}{2}\ , \tag{II. 3}$$

and the effective interaction is:

$$S = \frac{1}{4g^2}\ T_r \int F_{\mu\nu} F^{\mu\nu} d^4x\, . \tag{II. 4}$$

It has been shown recently by Belavin et al.[1] that there exist nontrivial solutions to the classical equations of motion, characterized by a topological quantum number, known as the Pontryagin index, and given by:

$$q = \frac{1}{8\pi^2}\ T_r \int F_{\mu\nu} \tilde{F}^{\mu\nu} d^4x\, . \tag{II.5}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\ \epsilon_{\mu\nu\rho\sigma}\ F^{\rho\sigma}$ is the dual of $F_{\mu\nu}$. This index q may be shown to take always integer values.

An important inequality holds for the effective action. For since:

$$T_r \int (F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 \ d^4x \ \geq 0 \ , \qquad \text{(II.6)}$$

then:

$$S \geq \frac{2\pi^2}{g^2} \ |q| \ . \qquad \text{(II.7)}$$

It is clear that if there exists a field such that for $q \neq 0$: $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, then the inequality is saturated and this $F_{\mu\nu}$ is a non-trivial solution to the classical field equations.

Belavin et al. construct such a solution for $q = 1$. This is given by:

$$A_\mu = \frac{x^2}{x^2 + \lambda^2} \ g^{-1} \ \partial_\mu g \ ; \ g = \frac{x_4 + i\,\vec{x}\cdot\vec{\sigma}}{|x|} \ , \qquad \text{(II.8)}$$

where λ is an arbitrary parameter, or using the Jackiw-Rebbi gauge transformation[5]

$$A_\mu = - \frac{\lambda^2}{x^2 + \lambda^2} \ (\partial_\mu g) \ g^{-1} \ . \qquad \text{(II.9)}$$

If we write this solution in the form $A_\mu = A^i_\mu(x) \frac{\sigma_i}{2}$ leading to the field strength: $F_{\mu\nu} = F^i_{\mu\nu}(x) \frac{\sigma_i}{2}$, then if $T_i, i = 1, 2, 3$ form any representation of SU(2): $[T_i, T_j] = 2_i \epsilon_{ijk} T_k$, the potential $A_\mu = A^i_\mu(x) \frac{T_i}{2}$ will give the field strength: $F^i_{\mu\nu}(x) \frac{T_i}{2}$, which is self dual and hence saturates the inequality for some q. Thus, if the gauge group is G, all such instanton solutions in G may be obtained in this manner by embedding SU(2) into G in all inequivalent ways.[1,2] All such solutions are thus characterized by all inequivalent embeddings of SU(2) in G.

III. EMBEDDING OF THE SU(2) ALGEBRA IN A (SEMI) SIMPLE ALGEBRA G

The mathematical problem of embedding a simple subalgebra $\tilde{G}$ in a simple algebra G has been discussed by Dynkin.[4]

Consider a semisimple complex algebra G and K its Cartan subalgebra. Let $\tilde{G}$ be a subalgebra of G, with $\tilde{K}$ its Cartan subalgebra. An embedding of $\tilde{G}$ into G is completely defined by a mapping from $\tilde{K}$ into K

$$f(\tilde{H}_i) = \sum_{k=1}^{n} f_{ik} H_k \qquad i = 1,-----,\ell \ , \tag{III.1}$$

with $\tilde{H}_i$ and H_i the elements of $\tilde{K}$ and K respectively, where ℓ is the rank of $\tilde{G}$ and n the rank of G.

The relation:

$$\left(f(\tilde{X}),\ f(\tilde{Y})\right) = j_f(\tilde{X},\tilde{Y}) \qquad \tilde{X}, \tilde{Y} \in \tilde{G} \ , \tag{III.2}$$

in which $(\tilde{X},\tilde{Y}) = T_r\, \mathrm{ad}\tilde{X} \cdot \mathrm{ad}\tilde{Y}$ is the killing form relative to $\tilde{X}$ and $\tilde{Y}$, determines a scalar factor j_f independent of $\tilde{X}, \tilde{Y}$, and is called the index of the embedding.

The set of numbers f_{ik} form the matrix of the embedding. The embedding of the shift operators $\tilde{E}_{\tilde{\alpha}}$ of the subalgebra $\tilde{G}$ in G is given as:

$$f(\tilde{E}_{\tilde{\alpha}}) = \sum_{\alpha \in \Gamma_{\tilde{\alpha}}} C_{\tilde{\alpha}\alpha} \ E_{\alpha} \ , \tag{III.3}$$

with the $C_{\tilde{\alpha}\alpha}$ complex numbers determined through f_{ik} by the embedding of the roots $\tilde{\alpha}$ into the subset $\Gamma_{\tilde{\alpha}}$ of the root system Γ of the algebra G, defined by:

$$\Gamma_{\tilde{\alpha}} = \left\{ \alpha \in \Gamma \middle| f^{*}(\alpha) = \tilde{\alpha} \right\} , \tag{III.4}$$

the map f^{*} being defined through the relation:

$$(H, f(\tilde{H})) = (f^{*}(H), \tilde{H}) , \tag{III.5}$$

for any two elements H and $f(\tilde{H})$ in K. The index j_f in the embedding f may also be defined by:

$$j_f = \sum_{\alpha \in \Gamma_{\tilde{\alpha}}} |C_{\tilde{\alpha}\alpha}|^2 , \tag{III.6}$$

as long as G and $\tilde{G}$ are different from C_n.[6] The index j_f is a non-negative integer. Embeddings of the same algebra $\tilde{G}$ in an algebra G which are related through an inner automorphism of G are equivalent embeddings and the subalgebras of G which correspond to these embeddings are conjugate subalgebras of G. (Two algebras are called conjugate if they are related by an inner automorphism.) Two equivalent embeddings have the same matrix (f_{ik}) and hence the same index j_f.

In most cases the indices j_f corresponding to inequivalent embeddings are different and hence may be used to label the embeddings. However, the indices of inequivalent embeddings may coincide, in which case one must refer to the defining matrix (f_{ik}).

If $\tilde{G}$ is isomorphic to SU(2) then the matrix (f_{ik}) bedomes a vector (f_k) $k = 1 \ldots . n$ called the defining vector of the embedding. The index is in this case given by:

$$j_f = \frac{1}{2} \sum_{k=1}^{n} f_k^2 , \qquad \text{(III.7)}$$

which is half the square of the length of the defining vector, in any m-dimensional Euclidean space with $m \geq n$.

It is well known that one can choose among the set of (positive) roots of an algebra G a subset of n roots $\alpha^{(1)} \ldots . \alpha^{(n)}$ such that any root is a linear combination of $\alpha^{(i)}$ with either all non-negative or all non-positive integers. This subset of n roots is called a system of simple roots for G. It holds that:

$$\frac{(\alpha^{(i)}, \alpha^{(j)})^2}{(\alpha^{(i)}, \alpha^{(i)})(\alpha^{(j)}, \alpha^{(j)})} = \frac{r}{4} \quad r = 0,1,2,3 \quad i \neq j , \qquad \text{(III.8)}$$

and $$(\alpha^{(i)}, \alpha^{(j)}) \leq 0.$$

The longest root is normalized through the usual scalar product

$$(\alpha, \alpha) = 2, \qquad \text{(III.9)}$$

except for C_n.[6]

A root system Γ of a simple Lie algebra G contains at most roots of two different lengths.

A graphical representation of the system of simple roots is given by a Dynkin diagram[4]: Every simple root is represented by a circle with a number of lines joining two circles equal to r, as defined in Eq. (II.8). When roots of different lengths exist, shorter roots are represented by filled in circles. The simple roots of $SU(N) = A_{N-1}$ are given for example as:

$$\vec{\alpha}^{(i)} = \hat{e}_i - \hat{e}_j \quad (i, j = 1,2,\ldots N;\ i \neq j)$$

where $\hat{e}_i$ for a set of orthonormal unit vectors in the N-dimensional Euclidean space. The corresponding Dynkin diagram is shown in Fig. 1a

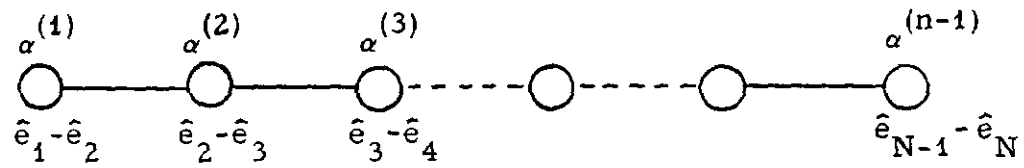

Fig. 1a

Dynkin Diagram for SU(N).

Fig. 1b shows the Dynkin Diagram for the exceptional algebra G_2 as another example.

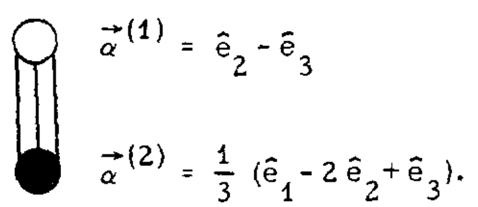

Fig. 1b

Dynkin Diagram for G_2.

If we label the circles of $\vec{\alpha}^{(i)}$ in the Dynkin diagram of G by the numbers $(\vec{f}, \vec{\alpha}^{(i)})$, where $\vec{f}$ is the defining vector of an embedding of SU(2) in G, we obtain the Dynkin characteristic of this embedding.

The following two theorems, due to Dynkin, will be very useful for our discussion.

Theorem D. I.[7]

> A necessary and sufficient condition that two three dimensional subalgebras of the semi-simple algebra G should be transformable into each other by an automorphism of G is that their characteristics coincide.

In the case of simple algebras G the word "automorphism" above may be replaced by "inner-automorphism".

It can be shown furthermore that every number which is written into the characteristic diagram is 0, 1, or 2.

Theorem D. II.[8]

> Let $f_1, f_2, \ldots f_3$ be embeddings of a simple algebra $\tilde{G}$ into the simple algebra G and let
>
> $$\left[f_i(X) , f_j(Y)\right] = 0 \quad (i \neq j; X, Y \epsilon \tilde{G}) ,$$
>
> then $f = f_1 + f_2 + \ldots f_3$ is likewise an embedding and

$$j_f = j_{f_1} + j_{f_2} + \ldots + j_{f_3} . \tag{III.10}$$

Theorem D.I. is useful in classifying pseudo particle solutions and Theorem D.II indicates cases when two solutions may be added to form a third.

Now, let the matrices T^i, $i = 1, 2, 3$ with

$$\left[T^i, T^j\right] = 2i\epsilon_{ijk} T^k, \tag{III.11}$$

represent the embedding f of SU(2) in G. This embedding is characterized by the defining vector (f_k) such that $T_3 = \sum_{k=1}^{n} f_k H_k$, and its index is $j_f = \frac{1}{2} \sum_{k=1}^{n} f_k^2$. This is half the square of the "length of the vector" T_3 in any Euclidean space where H_k form a set of unit orthonormal vectors.

If G = SU(N), then it is convenient to consider the root space in a hyperplane of the N dimensional Euclidean space. The H_k are (N-1) N x N diagonal matrices which can be written as a sum over the N-dimensional orthonormal basis:

$$E_1 = \begin{pmatrix} 1 & & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \end{pmatrix} E_2 = \begin{pmatrix} 0 & & & \\ & 1 & & 0 \\ & & 0 \ddots & \\ & 0 & & 0 \end{pmatrix} E_N = \begin{pmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 0 & \\ & 0 & & 1 \end{pmatrix} \tag{III.12}$$

In this basis T_3 is represented by:

$$T_3 = \sum_{k=1}^{N} f'_k E_k . \tag{III.13}$$

It is clear from Eq. (III.12) that f'_k are simply the diagonal elements of T_3. If we denote these elements[9] by: $m_J = 2J, 2(J-1), \ldots, -2J$; $m_{J'} = 2J', \ldots, 3J' \ldots$ where $(2J+1) + (2J'+1) + \ldots = N$, then the index of the embedding becomes:

$$j_f = \frac{1}{2} \sum_{J,J',\ldots} m_J^2 \,, \qquad \text{(III.14)}$$

or:

$$j_f = \frac{1}{2} \; T_r \, T_3^2 \;. \qquad \text{(III.15)}$$

Note that Eq. (III.15) may be directly obtained from Eqs. (III.1 & 2), taking proper care of normalization factors.

It is obvious to generalize this property for any semi-simple algebra. Note that if the gauge group is of the form $G^{(1)} x G^{(2)} x \ldots x G^{(k)}$ with $G^{(1)}, G^{(2)}, \ldots G^{(k)}$ simple, then the corresponding defining vectors are sets of k vectors $f^{(1)}, \ldots f^{(k)}$, with $f^{(i)}$ characterizing the embedding of SU(2) in $G^{(i)}$. In this case the index is the sum of the indices $j_{f^{(i)}}$.

IV. CLASSIFICATION OF PSEUDO-PARTICLE SOLUTIONS

Let $F^i_{\mu\nu} T_i$ represent solutions obtained by embedding SU(2) into G. The Pontryagin index is given by:

$$q = \frac{1}{8\pi^2} T_r \int F_{\mu\nu} F^{\mu\nu} d^4x = \frac{1}{8\pi^2} T_r (T_3)^2 \int F^i_{\mu\nu} F^{\mu\nu}_i d^4x$$

$$= j_f \frac{1}{4\pi^2} \int F^i_{\mu\nu} F^{\mu\nu}_i d^4x . \tag{IV.1}$$

where j_f is the index of the embedding.

For the solution (III.8) of Belavin et al., $q = j_f = 1$. In other words the integral $\frac{1}{4\pi^2} \int F^i_{\mu\nu} F^{\mu\nu}_i d^4x = 1$. A new set of solutions[10] where this integral takes all integral values has been obtained by Witten, i.e. $q = j_f \cdot w$ where $j_f = 1$, w any integer.

For any embedding of SU(2) in G, we get a generalization of the Belavin et al., solutions with:

$$q = j_f \cdot w , \tag{IV.2}$$

where j_f is the index of the embedding.

It must be clear from our discussion of Section III that the index j_f is not enough to classify the embedding, and hence the solutions. One must refer to the Dynkin characteristic of the embeddings. In the following, we discuss the solutions for G = SU(N) N = 3,4,... 8

and for E_7, one of the exceptional groups, which are also of physical interest.

Let us mention at this point that the classification of the embeddings of SU(2) into simple algebras of rank ≤ 6 are given in Ref. (11), and into the exceptional algebras in Ref. (4).

Case G = SU(N):

The inequivalent embeddings of SU(2) into SU(N), N = 3,4,5,6,7,8 are given in Table I.

In Table II we show the characteristics for the cases SU(3) and SU(4). Let us recall that the Dynkin characteristic is simply a graphical description of the defining vector. The second column shows the minimal including regular subalgebra. One calls G' a regular subalgebra of G if the set $\Gamma_{\alpha'}$ contains one and only one root α of G for every root $\alpha' \in \Gamma'$ in G' (see Eqs. III.4 & 5). The knowledge of these subalgebras will be useful in the discussion of Section V to visualize the positions of the different SU(2) in the matrix adjoint representation of G.

We notice that when N is odd, the $\frac{N+1}{2}^{\text{th}}$ coordinate of the defining vector is 0, and the other coordinates are symmetric with respect to it. When N is even, the coordinates are symmetric with respect to the $\frac{N}{2}^{\text{th}}$ comma.

Every embedding specifies the transformation properties of the pseudo-particle solution under G. For example, in the case of SU(3), the solutions with $j_f = 1$ and 4 transform like λ_3; in the case of SU(4) they transform like: $\lambda_3 - \frac{1}{\sqrt{3}}\lambda_8 + \sqrt{\frac{2}{3}}\lambda_{15}$ for $j_f = 2$, and $\lambda_3 + \sqrt{3}\lambda_8 + \sqrt{6}\lambda_{15}$ for $j_f = 10$.

Note that in the case of SU(8), one obtains two solutions with the same index but different defining vector. The degeneracy of j_f, already seen by Wilczek[2], emphasizes the need of the full defining vector (or characteristic) to classify solutions.

Case G = E(7):

As another particular case of different characteristics corresponding to the same index, we mention in Table III few of the embeddings of SU(2) into E(7). A complete list is given in Ref. (4).

For all other simple groups B_ν, C_ν, D_ν and G_2, F_4, E_6, E_8, complete listings of all SU(2) embeddings are given in Ref. 4 and 11.

Table I: Embeddings of SU(2) into SU(N) $3 \leq N \leq 8$.

Algebra	Minimal Including Regular Subalgebra	Index j_f	Defining Vector
SU(3)	A_1	1	(1,0,-1)
	A_2	4	(2,0,-2)
SU(4)	A_1	1	(1,0,0,-1)
	$2A_1$	2	(1,1,-1,-1)
	A_2	4	(2,0,0,-2)
	A_3	10	(3,1,-1,-3)
SU(5)	A_1	1	(1,0,0,0,-1)
	$2A_1$	2	(1,1,0,-1,-1)
	A_2	4	(2,0,0,0,-2)
	$A_2 + A_1$	5	(2,1,0,-1,-2)
	A_3	10	(3,1,0,-1,-3)
	A_4	20	(4,2,0,-2,-4)
SU(6)	A_1	1	(1,0,0,0,0,-1)
	$2A_1$	2	(1,1,0,0,-1,-1)
	$3A_1$	3	(1,1,1,-1,-1,-1)
	A_2	4	(2,0,0,0,0,-2)
	$A_2 + A_1$	5	(2,1,0,0,-1,-2)
	$2A_2$	8	(2,2,0,0,-2,-2)
	A_3	10	(3,1,0,0,-1,-3)

Table I: Embeddings of SU(2) into SU(N) $3 \le N \le 8$. (Cont.)

Algebra	Minimal Including Regular Subalgebra	Index j_f	Defining Vector
SU(6) (Cont.)	$A_3 + A_1$	11	(3,1,1,-1,-1,-3)
	A_4	20	(4,2,0,0,-2,-4)
	A_5	35	(5,3,1,-1,-3,-5)
SU(7)	A_1	1	(1,0,0,0,0,0,-1)
	$2A_1$	2	(1,1,0,0,0,-1,-1)
	$3A_1$	3	(1,1,1,0,-1,-1,-1)
	A_2	4	(2,0,0,0,0,0,-2)
	$A_2 + A_1$	5	(2,1,0,0,0,-1,-2)
	$A_2 + 2A_1$	6	(2,1,1,0,-1,-1,-2)
	$2A_2$	8	(2,2,0,0,0,-2,-2)
	A_3	10	(3,1,0,0,0,-1,-3)
	$A_3 + A_1$	11	(3,1,1,0,-1,-1,-3)
	$A_3 + A_2$	14	(3,2,1,0,-1,-2,-3)
	A_4	20	(4,2,0,0,0,-2,-4)
	$A_4 + A_1$	21	(4,2,1,0,-1,-2,-4)
	A_5	35	(5,3,1,0,-1,-3,-5)
	A_6	56	(6,4,2,0,-2,-4,-6)
SU(8)	A_1	1	(1,0,0,0,0,0,0,-1)
	$2A_1$	2	(1,1,0,0,0,0,-1,-1)
	$3A_1$	3	(1,1,1,0,0,-1,-1,-1)

Table I: Embeddings of SU(2) into SU(N) $3 \leq N \leq 8$. (Cont.)

Algebra	Minimal Including Regular Subalgebra	Index j_f	Defining Vector
SU(8) (Cont.)	$4A_1$	4'	(1,1,1,1,-1,-1,-1,-1)
	A_2	4''	(2,0,0,0,0,0,0,-2)
	$A_2 + A_1$	5	(2,1,0,0,0,0,-1,-2)
	$A_2 + 2A_1$	6	(2,1,1,0,0,-1,-1,-2)
	$2A_2$	8	(2,2,0,0,0,0,-2,-2)
	$2A_2 + A_1$	9	(2, 2, 1, 0,0,-1,-2,-2)
	A_3	10	(3,1,0,0,0,0,-1,-3)
	$A_3 + A_1$	11	(3,1,1,0,0,-1,-1,-3)
	$A_3 + 2A_1$	12	(3,1,1,1,-1,-1,-1,-3)
	$A_3 + A_2$	14	(3,2,1,0,0,-1,-2,-3)
	$2A_3$	20'	(3,3,1,1,-1,-1,-3,-3)
	A_4	20''	(4,2,0,0,0,0,-2,-4)
	$A_4 + A_1$	21	(4,2,1,0,0,-1,-2,-4)
	$A_3 + A_2$	24	(4,2,2,0,0,-2,-2,-4)
	A_5	35	(5,3,1,0,0,-1,-3,-5)
	$A_5 + A_1$	36	(5,3,1,1,-1,-1,-3,-5)
	A_6	56	(6,4,2,0,0,-2,-4,-6)
	A_7	86	(7,5,3,1,-1,-3,-5,-7)

Table II: Dynkin Characteristics for SU(2) ⊂ SU(3) and SU(4).

Algebra		Index	Dynkin Characteristic
SU(3)	SU(2) case:	1	$\overset{1}{\circ}\!-\!\!-\!\!-\!\overset{1}{\circ}$
	SO(3) case:	4	$\overset{2}{\circ}\!-\!\!-\!\!-\!\overset{2}{\circ}$
SU(4)		1	$\overset{1}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\!-\!\overset{1}{\circ}$
		2	$\overset{0}{\circ}\!-\!\!-\!\!-\!\overset{2}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}$
		4	$\overset{2}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ}\!-\!\!-\!\!-\!\overset{2}{\circ}$
		10	$\overset{2}{\circ}\!-\!\!-\!\!-\!\overset{2}{\circ}\!-\!\!-\!\!-\!\overset{2}{\circ}$

Table III: Embeddings of SU(2) into E(7) for $j_f \leq 5$.

Index	Minimal Including Regular Subalgebra	Characteristic					
		0	0	0	0	0	0
					0		
1	A_1	0	0	0	0	0	1
					0		
2	$2A_1$	0	1	0	0	0	0
					0		
3'	$[3A_1]'$	0	0	0	0	1	0
					0		
3"	$[3A_1]''$	2	0	0	0	0	0
					0		
4'	$[4A_1]'; A_2$	0	0	0	0	0	2
					0		
4"	$[4A_1]''$	1	0	0	0	0	0
					1		
5	$5A_1; A_2 + A_1$	0	1	0	0	0	1
					0		

V. SUMS OF PSEUDOPARTICLE SOLUTIONS

We consider now the problem of adding two or more solutions to get a third. It is clear that when two or more solutions are simply added, the sum does not form in general a solution, except finally when their positions coincide. This later possibility arises when the sum of two --or more-- embeddings of SU(2) in G is itself an embedding.

Consider the sum of solutions

$$A_\mu = A^i_\mu \,(x - t_a)^{(a)}\, T_i + A^i_\mu \,(x - t_b)^{(b)}\, T_i + \dots \quad .$$

From the general form of the action, one clearly sees that:

$$S > \frac{2\pi^2}{g^2} \left(|q_{(a)}| + |q_{(b)}| + \dots \right) .$$

If A_μ were a solution the equality sign holds. When we let all $t_{(a)}, t_{(b)}, \dots$ go to zero, the form A_μ becomes a solution only if the sum

$${}^{(a)}T_i + {}^{(b)}T_i + \dots = T_i$$

with $[T_i, T_j] = 2i\epsilon_{iyk} T_k$. This clearly would be the case if the sum of the embeddings (a), (b)... is itself an embedding of SU(2) in G. Then the action becomes:

$$S = \frac{2\pi^2}{g^2} \left(|q_{(a)}| + |q_{(b)}| + \dots \right) .$$

Using Theorem D.2 of Section III, we can see that this would happen in the following cases.

i) SU(4):

The embedding with $j_f = 2$ may be obtained from the sum of two embeddings of $j_f = 1$. This is because one can add two equivalent embeddings of the form $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$, with T being 2×2 matrices corresponding to $j_f = 1$, to form the inequivalent embedding $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ with $j_f = 2$ in SU(4).

ii) SU(5):

For the same reason as in the case of SU(4) above, $j_f = 2$ solutions may be obtained from the sum of two $j_f = 1$ solutions. Moreover, the $j_f = 5$ solution can be obtained as the sum of the $j_f = 1$ and $j_f = 4$ solutions. This is because for the $j_f = 5$ solutions the corresponding matrices are of the form: $\begin{pmatrix} T & 0 \\ 0 & T' \end{pmatrix}$ with T 2×2 matrices and T' 3×3 matrices, whereas for $j_f = 4$, these are of the form: $\begin{pmatrix} 0 & 0 \\ 0 & T' \end{pmatrix}$ and for $j_f = 1$ of the form $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$.

iii) SU(6):

In this case, just as in SU(5), the $j_f = 2$ may be obtained as the sum of two $j_f = 1$ and $j_f = 5$ as the sum of $j_f = 4$ and $j_f = 1$. Moreover, the $j_f = 3$ solution may be considered as the sum of three $j_f = 1$ solutions, or of one $j_f = 1$ and one $j_f = 2$ solutions, for now the matrices for $j_f = 3$ are of the form $\begin{pmatrix} T & & \\ & T & \\ & & T \end{pmatrix}$. Similarly the $j_f = 8$ may be the sum of two $j_f = 4$, and $j_f = 11$ the sum of $j_f = 10$ and $j_f = 1$.

iv) SU(7):

The list grows and in addition to the case of SU(6) we have: $j_f = 6$ the sum of $j_f = 4$ and $j_f = 2$, or $j_f = 5$ and $j_f = 1$, or $j_f = 4$ and $j_f = 1$ and $j_f = 1$; $j_f = 14$ the sum of $j_f = 10$ and $j_f = 4$; and $j_f = 21$ the sum of $j_f = 20$ and $j_f = 1$.

v) SU(8):

In SU(8), the list goes further. However, we note that $j_f = 4'$ (following notation of Table I) may be obtained as the sum of four $j_f = 1$, or two $j_f = 2$, or $j_f = 1$ and $j_f = 3$, or two $j_f = 1$ and one $j_f = 2$, while for the inequivalent embedding $j_f = 4''$ Theorem D.2 does not apply. Nevertheless, as shown by Wilczek[2] in SU(3) and hence in all SU(N), $N > 3$, one can consider $j_f = 4''$ as the sum of four $j_f = 1$.

A similar discussion can be done in the case of all other semi-simple groups, see for example Table III for the case of E(7).

Another set of cases where solutions can be seen as the sum of two or more, may be obtained using the sum constructed by Wilczek in the case of SU(3). For example one can see that the $j_f = 8$ solution in SU(6) can be obtained as the sum of $4\ j_f = 2$ solutions. We notice however that the defining vectors corresponding to $j_f = 1$ and $j_f = 4$ are parallel, so are the defining vectors for $j_f = 2$ and $j_f = 8$ in SU(6). We conjecture that in any semi-simple algebra G, when two embeddings f and f' are such that their defining vectors are parallel, i.e. $f' = \alpha f$, $\alpha > 1$, then it is possible to sum α^2 embeddings f and obtain an embedding f'.

We remark here that the solutions[10] in SU(2) with $9>1$ appear to be a sum of separated solutions of $j_f = 1$ which is itself a solution because of the special field configuration it has in space. The individual instantons in this case are in a sense free as there is no interaction energy. These solutions however are of the general form $A^i_\mu(x)\frac{\sigma_i}{2}$ and hence similar solutions may be obtained in G by replacing σ_i by T_i, the various embeddings of SU(2) in G, and the interaction energy would still be zero.

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[6] In the case of the algebras C_n, the longest root is normalized such that $(\alpha, \alpha) = 4$. When $\tilde{G} = SU(2)$, the index j_f is $= 2 \sum_{\alpha \in \Gamma_{\tilde{\alpha}}} |C_{\alpha'\alpha}|^2 = \sum_k f_k f_k$.

[7] Reference (4). Theorem 8.2.

[8] Reference (4). Theorem 2.3.

[9] Note that the C. R. defined in (III.9) imply matrices which are twice the usual one.

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$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & 1 & 0 \cdots & 0 \\ & 0 & & \vdots \\ & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$E_N = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & 0 & \cdots 0 & \vdots \\ & \vdots & \vdots & \vdots \\ & \vdots & 0 & 0 \\ 0 & 0 & \cdots 0 & 1 \end{pmatrix}$$

$$E_1 = \begin{matrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \cdot & & & \\ \cdot & & 0 & \\ 0 & & & \end{matrix} \qquad E_2 = \begin{matrix} 0 & . & . & & 0 \\ 0 & 1 & 0 & . & . \\ \cdot & & & & \\ \cdot & 0 & & & 0 \\ 0 & . & & & \end{matrix} \qquad E_N = \begin{matrix} 0 & . & . & . & 0 \\ 0 & \ddots & & & 0 \\ 0 & & & \ddots & 0 \\ 0 & . & . & 0 & 1 \end{matrix}$$