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Phase Transition in the Nonlinear σ Model in a $2 + \epsilon$ Dimensional Continuum

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ABSTRACT

We study the phase transition in the nonlinear O(N) σ model in $2 + \epsilon$ dimensions. Our analysis is of the continuum theory and does not rely upon the artifice of a lattice. This phase transition occurs at a critical value of the coupling constant λ_c, which is an ultraviolet stable fixed point of the renormalization group. In the "low temperature" phase the O(N) symmetry is realized nonlinearly with N - 1 massless pions. By solving the theory in the large N limit, to leading order in 1/N, we show that in the "high temperature" phase the pions gain mass and there appears a new particle, σ, which is a bound state of the π's and is degenerate with them. Furthermore, by a general steepest descent approximation to the generating functional and by explicit calculations it is shown that this upper phase is fully O(N) symmetric and can be described by a linear σ model Lagrangian. The unitarity of the theory is demonstrated and analogies with quark confinement in quantum chromodynamics are discussed. We prove the renormalizability of the theory, taking special care to separate infrared and ultraviolet divergences.

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INTRODUCTION

In this paper we propose to study the phase transition in the nonlinear σ-model in a $2+\epsilon$ dimensional continuum. While a system of this kind is unphysical, and is purely of academic interest, we are motivated in this study by the following considerations.

There are some similarities between nonabelian gauge theories in $4+\epsilon$ dimensions and the nonlinear σ-model in $2+\epsilon$ dimensions. They are both asymptotically free at $\epsilon = 0$. Their similarity transcends even this straightforward observation, when a nonabelian gauge theory is placed on a lattice. The lattice gauge theory as formulated by Wilson[1] and extensively studied by others[2] is invariant under $\prod_n [SU(3)]_n$ where n denotes lattice sites. The gauge linkage $U_{\hat{\mu}}(n) = \exp[iga\underset{\sim}{\lambda}\cdot\underset{\sim}{A}_\mu(n)]$ in Wilson's theory is a nonlinear realization of the group $[SU(3)]_n \otimes [SU(3)]_{n+a\hat{\mu}}$. The nonlinear σ model is a dynamical model in which the chiral symmetry $SU(2)\times SU(2) \approx O(4)$ is realized in a nonlinear manner in terms of the Goldstone fields (pions). We believe that this similarity, i.e., nonlinear realization of the underlying symmetry, is at the heart of the isomorphism in the block spin recursion relations in the above two theories observed and exploited by Migdal and Polyakov.[3] Brézin and Zinn-Justin[4] have shown that the $2+\epsilon$ dimensional nonlinear σ- model undergoes a phase transition at a critical coupling constant. Above the critical point, the

Goldstone bosons acquire a mass, and from their work one presumes that the underlying symmetry is fully restored; the fields which were Goldstone bosons, together with some "bound state(s)" transform linearly under the group. This observation forms a part of the basis of the theory of color confinement by Bardeen and Pearson[5] in quantum chromodynamics.

The purpose of this paper is to investigate the dynamics of spontaneous mass generation and the formation of a new state (which we shall call σ) above the critical point, and to explore the dynamics of particle interactions in the fully symmetric phase. Our work was stimulated by the work of Brézin and Zinn-Justin,[4] but, in contrast to these authors, we choose to study the problem in a $2+\epsilon$ dimensional space-time continuum rather than relying upon a lattice. We shall concentrate on a class of models which has O(N) symmetry realized nonlinearly in terms of N-1 Goldstone massless bosons. We find that in the limit of large N, the model is sufficiently tractable so that we can indeed ascertain the nature of the phase above the critical point.[6] As expected, the "high temperature" or strong coupling phase can be described by a linear σ-model. Implications of our findings to a similar many body problem are not completely lost on us, but it is not within the scope of this paper to elaborate on these. We would rather regard the present study as a testing ground for tools to be used in the quark confinement problem in quantum chromodynamics without the artifice of a lattice. Only time can tell whether what we learned in this study will prove useful in understanding the structure of hadrons, however.

A brief survey of the nonlinear σ-model is in order. We consider a N-component boson theory defined by the generating functional:

$$Z_L[\underset{\sim}{J}] = \int \prod d\underset{\sim}{\chi}(x) \; \exp i \int d^dx \left\{ \frac{1}{2}(\partial\underset{\sim}{\chi})^2 - \frac{\alpha}{4}(\underset{\sim}{\chi}^2 - f_\pi^2) + \underset{\sim}{J}\cdot\underset{\sim}{\chi} \right\} . \tag{1.1}$$

If f^2 is positive and $\alpha\to\infty$, the long-wave behavior of the theory is described by

$$Z_{NL}[\underset{\sim}{J}] = \int \prod d\underset{\sim}{\chi}(x) \prod_x \delta(\underset{\sim}{\chi} - f_\pi^2) \exp i \int d^dx \left\{ \frac{1}{2}(\partial\underset{\sim}{\chi})^2 + \underset{\sim}{J}\cdot\underset{\sim}{\chi} \right\} . \tag{1.2}$$

We label the sources $\underset{\sim}{J}$ and the fields $\underset{\sim}{\chi}$ by an index i which runs from 0 to N-1. We may set $J_0 = 0$, denote $\chi_i = \pi_i$, i = 1 to N-1, and perform functional integrations over $\chi_0(x) = \sigma(x)$. We obtain [7]

$$Z_{NL}[\underset{\sim}{J}] = \int \prod d\underset{\sim}{\pi}(x) \exp i \int d^dx \left\{ \frac{1}{2}(\partial\underset{\sim}{\pi})^2 + \frac{\lambda_0}{2} \frac{(\underset{\sim}{\pi}\cdot\partial\underset{\sim}{\pi})^2}{1-\lambda_0\pi^2} + \underset{\sim}{J}\cdot\underset{\sim}{\pi} \right\} , \tag{1.3}$$

where $\lambda_0 = 1/f_\pi^2$. There is an additional term in the exponent arising from the integrations over $\sigma(x)$. It is

$$- \frac{1}{2} \delta^d(0) \int d^d x \ \ln [1 - \lambda_0 \underset{\sim}{\pi}^2(x)] . \tag{1.4}$$

By the rule of the dimensional regularization $\delta^d(0)$ is equal to zero, and we need not worry about this term. [Actually, the role of the terms in Eq. (1.4) is to cancel[8] symmetry breaking divergent terms elsewhere in the S-matrix proportional to $\delta^d(0) = \int (dk/2\pi)^d$. But these terms are also zero by dimensional regularization.[9]] The fields $\underset{\sim}{\pi}$ represent degrees of freedom associated with Goldstone bosons in a spontaneously broken O(N) symmetry. The set $(\sigma = \sqrt{1/\lambda_o - \underset{\sim}{\pi}^2}, \underset{\sim}{\pi})$ transforms vectorially under O(N). The vacuum is invariant under the subgroup O(N-1), which leaves the vector (1,0,. . . ,0) invariant. We label the generators of the coset O(N)/O(N-1) by $T_{oi} = - T_{io}$. We use the notation $\delta_i X = i[T_{oi}, X]$. We then have

$$\delta_i \pi_j = \delta_{ij} \sqrt{\frac{1}{\lambda_o} - \underset{\sim}{\pi}^2} = \delta_{ij} \, \sigma \, (\underset{\sim}{\pi}^2) ,$$

and (1.5)

$$\delta_i \sigma(\underset{\sim}{\pi}^2) = - \pi_i ,$$

i.e., transformations by the coset O(N)/O(N-1) are represented non-linearly over the manifold $\underset{\sim}{\pi}(x)$.

As explained elsewhere,[10] the nonlinear transformation law under O(N)/O(N - 1) need not be restricted to the form (1.5). More generally we can have

$$\delta_i \phi_j = \delta_{ij} f(\underset{\sim}{\phi}^2) + \phi_i \phi_j g(\underset{\sim}{\phi}^2), \tag{1.6}$$

provided that $f(x)$ and $g(x)$ satisfy

$$fg - 2ff' - 2xgf' = 1, \tag{1.7}$$

where $f' = df(x)/dx$, which follows from the Jacobi identity over O(N). The correspondence between $\underset{\sim}{\pi}$ and $\underset{\sim}{\phi}$ is given by

$$\pi_i \sim \phi_i f(0)[f^2(\underset{\sim}{\phi}^2) + \underset{\sim}{\phi}^2]^{-1/2}, \tag{1.8}$$

i.e., $\underset{\sim}{\pi}$ and $\underset{\sim}{\phi}$ are related by a nonlinear canonical transformation. A convenient form of the action is obtained by choosing $f(\phi^2) = f_\pi(1 - \underset{\sim}{\phi}^2/4f_\pi^2)$ and $g(\underset{\sim}{\phi}^2) = 2/f_\pi$, a choice first made by Schwinger.[11] The generating functional of Green's functions for this version of the nonlinear σ-model is given by

$$Z_S[\underset{\sim}{J}] = \int \prod d\underset{\sim}{\phi}(x) \exp i\int \left\{ \frac{1}{2}(\partial\underset{\sim}{\phi})^2 (1 + \frac{\lambda_0}{4}\underset{\sim}{\phi}^2)^{-2} + \underset{\sim}{J}\cdot\underset{\sim}{\phi} \right\}. \tag{1.9}$$

Since the actions in Eqs. (1.3) and (1.9) are related by a nonlinear canonical transformation of fields, Eq. (1.8), the renormalized S-matrix is identical in the two versions of the theory.[12] The action of Eq. (1.9) is most convenient in evaluating the S-matrix of the theory in the large N limit, as we shall see.

Thoughout the paper, we shall make use of the dimensional regularization method of 't Hooft and Veltman.[9] There is one inconvenient aspect of this method, and it may cause confusion in the results we shall obtain. To illustrate this, we evaluate the vacuum expectation value of ϕ^2, ϕ being a free hermitian scalar field of mass μ^2. We have

$$
\begin{aligned}
<\underset{\sim}{\phi}^2>_\epsilon &= i\int \left(\frac{dk}{2\pi}\right)^d \frac{1}{k^2 - \mu^2 + i\epsilon} \\
&= \frac{1}{4\pi}\left(\frac{\mu^2}{4\pi}\right)^{\epsilon/2} \Gamma\left(-\frac{\epsilon}{2}\right) \equiv I(\epsilon, \mu^2) ,
\end{aligned}
\tag{1.10}
$$

where $< \;\; >_\epsilon$ denotes the vacuum expectation value evaluated in the dimensional regularization scheme, with $d = 2 + \epsilon$. Note that I is negative for $\epsilon > 0$. This has to do with the way the integral is regulated (or defined by analytic continuation in ϵ), and this result is quite the opposite of our intuition based on the ultraviolet cutoff procedure. We do not know how to reconcile this to the presumably positive definite character of the underlying Hilbert space. We are however certain that the S-matrix evaluated with this method is sensible and will proceed pragmatically. In a later section we shall have occasion to check the unitarity of the S-matrix specifically because of this peculiarity of the regularization method used. It is an opportune juncture to comment on the limiting properties of $I(\epsilon, \mu^2)$: for $\epsilon > 0$, and $\mu^2 \to 0$, we have

$$\lim_{\mu^2 \to 0} I(\epsilon, \mu^2) = 0, \ \epsilon > 0 . \tag{1.11}$$

On the other hand, we can write

$$I(\epsilon, \mu^2) = -\frac{1}{2\pi\epsilon} + \left[\frac{1}{2\pi\epsilon} + \frac{1}{4\pi}\left(\frac{\mu^2}{4\pi}\right)^{\epsilon/2} \Gamma\left(-\frac{\epsilon}{2}\right)\right], \tag{1.12}$$

where the first term on the right represent the ultraviolet divergence in the limit $\epsilon \to 0_+$, and the second term on the right hand side is finite in the same limit as long as $\mu^2 \neq 0$. On the other hand if the limit $\mu^2 \to 0$ is taken before $\epsilon \to 0_+$, the second term becomes $1/2\pi\epsilon$, which represents the infrared divergence of the integral at $\epsilon = 0$. The result (1.11) may be regarded as due to the cancellation of the ultraviolet singularity $[-(2\pi\epsilon)^{-1}]$ and the infrared one $[(2\pi\epsilon)^{-1}]$.

The rest of this paper is organized as follows. In Section II, we discuss the renormalization procedure of the theory given by Eq. (1.9). Up until now the renormalizability of the nonlinear σ model in two dimensions was not a matter for concern. The model was used simply as a nonrenormalizable phenomenological lagrangian in four dimensions which, when used in tree approximation, reproduced the results of current algebra. This section is largely independent of the rest of the paper and may be omitted by the less dedicated reader. To make absolutely certain that we are isolating ultraviolet divergences from infrared ones, at $\epsilon = 0_+$, we add a mass term to the action

$$S = \int d^d x \left\{ \frac{1}{2} (\partial \underset{\sim}{\phi})^2 \left(1 + \frac{\lambda_0}{4} \underset{\sim}{\phi}^2\right)^{-2} - \frac{\mu^2}{2} \underset{\sim}{\phi}^2 \right\}. \tag{1.13}$$

The mass term, of course, breaks the symmetry, and generates new counter terms in each order of perturbation theory. That this symmetry breaking term can be controlled, and can be eliminated after renormalization by letting $\mu^2 \to 0$ is shown by the use of the Ward-Takahashi identity. We find that wave function and coupling constant renormalization eliminate all ultraviolet divergences as $\epsilon \to 0_+$. A novel feature in this theory is that the wave function renormalization is not just a scale transformation of the field, but entails a nonlinear canonical transformation of fields as well. This is perhaps not as strange as it might sound at first, when we realize that at $d = 2$ the scalar field $\underset{\sim}{\phi}$ is dimensionless, and $(\underset{\sim}{\phi}^2)^n \underset{\sim}{\phi}$ has the same dimension as $\underset{\sim}{\phi}$. This fact however has no profound effects in our ensuing discussions in which we deal exclusively with the renormalized S-matrix. The general renormalization procedure is illustrated in the one-loop approximation; the generating functional of irreducible vertices is evaluated in Appendix A.

In Section III, we solve the model in the large N limit, i.e., to lowest order in λ_0 or equivalently, to lowest order in $1/N$, with $\lambda_0 N$ fixed. We show here that the normalization conditions of the two-point irreducible vertex function (i.e., inverse propagator) lead to an eigenvalue equation for the mass of the $\underset{\sim}{\phi}$-field, which is similar to the gap equation in the BCS theory. We prove that for $\lambda < \lambda_c$ where λ is the renormalized coupling constant, and λ_c is the zero of the β-function, an ultraviolet fixed point of the renormalization group, the mass of the $\underset{\sim}{\phi}$ field is zero; for $\lambda > \lambda_c$, we must choose the branch of solutions which corresponds to a nonvanishing pion mass.

Sections IV and V are devoted to showing that the high temperature phase corresponds to the restoration of the full O(N) symmetry, that the vacuum is invariant under O(N). This is done by demonstrating that there exists a bound state degenerate with the $\underset{\sim}{\phi}$'s, and that the interactions among the $\underset{\sim}{\phi}$'s and the bound state are O(N) symmetric. Specifically in Section IV we consider the process $\phi\phi \rightarrow \phi\phi$ and some correlation functions which exhibit a pole corresponding to the σ particle. Special attention is given to the positive definite norm of this state which is somewhat obscured by the dimensional regularization procedure. In Section V we evaluate the amplitude for $\sigma\sigma \rightarrow \phi\phi$ to show that the O(N) symmetry is restored. On the basis of these calculations, we deduce a linear σ-model Lagrangian which describes the dynamics of the high temperature phase in terms of an N-component field. Appendix B gives a somewhat more systematic derivation of this result.

Section VI briefly describes some remaining problems in this model. Certain mathematical details are relegated to Appendix C.

II. RENORMALIZATION IN PERTURBATION THEORY

We begin by deriving the Ward-Takahashi identity for Z_S of Eq. (1.9) with the addition of mass term as in Eq. (1.13). By a change of integration variables, we define a new version of Z_S

$$Z[\underset{\sim}{j}] = \int \prod d\underset{\sim}{\phi}(x) \exp\, i\int d^dx \{ \frac{2}{\lambda_o}(\partial \underset{\sim}{\phi}^2)(1+\underset{\sim}{\phi}^2)^{-2} - \frac{2\mu^2}{\lambda_o}\underset{\sim}{\phi}^2 + \underset{\sim}{j}\cdot\underset{\sim}{\phi}\} \quad (2.1)$$

(Here we assume $\lambda_o > 0$). We imagine changing variables of functional integrations as in Eq. (1.6):

$$\delta_i \phi_j \sim \delta_{ij}(1 - \underset{\sim}{\phi}^2) + 2\,\phi_i \phi_j .$$

Under this transformation the exponent in Eq. (2.1) remains invariant except the mass term and the source term. Since the integral should remain invariant under any change of integration variables, we have

$$\int d^dx [j_i + 2\nu \frac{\delta}{i\delta j_i}] [\left(1 - \left(\frac{1}{i}\frac{\delta}{\delta \underset{\sim}{j}}\right)^2\right)\delta_{ij} + 2\left(\frac{1}{i}\frac{\delta}{\delta \underset{\sim}{j}_i}\right)\left(\frac{1}{i}\frac{\delta}{\delta j_j}\right)] Z[\underset{\sim}{j}] = 0. \quad (2.2)$$

where $\nu = 2\mu^2/\lambda_o$. Upon defining

$$Z[\underset{\sim}{j}] = e^{iW[\underset{\sim}{j}]} ,$$

$$\chi_i(x) = \delta W/\delta j_i(x) \quad (2.3)$$

and the generating functional of proper vertices $\tilde{\Gamma}[\underset{\sim}{\chi}]$ by

$$\tilde{\Gamma}[\underset{\sim}{\chi}] = W[\underset{\sim}{j}] - \int d^d x \underset{\sim}{j}(x) \cdot \underset{\sim}{\chi}(x) \tag{2.4}$$

We can write Eq. (2.2) in a more useful form:

$$\begin{aligned}\int d^d x \, [(1 - \underset{\sim}{\chi}^2)\delta_{ij} + 2\chi_i \chi_j] \frac{\delta\Gamma}{\delta\chi_j} \\ = \int d^d x \, \frac{i\delta^2 W[\underset{\sim}{j}]}{[i\delta \underset{\sim}{j}(x)]^2} \left(\frac{N-3}{N-1}\right)\left[\frac{\delta\Gamma}{\delta\chi_i} - 2\left(\frac{N-1}{N-3}\right)\nu\chi_i\right]\end{aligned} \tag{2.5}$$

The function

$$\frac{i\delta^2 W[j]}{i\delta j_i(x) i\delta j_j(x)} = \langle T(\phi_i(x)\phi_j(x))\rangle_\epsilon^{j = j[\chi]}$$

is a two point function at the same point x, in the presence of external sources $\underset{\sim}{j}$, so as to ensure $\langle\phi_i(x)\rangle_\epsilon = \chi_i(x)$, and Γ is

$$\tilde{\Gamma} = \Gamma - \nu\underset{\sim}{\chi}^2. \tag{2.6}$$

We define

$$\mathscr{H}(x;\underset{\sim}{\chi}) = \left(\frac{N-3}{N-1}\right)\frac{i\delta^2 W[\underset{\sim}{j}]}{[i\delta \underset{\sim}{j}(x)]^2} \tag{2.7}$$

and write the Ward-Takahashi identity as

$$\begin{aligned}\int d^d x [(1 - \underset{\sim}{\chi}^2 - \mathscr{H})\delta_{ij} + 2\chi_i\chi_j] \frac{\delta\Gamma}{\delta\chi_j} \\ + 2\nu\left(\frac{N-1}{N-3}\right)\int d^d x \mathscr{H}\chi_i = 0\end{aligned} \tag{2.8}$$

We shall now describe the renormalization procedure. The purpose here is to construct a finite Γ in terms of renormalized field and coupling constant in the limit $\mu^2 \to 0$ ($\nu \to 0$), for $\epsilon \geq 0$. We do not claim that terms which are explicitly proportional to ν can be rendered finite thereby. The device of inserting the mass term in intermediate steps is to insure an unambiguous separation of ultraviolet and infrared divergences in the limit $\epsilon \to 0_+$. The renormalization procedure described herein is intended to remove ultraviolet divergences from the theory.

First we transform the field $\underset{\sim}{\chi}$ by a nonlinear renormalization transformation:

$$\chi_i = \xi_i X^{\frac{1}{2}}(\underset{\sim}{\xi}^2), \tag{2.9}$$

where

$$X(\underset{\sim}{\xi}^2) = 1 + \alpha A^{(1)}(\underset{\sim}{\xi}^2) + \alpha^2 A^{(2)}(\underset{\sim}{\xi}^2) + \dots, \tag{2.10}$$

α being a fictitious loopwise expansion parameter. We expand Γ and $\mathscr{H}$ similarly:

$$\Gamma[\underset{\sim}{\chi}] = \Gamma_r[\underset{\sim}{\xi}] = \sum_{m=0} \alpha^m \Gamma^{(m)}[\underset{\sim}{\xi}] \quad ; \tag{2.11}$$

$$\mathscr{H}[x,\underset{\sim}{\chi}] = X \mathscr{H}_r[x,\underset{\sim}{\xi}] \quad ;$$

$$\mathscr{H}_r[x,\underset{\sim}{\xi}] = \sum_{\alpha=1} \alpha^m \mathscr{H}^{(m)}[x,\underset{\sim}{\xi}] \quad . \tag{2.12}$$

Now we define

$$X_{ij}[x,\underset{\sim}{\xi}] = [(1-\xi^2 X - \mathscr{H}_r X)\delta_{ik} + 2\xi_i\xi_k X]\frac{\delta\xi_j}{\delta\chi_k}$$

$$= [(1-\xi^2 X - \mathscr{H}_r X)\delta_{ik} + 2\xi_i\xi_k X]\, X^{-\frac{1}{2}}[\delta_{kj} - X'(X+\underset{\sim}{\xi}^2 X')\xi_k\xi_j] \qquad (2.13)$$

where $X'(z) = dX/dz$. $X_{ij}[x,\underset{\sim}{\xi}]$ can be expanded in α:

$$X_{ij}[x,\underset{\sim}{\xi}] = \sum_{m=0}\alpha_m X_{ij}^{(m)}[x,\underset{\sim}{\xi}], \qquad (2.14)$$

with

$$X_{ij}^{(0)} = (1-\underset{\sim}{\xi}^2)\delta_{ij} + 2\xi_i\xi_j \quad , \qquad (2.14a)$$

$$X_{ij}^{(1)} = [-\tfrac{1}{2}A^{(1)}(1+\underset{\sim}{\xi}^2) - \mathscr{H}^{(1)}]\,\delta_{ij}$$

$$+ \xi_i\xi_j\,[A^{(1)} - A^{(1)'}(1+\xi^2)] \quad , \qquad (2.14b)$$

etc.

Lastly, we can write Eq. (2.8) in a renormalized form:

$$\int d^dx\; X_{ij}[x,\underset{\sim}{\xi}]\,\frac{\delta\Gamma_r}{\delta\xi_j} + 2\nu_r\left(\frac{N-1}{N-3}\right)\int d^dx\,\mathscr{H}_r[x,\underset{\sim}{\xi}]\xi_i(x) = 0 \qquad (2.15)$$

where $\nu_r = \nu X^{3/2}$.

Before giving an inductive description of renormalization, the following observation is crucial. We denote by $[\mathcal{O}]^{\rm div}$ the divergent part of the quantity $\mathcal{O}$, defined for example as pole terms in ϵ for finite μ^2. By power counting we observe that $\left[\mathscr{H}_r[x,\underset{\sim}{\xi}]\right]^{\rm div}$ is a local function of $\underset{\sim}{\xi}^2(x)$:

$$\left[\mathscr{H}^{(n)}[x, \underset{\sim}{\xi}]\right]^{div} = H^{(n)}(\underset{\sim}{\xi}^2(x)) . \tag{2.16}$$

By examination of Feynman diagrams, we find also

$$\lim_{z \to -1} (1 + z)^{-2 + \delta} H^{(n)}(z) = 0 \tag{2.17}$$

for positive δ, however small, for finite n. Should Eq. (2.17) be invalid, the theory is "anomalous," and it is not renormalizable in the sense described previously.

Let us consider the implications of Eq. (2.15) in each order in α. To zeroth order, we have

$$\int d^d x \; X_{ij}^{(0)}(x, \underset{\sim}{\xi}) \frac{\delta \Gamma^{(0)}}{\delta \xi_j} = 0 \tag{2.18}$$

where $X_{ij}^{(0)}$ is explicitly shown in Eq. (2.14a). The physical solution is of course

$$\Gamma^0 = \frac{2}{\lambda_0} \frac{(\partial \underset{\sim}{\xi})^2}{(1 + \underset{\sim}{\xi}^2)^2} . \tag{2.19}$$

To first order, we examine the divergent part of the equation. We have

$$\int d^d x \left\{ X_{ij}^{(0)} \left[\frac{\delta \Gamma^{(1)}}{\delta \xi_j}\right]^{div} + \left[X_{ij}^{(1)}\right]^{div} \frac{\delta \Gamma^{(0)}}{\delta \xi_j} \right\}$$

$$= -2 \nu_r \left(\frac{N - 1}{N + 3}\right) \int d^d x \; H^{(1)}(\underset{\sim}{\xi}^2(x)) \xi_i(x), \tag{2.20}$$

where $\left[X_{ij}^{(1)}\right]^{div}$ is, from Eqs. (2.14b) and (2.16),

$$\left[X_{ij}^{(1)}\right]^{div} = \left[-\tfrac{1}{2}A^{(1)}(1+\xi^2) - H^{(1)}(\xi^2)\right]\delta_{ij} + \xi_i\xi_j\left[A^{(1)} - A^{(1)'}(1+\xi^2)\right] \quad (2.21)$$

The general structure of the divergent part of $\Gamma^{(n)}$ is, by power counting,

$$\left[\Gamma^{(n)}\right]^{div} = \tfrac{1}{2}(\theta\xi_i)(\theta\xi_j)\left[f_1^{(n)}(\xi^2)\delta_{ij} + \xi_i\xi_j f_2^{(n)}(\xi^2)\right] + g^{(n)}(\xi^2) \quad (2.22)$$

For brevity, we shall drop the superscript (n), n = 1 being understood in this and the next paragraphs. When Eqs. (2.21) and (2.22) are substituted in Eq. (2.20), there results a set of four differential equations, the first three of which are (see Appendix C)

$$\frac{\lambda_o}{2}(1+z)^{-1}\frac{d}{dz}\left[(1+z)^2 f_1(z)\right] = 2(1+z)^{-3}\left[-A + (1+z)A' - H - \tfrac{1}{2}(1+z)^2 A'\right] \quad (2.22a)$$

$$\frac{\lambda_o}{2}(1+z)f_2 = (1+z)^{-2}\, 2\left[(1+z)A' + H'\right] \quad (2.23b)$$

$$\frac{\lambda_o}{2}(1+z)^{-1}\frac{d}{dz}\left[(1+z)^2 f_2(z)\right] = 2(1+z)^{-1}A'' \quad (2.23c)$$

Combining Eqs. (2.23b - c), we find that

$$\frac{d}{dz}\left[(1+z)^{-1}H'\right] = 0 \quad (2.24)$$

This can be solved trivially. One constant of integration is determined from Eq. (2.17):

$$H(z) = a(1 + z)^2 \tag{2.25}$$

where a is an ϵ-dependent constant. Our next task is to see if it is possible to make $f_2 = 0$ by a judicious choice of A. From Eq. (2.23b), we learn that we must have $A = -a(1 + z) + b$, where b is at our disposal. We choose $b = 0$, i.e.,

$$A(z) = -2a(1 + z) \tag{2.26}$$

With this choice, the right hand side of Eq. (2.23a) vanishes; we have

$$f_1(z) = \frac{2f_1}{(1 + z)^2} \tag{2.27}$$

where f_1 is ϵ-dependent. Consider now $\Gamma^{(0)} + \Gamma^{(1)}$.

$$\Gamma^{(0)} + [\Gamma^{(1)}]^{\text{div}} = \left(\frac{2}{\lambda_0} + \alpha f\right) (\partial\underset{\sim}{\xi})^2(1 + \underset{\sim}{\xi}^2)^{-2} + g(\underset{\sim}{\xi}^2) \tag{2.28}$$

The infinity in f_1 is removed by a renormalization of λ_0:

$$\lambda_0 = \lambda Z(\lambda) \tag{2.29}$$

To order α, we choose Z to be

$$Z^{-1} = 1 - \alpha \frac{\lambda}{2} f. \tag{2.30}$$

We next examine $g(\underset{\sim}{\xi}^2)$. From Eq. (2.20) we find that

$$(1 + z)g'(z) = -\nu_r \, H(z)\left(\frac{N - 1}{N + 3}\right)$$

or

$$g(z) = -\nu_r\left(\frac{N - 1}{N - 3}\right)\frac{a}{2}\,(1 + z)^2 \quad .$$

Note that g is explicitly proportional to ν_r, or μ^2.

It is of interest to verify the above result by an explicit calculation. $\Gamma^{(0)} + \Gamma^{(1)}$ is computed in Appendix A. We find that the two agree completely, with the identification

$$a = \frac{\lambda}{8}\,(N - 3)\,I\,(\epsilon,\ \mu^2) \rightarrow -\frac{\lambda}{8}\,(N - 3)(2\pi\epsilon)^{-1}$$

$$\frac{f}{2} = -(N - 2)\,I\,(\epsilon,\ \mu^2) \rightarrow (N - 2)(2\pi\epsilon)^{-1}$$

The most important results so far, which can be deduced from Eqs. (2.21), (2.25) and (2.26) are that

$$\left[X_{ij}{}^{(1)}[x, \underset{\sim}{\xi}]\right]^{div} = 0 \tag{2.31}$$

and

$$\left[\Gamma^{(1)}\right]^{div} = \mathcal{O}(\mu^2)$$

after the coupling constant renormalization (2.29). This forms the basis of our induction, to be given presently.

We assume $\left[X_{ij}^{(m)}\right]^{div} = 0$ for $1 \leq m \leq n-1$, and $\left[\Gamma^{(m)}\right]^{div} = 0$ except for terms $\mathcal{O}(\mu^2)$. We can then write Eq. (2.15) in the n-loop approximation:

$$\int d^d x \left\{ X_{ij}^{(0)} \left[\frac{\delta\Gamma^{(n)}}{\delta \xi_j}\right]^{div} + \left[X_{ij}^{(n)}\right]^{div} \frac{\delta\Gamma^{(0)}}{\delta \xi_j} + \nu_r N^{(n)}(\xi^2)\xi_j \right\} = 0 \quad , \qquad (2.32)$$

where

$$\left[X_{ij}^{(n)}\right]^{div} = K_1(\xi^2)\,\delta_{ij} + \xi_i\xi_j\, K_2(\xi^2) \quad , \qquad (2.33)$$

with

$$K_1(z) = -\tfrac{1}{2}(1+z)A^{(n)}(z) - H^{(n)}(z) + L^{(n)}(z) \quad , \qquad (2.33a)$$

and

$$K_2(z) = A^{(n)}(z) - (1+z)\,A^{(n)\prime}(z) + M^{(n)}(z) \quad . \qquad (2.33b)$$

Here $L^{(n)}(z)$, $M^{(n)}(z)$ and $N^{(n)}(z)$ are ϵ-dependent local functions of z, expressible in terms of the previously determined $A^{(m)}$ and $H^{(m)}$, $1 \leq m \leq n-1$. In particular, since $L^{(n)}$ is a sum over products of $A^{(m)}$'s and $H^{(k)}$'s, $1 \leq m, k \leq n-1$, and $(1+z)^{-1+\delta} A^{(m)} \to 0$ as $z \to -1$ for any positive δ, however small, as will become apparent in our inductive discussion, $L^{(n)}$ has the same behavior as $H^{(n)}$ in Eq. (2.17), viz.,

$$(1+z)^{-2+\delta} L^{(n)}(z) \to 0 \quad \text{as } z \to -1 \quad , \qquad (2.17')$$

for $\delta > 0$.

Again we shall drop the superscript (n) for economy. The analogues of Eqs. (2.23a - c) are [see Appendix C]

$$\frac{\lambda_o}{2}(1+z)^{-1}\frac{d}{dz}[(1+z)^2 f_1(z)] = (1+z)^{-3}[2(K_2(z)-K_1(z)) - (1+z)K_1(z)]$$

(2.34a)

$$\frac{\lambda_o}{2}(1+z)f_2(z) = -(1+z)^{-2}[2K_1'(z)+K_2(z)] \qquad (2.34b)$$

$$\frac{\lambda_o}{2}(1+z)^{-1}\frac{d}{dz}[(1+z)^2 f_2(z)] = -2(1+z)K_2' \qquad (2.34c)$$

We shall use a tactic to handle this set of equations, which is somewhat different from the one used in the one-loop case. Since K_2 involves A and not H, we may choose A so that

$$K_2(z) = 0 \qquad (2.35)$$

which requires A to be of the form [see Eq. (2.33b)]

$$A(z) = (1+z)\left[\int_{z_0}^{z}\frac{dy}{(1+y)^2}M(y) + \alpha\right] \qquad (2.36)$$

where α is yet to be determined. It shows $(1+z)^{-1+\delta}A(z) \to 0$ as $z \to -1$ for $\delta > 0$ as assumed. Equation (2.34c) can now be solved:

$$f_2(z) = \frac{\beta}{(1+z)^2} \qquad (2.37)$$

where β is an arbitrary constant. Eq. (2.34b) can now be written as

$$\beta(1 + z) = 2 \frac{d}{dz} K_1(z) \qquad (2.38)$$

This is an equation for H(z) through Eq. (2.33a). We find for H(z) the following expression:

$$H(z) = -\frac{(1+z)^2}{2}\left[\int_{z_0}^{z} \frac{dy}{(1+y)^2} M(y) + \gamma\right] + L(z), \qquad (2.39)$$

where γ is an ϵ-dependent parameter, related to α and β above through

$$\gamma = \frac{\beta}{2} + \alpha. \qquad (2.40)$$

Since L(z) satisfies Eq. (2.17'), H(z) in Eq. (2.39) satisfies Eq. (2.17). We can choose α such that

$$\alpha = \gamma \quad , \qquad (2.41)$$

in which case $\beta = 0$, and we have from Eq. (2.37),

$$f_2(z) = 0 \qquad (2.37')$$

Combining Eqs. (2.36) and (2.39) in Eq. (2.33a), keeping in mind Eq. (2.41), we find

$$K_1(z) = 0 \qquad (2.42)$$

and, together with Eqs. (2.35) and (2.33), we see that

$$\left[X_{ij}{}^{(n)}\right]^{div} = 0. \qquad (2.43)$$

Further the solution of Eq. (2.34a) is

$$f_1{}^{(n)}(z) = \frac{2f^{(n)}}{(1+z)^2}$$

where $f^{(n)}$ is an ϵ-dependent number. As in the one-loop case [see Eqs. (2.28 - 2.30)] this divergence can be eliminated by a suitable choice of the n-th term in Z^{-1}:

$$Z^{-1} = 1 - \frac{\lambda}{2}\sum_{m=1} \alpha^m f^{(m)} \quad . \qquad (2.44)$$

We have thus proved that the wave function redefinition and the coupling constant renormalization make $\Gamma^{(n)}$ finite for terms not explicitly proportional to ν_r. This closes our induction loop. The form of $g^{(n)}(z)$ can be worked out and expressed in terms of $N^{(n)}(z)$; the result is not particularly illuminating, except that $g^{(n)}$ is of order ν_r.

III. THE EXISTENCE AND NATURE OF THE PHASE TRANSITION

In this section we will show that the nonlinear σ model exhibits a second order[13] phase transition at a certain critical value of the (renormalized) coupling constant λ. The order parameter is $<\underset{\sim}{\phi}^2>_\epsilon$, or equivalently m_ϕ^2; these are both zero below the critical point and increase continuously to nonzero values above this point. This is obviously a nonperturbative phenomenon, which is found only when the renormalized theory is summed to all orders in (λN). Recall, however, that we work to lowest order in λ, or equivalently, in $(1/N)$.

In the large N limit the Lagrangian of Eq. (1.9) can be written

$$\mathscr{L} = \frac{\frac{1}{2}\left(:(\partial\underset{\sim}{\phi}_0)^2: + <(\partial\underset{\sim}{\phi}_0)^2>_\epsilon\right)}{\left((1 + \frac{\lambda_0}{4}\left[<\underset{\sim}{\phi}_0{}^2>_\epsilon + :\underset{\sim}{\phi}_0{}^2:\right]\right)^2} \tag{3.1}$$

Here ϕ_0 and λ_0 are the unrenormalized field and coupling constant. For brevity of notation we use $<0|\underset{\sim}{\phi}_0{}^2|0>_\epsilon \equiv <\underset{\sim}{\phi}_0{}^2>_\epsilon$ and similarly for other vacuum expectation values. The expression for $\mathscr{L}$ in Eq. (3.1) is, like any nonpolynomical Lagrangian, to be understood as its power series expansion. Eq. (3.1) is derived by noting that in the large N limit the leading terms in the Wick expansion of a product of fields of the form

$$(\partial\underset{\sim}{\phi})^2(\underset{\sim}{\phi}^2)^n \qquad \text{are} \qquad (\underbrace{\partial\underset{\sim}{\phi}\cdot\partial\underset{\sim}{\phi}})(\underbrace{\underset{\sim}{\phi}\cdot\underset{\sim}{\phi}})(\underbrace{\underset{\sim}{\phi}\cdot\underset{\sim}{\phi}})\ldots(\underbrace{\underset{\sim}{\phi}\cdot\underset{\sim}{\phi}})$$

These dominate over other possible contractions, such as

$$(\partial \underset{\sim}{\phi} \cdot \partial \underset{\sim}{\phi})(\underset{\sim}{\phi} \cdot \underset{\sim}{\phi})(\underset{\sim}{\phi} \cdot \underset{\sim}{\phi}) \ldots (\underset{\sim}{\phi} \cdot \underset{\sim}{\phi})$$

by powers of N. Furthermore, for the reason just cited, $(:\underset{\sim}{\phi}^2:)^n = :(\underset{\sim}{\phi}^2)^n:$ for large N.

We first calculate the propagator, given by

$$\Delta_{F\,ij}^{-1} = \Gamma_{ij}^{(2)} = \left.\frac{\delta^2 \Gamma}{\delta \phi_i \delta \phi_j}\right|_{\phi = 0}$$
$$= \frac{\delta_{ij}}{N-1} \left.\frac{\delta^2 \Gamma}{\delta \underset{\sim}{\phi} \cdot \delta \underset{\sim}{\phi}}\right|_{\underset{\sim}{\phi} = 0} \tag{3.2}$$

where the latter equality is a result of the large N limit, as discussed above. We have

$$\Delta_{F\,ij}^{-1}(k^2) = \Delta_F^{-1}(k^2)\delta_{ij} \tag{3.3}$$

with

$$\Delta_F^{-1}(k^2) = \frac{k^2}{\left(1 + \frac{\lambda_0}{4} \langle \underset{\sim}{\phi}_0^{\,2} \rangle_\epsilon\right)^2} - \frac{\frac{\lambda_0}{2} \langle (\partial \underset{\sim}{\phi}_0)^2 \rangle_\epsilon}{\left(1 + \frac{\lambda_0}{4} \langle \underset{\sim}{\phi}_0^{\,2} \rangle_\epsilon\right)^3} \tag{3.4}$$

This equation already contains a summation of an infinite number of the leading graphs which contribute to the propagator. These graphs are shown in Fig. 3.1; they consist first of the bare propagator, then the set of (daisy) graphs with n loops attached at the same point, next the set with one loop attached to the original line and an n-daisy loop complex connected to the first loop, and so forth. Alternatively one could have started with the Lagrangian in the form of Eq. (1.9), where $\underset{\sim}{\phi}^2$ is not normal ordered, and calculated the graphs of Fig. 3.1; one would thereby have obtained the same result.

The simplicity of the set of dominant graphs in the large N limit and the resulting ability to sum those contributing to $\Delta_{F\ ij}^{-1}$ at the Lagrangian level are the reasons for our use of the particular form of the nonlinear σ model Lagrangian given in Eq. (1.9). If, instead, we had used the standard form of Eq. (1.3) the large N approximation would have required summing over considerably more complicated sets of diagrams. To illustrate this point we show in Fig. 3.2 the three infinite sets of graphs which would contribute to $\Delta_{F\ ij}^{-1}$ for large N with the Lagrangian (1.3). The set in Fig. 3.2a consists of diagrams topologically identical to those of Fig. 3.1 but of course with different vertices. The additional complexity comes with the sets in Figs. (3.2b, c), especially the latter since it introduces a momentum dependent correction to the propagator. (The heavy dots in Figs. 3.2b and 3.2c represent the full four and six point vertices including daisy corrections.) The explanation for this difference in what constitute the leading large N graphs is as follows. With the vertices resulting from the Lagrangian of Eq. (1.9) the graphs of Fig. 3.2 (aside from the bare propagator) are of order $(\lambda_0 N)^m$, m = 1, 2, ..., whereas the analogues of those shown in Figs. 3.2b and 3.2c are of order $\lambda_0(\lambda_0 N)^m = \frac{1}{N}(\lambda_0 N)^{m+1}$, m = 1, 2, ..., and hence are negligible in comparison.

However, with the standard Lagrangian (1.3) the leading N part of the set of diagrams in Fig. 3.2a vanishes, so that all three sets in Fig. 3.2 are of the same order, $\lambda_0(\lambda_0 N)^m$.

Returning to the theory based on the Lagrangian of Eqs. (1.9), (3.1), we define a renormalized field ϕ by

$$\underset{\sim}{\phi}_0 = Z_{\underset{\sim}{\phi}}^{\frac{1}{2}} \underset{\sim}{\phi} \tag{3.5}$$

Requiring that the renormalized propagator satisfy

$$\left. \frac{\partial \tilde{\Delta}^{-1}_{F\,ij}(k^2)}{\partial k^2} \right|_{k^2 = m_\phi^2} = 1 \tag{3.6}$$

we find

$$Z_\phi^{\frac{1}{2}} = \frac{2}{\left[1 + \sqrt{1 - \lambda_0 \; \langle \underset{\sim}{\phi}^2 \rangle_\epsilon}\right]} \tag{3.7}$$

It should be emphasized that this renormalization is a trivial finite rescaling in contrast with the case dealt with in section II, where wavefunction renormalization introduces higher powers of the field. Moreover, since the quantity $(1 + \frac{\lambda_0}{4} \langle \underset{\sim}{\phi}_0{}^2 \rangle_\epsilon)$ is independent of k^2 we could just as well have used $k^2 = 0$ or some other value in the renormalization point in Eq. (3.6) and we would have derived the same result for Z_ϕ.

The Lagrangian expressed in terms of the rescaled fields is

$$\mathscr{L} = \frac{\frac{1}{2}[:(\partial\underset{\sim}{\phi})^2: + <(\partial \underset{\sim}{\phi}^2>]}{\left[1 + \frac{\lambda_0}{4} Z_\phi^{\frac{1}{2}} :\underset{\sim}{\phi}^2:\right]^2} \tag{3.8}$$

and the full renormalized propagator is, to order $\frac{1}{N}$,

$$\tilde{\Delta}_F(k^2)_{ij} = \frac{\delta_{ij}}{k^2 - m_\phi^2} \tag{3.9}$$

where

$$m_\phi^2 = \frac{\lambda_0}{2} Z_\phi^{\frac{1}{2}} <(\partial\underset{\sim}{\phi})^2>_\epsilon \tag{3.10}$$

$$\equiv m^2$$

is the renormalized mass (as defined by $\tilde{\Delta}_F^{-1}(k^2 = m^2) = 0$, and still expressed in terms of the bare coupling constant).

Using this propagator we find

$$<\underset{\sim}{\phi}^2> = (N - 1)\int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2} = \frac{(N - 1)}{(4\pi)^{1 + \epsilon/2}} \Gamma(-\epsilon/2)(m^2)^{\epsilon/2} \tag{3.11}$$

and

$$<(\partial \underset{\sim}{\phi})^2>_\epsilon = m^2 <\underset{\sim}{\phi}^2>_\epsilon \tag{3.12}$$

From Eqs. (3.10), (3.11), and (3.12), we derive the important equation

$$<(\partial\underset{\sim}{\phi})^2>_\epsilon \left[1 - \frac{\lambda_0}{2} Z_\phi^{\frac{1}{2}} <\underset{\sim}{\phi}^2>_\epsilon\right] = 0 \tag{3.13}$$

This equation signals the existence of two separate phases, corresponding to its two possible solutions:

(1) $$<(\partial \underset{\sim}{\phi})^2>_\epsilon = 0 \tag{3.14}$$

which implies

$$m^2 = 0, \tag{3.15}$$

$$<\underset{\sim}{\phi}^2>_\epsilon = 0, \tag{3.16}$$

and $$Z_\phi = 1. \tag{3.17}$$

(2) $$<\underset{\sim}{\phi}^2>_\epsilon = \frac{2}{\lambda_0 \sqrt{Z_\phi}} \tag{3.18}$$

$$= \frac{1}{\lambda_0} \quad ,$$

($Z_\phi = 4$), which implies that (with $(N - 1) \simeq N$)

$$1 = \lambda_0 N I(\epsilon, m^2), \qquad \text{i.e.} \tag{3.19}$$

$$m^2 = \left[\frac{(4\pi)^{1+\epsilon/2}}{\lambda_0 N \Gamma(-\epsilon/2)}\right]^{\frac{2}{\epsilon}} \tag{3.20}$$

We shall next show that one must pick solution (1) if $\lambda < \lambda_c$ and solution (2) if $\lambda > \lambda_c$, where λ is the renormalized running coupling constant and[14]

$$\lambda_c = 2\frac{(4\pi)^{1+\epsilon/2}}{N} \frac{1}{\Gamma(1-\epsilon/2)B(\epsilon/2, \epsilon/2)} \tag{3.21}$$

$$\simeq \frac{2\pi\epsilon}{N} \qquad \text{for } \epsilon << 1$$

is the critical value of the coupling constant. Here $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. The critical coupling constant λ_c here is an ultraviolet stable fixed point of the renormalization group. The relation between λ and λ_0 is determined by the definition

$$\lambda_0 = \lambda Z_\lambda M^{-\epsilon} \tag{3.22}$$

which renders λ dimensionless by introducing an arbitrary scaling mass M, and the condition (for $\lambda < \lambda_c$)

$$\Gamma_s^{(4)}(\lambda_0;\ s = -M^2,\ p_i^{\,2} = 0) = Z_\phi^2[\Gamma_s]_r^{(4)}(\lambda;\ s = -M^2,\ p_i^{\,2} = 0) \tag{3.23}$$

In Eq. (3.23), Γ_s is defined by

$$\Gamma^{(4)}_{ijk\ell} = \Gamma_s^{(4)}\delta_{ij}\delta_{k\ell} + \Gamma_t^{(4)}\delta_{ik}\delta_{j\ell} + \Gamma_u^{(4)}\delta_{i\ell}\delta_{jk} \tag{3.24}$$

Explicitly, for $\lambda < \lambda_c$, $[\Gamma_s]_r^{(4)} = \lambda(p_1 \cdot p_2 + p_3 \cdot p_4)$, where the O(N) indices and momenta are defined as in Fig. 4.2. The condition (3.23) also applies, with appropriate changes ($s \to t$, $s \to u$) to $\Gamma_t^{(4)}$ and $\Gamma_u^{(4)}$. From the renormalization of the four-point proper vertex in the phase where $m^2 = 0$ and $\langle \underset{\sim}{\phi}^2 \rangle_\epsilon = 0$ we find

$$Z_\lambda^{-1} = \left(1 - \frac{\lambda}{\lambda_c}\right) \tag{3.25}$$

It should be remarked that the renormalization point chosen in Eq. (3.23) is somewhat arbitrary. Indeed in principle the on-shell point chosen could produce infrared singularities since it is an exceptional point in momentum space. However these do not appear in leading order in (1/N). As will be evident from the calculation in section IV, Z_λ represents an ultraviolet divergence. This is in contrast to Z_ϕ, which arises from the infinite daisy sum of Fig. 3.1. The value of a daisy loop integral changes from zero in the phase where $m^2 = 0$, to a nonzero constant ($\propto \Gamma(-\epsilon/2)$) in the phase where $m^2 \neq 0$. The fact that it is zero in the $m^2 = 0$ phase really is the consequence of the exact cancellation of infrared and ultraviolet divergences. When $m^2 \neq 0$, only the ultraviolet piece remains. It is for this reason that $Z_\phi^{\frac{1}{2}}$ changes discontinuously from $Z_\phi^{\frac{1}{2}} = 1$ to $Z_\phi^{\frac{1}{2}} = 2$ between the two phases. Stated differently, in this phase transition the short distance (ultraviolet) properties of the theory are continuous across the critical point, whereas the long distance (infrared) properties are drastically different. Accordingly, a function such as Z_λ which depends only on th ultraviolet divergences of the theory can be expected to be continuous across the critical point, but a function such as Z_ϕ in which both ultraviolet and infrared properties enter, will in general be discontinuous across the transition. From Eq. (3.25) we then have that $\lambda_0 > 0$ in phase 1 and $\lambda_0 < 0$ in phase 2. It follows that for $\lambda < \lambda_c$ we must choose phase 1 as the solution to Eq. (3.13) since the expression for the mass in phase 2, Eq. (3.20) would be self-contradictory. For $\lambda > \lambda_c$ it might appear that one could choose either phase. However, solution 1 leads to the appearance of a tachyon in the $\phi\phi \to \phi\phi$ scattering amplitude (see section IV) and must therefore be rejected in favor of solution 2. We shall accordingly call solution 1 the weak coupling or lower phase and solution 2 the strong coupling or upper phase.

The order parameter in this phase transition is $<\underset{\sim}{\phi}^2>$ or equivalently m^2. Using Eqs. (3.20) and (3.25) we find the mass in the strong coupling phase to be

$$m^2 = M^2 \left[\left(1 - \frac{\lambda_c}{\lambda}\right) \frac{\Gamma^2(1+\epsilon/2)}{\Gamma(1+\epsilon)}\right]^{\frac{2}{\epsilon}}$$

$$\simeq M^2 \left(1 - \frac{\lambda_c}{\lambda}\right)^{\frac{2}{\epsilon}} \quad \text{as } \epsilon \to 0_+ \tag{3.26}$$

For $\epsilon \neq 0$, m^2 has branch point singularities at $\lambda = \lambda_c$ and $\lambda = 0$ (unless $\frac{2}{\epsilon}$ happens to be an integer). As claimed, the phase transition is second order since m^2 is continuous at $\lambda = \lambda_c$. Indeed

$$\lim_{\lambda - \lambda_c \to 0_-} \frac{d^n m^2}{d\lambda^n} = \lim_{\lambda - \lambda_c \to 0_+} \frac{d^n m^2}{d\lambda^n} = 0 \tag{3.27}$$

for $n < \frac{2}{\epsilon}$. The behavior of m^2 in a function of λ and ϵ is shown in Fig. 3.3. Taking the limit $\epsilon \to 0$ and using Eq. (3.21) for λ_c we find (see Fig. 3.4)

$$\lim_{\epsilon \to 0} m^2 = M^2 e^{-\frac{4\pi}{\lambda N}} \tag{3.28}$$

This expression, with its essential singularity at zero coupling constant, is typical of phenomena in which an energy or mass gap is generated nonperturbatively. Note that λ appears not alone but in the combination

λN, a reflection of the fact that the variable in which we have summed the renormalized perturbation theory to all orders is λN. From the $\epsilon \neq 0$ form, Eq. (3.26), one reads off the critical exponent for m^2 as[15]

$$\beta = \frac{2}{\epsilon} \tag{3.29}$$

Since m^2 is a physical parameter it must of course be independent of the dimensional scaling mass, introduced in renormalization, i.e.

$$M \frac{d}{dM} m = \left(M \frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda}\right) m = 0 \tag{3.30}$$

where

$$\beta(\lambda) = M \frac{\partial}{\partial M} \lambda(\lambda_0, M) \tag{3.31}$$

Calculating β, we find[14]

$$\beta(\lambda) = \epsilon\lambda\left[1 - \frac{\lambda}{\lambda_c}\right] \tag{3.32}$$

$$\simeq \lambda\left[\epsilon - \frac{\lambda N}{2\pi}\right] \quad \text{for } \epsilon \to 0$$

The behavior of $\beta(\lambda)$ is shown in Fig. 3.5. For $\lambda < \lambda_c$ $\beta(\lambda) > 0$; in this phase, the infrared structure of the theory is determined by the origin. As the scaling mass M is increased the coupling constant is driven to the critical value λ_c, where β has an ultraviolet stable zero. For $\lambda > \lambda_c$ but $\left(\frac{\lambda - \lambda_c}{\lambda_c}\right) << 1$, according to the argument given before, Z_λ and hence also $\beta(\lambda)$ have the same form as they do in the weak coupling phase. The fact that Z_ϕ changes at the phase transition does not affect $\beta(\lambda)$ (or the renormalized S-matrix). Using Eq. (3.32) one easily verifies that m is indeed independent of M.

In addition to m^2, as was remarked before, an equivalent order parameter is $<\phi_0{}^2>_\epsilon$, or $<\left(1 - \frac{\lambda_0}{4}\phi_0{}^2\right)>_\epsilon$. The significance of this order parameter is clear from the role which it plays in the transformation formula for $\underset{\sim}{\phi}_0$. The O(N) symmetry is spontaneously broken if and only if $<[T_{0i}, (\phi_0)_j]>_\epsilon \neq 0$. In the large N limit the $<(\phi_0)_i (\phi_0)_j>_\epsilon$ term is negligible compared with the first, $<\left(1 - \frac{\lambda_0}{4}\phi_0{}^2\right)>_\epsilon \delta_{ij}$. Thus the latter quantity plays the same role for our choice of nonlinear Lagrangian, Eq. (1.9), as $<\sigma_0>_\epsilon \propto \sqrt{1 - \lambda_0 \pi_0{}^2}$ does for the standard form, Eq. (1.3). In the lower phase, where $<\phi_0{}^2>_\epsilon = 0$ the O(N) symmetry is spontaneously broken, whereas in the upper phase, where $<\phi_0{}^2>_\epsilon = \frac{4}{\lambda_0}$, i.e. $<\phi^2>_\epsilon = \frac{1}{\lambda_0} = \frac{-M^\epsilon}{\lambda}\left(1 - \frac{\lambda_c}{\lambda}\right)$, $<[T_{0i}, (\phi_0)_j]> = 0$ and this symmetry is restored. This restoration will be further demonstrated in later sections.

As $\epsilon \to 0_+$, it is evident from Eq. (3.32) that the theory is asymptotically free. Moreover, Coleman's theorem forbids spontaneous symmetry breakdown via the Goldstone mechanism in two dimensions.[16] Accordingly, the weak coupling phase, in which the O(N) symmetry is spontaneously broken down to O(N - 1) with the occurrence of N - 1 massless Goldstone particles (the π_i or ϕ_i) is forbidden at $\epsilon = 0$, and the theory can exist only in the strong coupling phase. This phase can be regarded as the analogue of the confinement phase in quantum chromodynamics, and the implications for QCD in $d = 4 + \epsilon$ dimensions are suggestive.

IV. THE ELEMENTARY PARTICLE σ AND THE $\phi\phi \to \phi\phi$ SCATTERING AMPLITUDE

4.1 Appearance of the σ Particle

In order to determine what are the proper collective modes of this theory in the strong coupling phase we shall examine the connected one-particle irreducible amputated four-point function, i.e. the four-point proper vertex $\Gamma^{(4)}_{ijk\ell}$. Henceforth we shall deal with $\Gamma_s^{(4)}$, the $\delta_{ij}\delta_{k\ell}$ part of the full four-point vertex as defined in Eq. (3.24). With obvious changes our results also determine $\Gamma_t^{(4)}$ and $\Gamma_u^{(4)}$. We shall prove that in the strong coupling phase there appears a new particle, for which the field operator is $\sigma \propto :\tilde{\phi}^2:$. In this section and the next we shall further show that in the strong coupling phase the full O(N) symmetry is restored and is realized linearly.

We define the following functions which contribute to $\Gamma_s^{(4)}$:

$$iT_{11} = \int d^d x\, e^{ik\cdot x} <T(\tilde{\phi}^2(x)\tilde{\phi}^2(0))>_\epsilon \tag{4.1}$$

$$iT_{12} = \int d^d x\, e^{ik\cdot x} <T\left(\tilde{\phi}^2(x)(\partial\tilde{\phi}(0))^2\right)>_\epsilon \tag{4.2}$$

$$iT_{21} = \int d^d x\, e^{ik\cdot x} <T\left((\partial\tilde{\phi}(x))^2\tilde{\phi}^2(0)\right)>_\epsilon \tag{4.3}$$

$$iT_{22} = \int d^d x\, e^{ik\cdot x} <T\left((\partial\tilde{\phi}(x))^2(\partial\tilde{\phi}(0))^2\right)>_\epsilon \tag{4.4}$$

The graphs which give the leading contribution in the large N limit to the T_{ij} functions are strings of bubbles, as shown in Fig. 4.1. It is useful to introduce a symbolic graphical notation, first for the parts of the bare four-point vertex, as depicted in Fig. 4.2. In this notation the dot indicates that the derivatives in $(\partial \underset{\sim}{\phi})^2$ act on the adjacent legs and the graph with no dots corresponds to the term $3m^2$ in the vertex. As was true of $\Gamma^{(2)}$, each four-point vertex is dressed by an infinite sum of daisy loops, the contributions of which have already been summed exactly in the Lagrangian (3.1).

In the infinite bubble sum it is useful to arrange the terms according to how the derivatives in the four-point vertex act. Fig. 4.3 shows this for an individual bubble. Analytically the Σ_{ij} are defined as

$$\Sigma_{11}(k^2) = -2\,i\,N \int \frac{d^d q}{(2\pi)^d} \; \frac{i}{(q^2 - m^2)} \; \frac{i}{(q-k)^2 - m^2} \tag{4.5}$$

$$\Sigma_{12}(k^2) = \Sigma_{21}(k^2)$$

$$= -2\,i\,N \int \frac{d^d q}{(2\pi)^d} \; q \cdot (q-k) \frac{i}{q^2 - m^2} \; \frac{i}{(q-k)^2 - m^2} \tag{4.6}$$

$$\Sigma_{22}(k^2) = -2\,i\,N \int \frac{d^d q}{(2\pi)^d} [q \cdot (q-k)]^2 \frac{i}{q^2 - m^2} \; \frac{i}{(q-k)^2 - m^2} \tag{4.7}$$

Evaluating the integrals, one finds

$$\Sigma_{11}(k^2) = -2\,N\,J\,(\epsilon,\, k^2,\, m^2) \tag{4.8}$$

$$\Sigma_{12}(k^2) = \Sigma_{21}(k^2) = N(k^2 - 2m^2) J(\epsilon,\, k^2,\, m^2) + 2\,N\,I\,(\epsilon,\, m^2) \tag{4.9}$$

$$\Sigma_{22}(k^2) = -\tfrac{1}{2}(k^2 - 2m^2) N J(\epsilon, k^2, m^2) - (k^2 - 4m^2) I(\epsilon, m^2) \tag{4.10}$$

where

$$J(\epsilon, k^2, m^2) = \frac{1}{(4\pi)^{1+\epsilon/2}} \Gamma(1 - \epsilon/2) \left(\frac{4}{4m^2 - k^2}\right)^{1-\epsilon/2} {}_2F_1\left(1 - \epsilon/2; \tfrac{1}{2}; \tfrac{3}{2}; \frac{k^2}{k^2 - 4m^2}\right) \tag{4.11}$$

and ${}_2F_1(a, b; c; z)$ is the hypergeometric function. For reference, as $\epsilon \to 0$,

$$J(0, k^2, m^2) = \frac{1}{2\pi\sqrt{k^2(k^2 - 4m^2)}} \ln \left[\frac{\sqrt{1 - 4m^2/k^2} - 1}{\sqrt{1 - 4m^2/k^2} + 1} \right] \tag{4.12}$$

for $(1 - 4m^2/k^2) > 0$, and

$$J(0, k^2, m^2) = \frac{1}{\pi\sqrt{k^2(4m^2 - k^2)}} \tan^{-1}\left(\frac{1}{\sqrt{4m^2/k^2 - 1}} \right) \tag{4.13}$$

for $(1 - 4m^2/k^2) < 0$.

Next, we write a matrix equation for the full T_{ij} functions in terms of the four basic bubble graphs Σ_{ij} and a matrix G which expresses how the Σ_{ij} are attached to the T_{ij}:

$$T_{ij} = \Sigma_{ij} + \frac{\lambda_0}{2} \Sigma_{ik} G_{k\ell} T_{\ell j} \tag{4.14}$$

where

$$G = \begin{pmatrix} -\frac{3}{2} m^2 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.15}$$

Fig. 4.4 illustrates the ij = 11 and ij = 12 components of Eq. (4.14). The solution of Eq. (4.14) is

$$T = \left(1 - \frac{\lambda_0}{2} \Sigma G\right)^{-1} \Sigma \tag{4.16}$$

Specifically,

$$T_{11} = \frac{4}{\lambda_0} \frac{1}{k^2 - m^2} \tag{4.17}$$

$$T_{12} = T_{21} = \frac{4}{\lambda_0} \frac{m^2}{k^2 - m^2} \tag{4.18}$$

$$T_{22} = \frac{4}{\lambda_0} \frac{m^4}{k^2 - m^2} + \frac{k^2}{\lambda_0} + \frac{2}{\lambda_0^2 NJ(\epsilon, k^2 m^2)} \tag{4.19}$$

The diagonal T_{ij}'s can be considered not as pieces of the four-point vertex, but as propagators for the operators $\underset{\sim}{\phi}^2$ and $(\partial \underset{\sim}{\phi})^2$. Viewing them in this way one sees a remarkable feature of Eq. (4.17) which is one of the main results of this paper: the $\underset{\sim}{\phi}^2$ propagator consists only of a simple pole. That is, (to leading order in 1/N) $\underset{\sim}{\phi}^2$ is the field operator for an elementary particle, and creates only a single particle state when acting on the vacuum. Thus in the strong coupling phase there appears a new particle, to be called σ which is a bound state of the $\underset{\sim}{\phi}$'s but nevertheless is elementary in the sense specified above. Furthermore, the σ is degenerate in mass with the N - 1 ϕ_i's, a consequence of the fact that the full O(N) symmetry is restored in the strong coupling phase. In Appendix B we present another formulation of the physics in the upper phase, in which the restored O(N) symmetry is manifest. We note in passing that T_{22} has a momentum space structure more typical of a usual operator product, viz. a pole, a polynomial term, and a function $(J(\epsilon, k^2, m^2))$ having a cut beginning at $k^2 = 4m^2$.

The inconvenient characteristic of dimensional regularization mentioned in the Introduction, that the positivity of operator products is not preserved, shows up in Eq. (4.17); the residue of the pole is $4\lambda_0^{-1}$, which is negative for $\lambda > \lambda_c$. That this negative residue is a direct consequence of the fact that $\langle\underset{\sim}{\phi}^2\rangle_\epsilon < 0$ in the dimensional regularization scheme is proved by using Eq. (3.19) to write Eqs. (4.1) and (4.17) as

$$T_{11} = -i\int d^dx\, e^{ik\cdot x}\langle T(\underset{\sim}{\phi}^2(x)\underset{\sim}{\phi}^2(0))\rangle_\epsilon = \frac{4}{k^2 - m^2}\langle\underset{\sim}{\phi}^2\rangle_\epsilon$$

Hence, if one uses dimensional regularization it would be wrong to consider σ to be a ghost just because the residue of the pole in its propagator is negative. In order to test the unitarity of the theory one must actually examine the S-matrix elements. The essential observation is that

$T_{11} = \frac{2}{\sqrt{\lambda_0}} \frac{1}{k^2 - m^2} \frac{2}{\sqrt{\lambda_0}}$ and that either factor $2\lambda_0^{-\frac{1}{2}}$ may be eliminated by an external line wave function renormalization, or it may be absorbed in a proper vertex. Because of the fact that the upper phase is fully O(N) symmetric (as is established in Appendix B), proper vertices are always connected to an even number of σ lines. Hence, any proper vertex absorbs an even number of $\lambda_0^{-\frac{1}{2}}$ factors and so does not acquire a factor i. This argument, together with our results for the $\phi\phi \to \phi\phi$, $\sigma\sigma \to \phi\phi$, and $\sigma\phi\phi \to \sigma\phi\phi$ scattering amplitudes, shows unequivocally that σ is a physical particle, and the S-matrix is unitary.

4.2 The $\phi\phi \to \phi\phi$ Scattering Amplitude

Let us next use the T_{ij} functions to calculate the four-point proper vertex and corresponding $\phi\phi \to \phi\phi$ scattering amplitude in the upper phase (specifically, the s-channel part, from which the t and u channel parts can be obtained by crossing). The sets of graphs which give the leading contribution to $\Gamma_s^{(4)}$ in the large N limit are shown in Fig. 4.5.

$$\begin{aligned}\Gamma_s^{(4)} = &+ \lambda_0 (2p_1 \cdot p_2 + 2p_3 \cdot p_4 + 3m^2) \\ &- \lambda_0^2 (p_1 \cdot p_2 + \tfrac{3}{2} m^2) T_{11} (p_3 \cdot p_4 + \tfrac{3}{2} m^2) \\ &+ \lambda_0^2 (p_1 \cdot p_2 + \tfrac{3}{2} m^2) T_{12} + \lambda_0^2 T_{21} (p_3 \cdot p_4 + \tfrac{3}{2} m^2) \\ &- \lambda_0^2 T_{22} \end{aligned} \qquad (4.24)$$

Substituting our results for the T_{ij} and using $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$ we get

$$\Gamma_s^{(4)} = -\lambda_0\left(s + m^2 - p_1{}^2 - p_2{}^2\right)\frac{1}{s - m^2}\left(s + m^2 - p_3{}^2 - p_4{}^2\right)$$

$$+ \lambda_0\left(s + 3m^2 - \sum_{i=1}^{4} p_i{}^2\right) - \frac{2}{NJ(\epsilon,\ s,\ m^2)} \qquad (4.25)$$

and similarly, via crossing, for $\Gamma_t^{(4)}$ and $\Gamma_u^{(4)}$. The physical $\phi\phi \to \phi\phi$ scattering amplitude, $A^{(s)}_{\phi\phi \to \phi\phi}$, related to the full amplitude $(A_{\phi\phi \to \phi\phi})_{ijk\ell}$ by the analogue of Eq. (3.24), is then

$$A^{(s)}_{\phi\phi \to \phi\phi} = -\frac{2}{NJ(\epsilon,\ s,\ m^2)} \qquad (4.26)$$

Several features of the amplitude $A_{\phi\phi \to \phi\phi}$ are important to notice. First, although the offshell proper vertex contains poles at $s = m^2$, $t = m^2$, and $u = m^2$, there are no such poles in the onshell amplitude. These poles represent σ exchange in the s, t, and u channels resulting from a $\sigma\phi\phi$ coupling. It is thus clear why they cannot appear in the onshell amplitude since the restoration of the full O(N) symmetry in the strong coupling phase forbids there from being a $\sigma\phi\phi$ vertex. Another interesting feature is that although the offshell amplitude explicitly depends on λ_0 the onshell amplitude does not. The latter does depend implicitly on λ_0 though the mass m appearing in the bubble integral $J(\epsilon,\ s,\ m^2)$. Furthermore, the $\phi\phi \to \phi\phi$ amplitude is real for $s < 4m^2$, and has a branch cut starting at $s = 4m^2$. Evaluation of the discontinuity across the cut shows that, to leading order in $1/N$, this amplitude satisfies unitarity.

It is of interest to compare the $\phi\phi \rightarrow \phi\phi$ scattering amplitude calculated in the strong coupling phase with that calculated in the weak coupling phase, as $\lambda \rightarrow \lambda_c$. For this purpose we start again with the Lagrangian of Eq. (3.8), with $\langle \underset{\sim}{\phi}^2 \rangle_\epsilon = 0$ and $Z_\phi = 1$. This yields a bare four-point vertex equal to that in the strong coupling phase but with $\lambda_0 \rightarrow \frac{\lambda_0}{2}$, $m \rightarrow 0$. The basic four bubble graph integrals are given by Eqs. (4.8) - (4.10) with $m^2 = 0$:

$$\Sigma_{11} = -2\,NJ(\epsilon, s, 0) \tag{4.27}$$

$$\Sigma_{12} = \Sigma_{21} = sNJ(\epsilon, s, 0) \tag{4.28}$$

$$\Sigma_{22} = -\frac{s^2}{2} NJ(\epsilon, s, 0) \tag{4.29}$$

Explicitly,

$$J(\epsilon, s, 0) = \frac{1}{(4\pi)^{1+\epsilon/2}} \, \Gamma(1-\epsilon/2)\, B(\epsilon/2, \epsilon/2)(-s)^{-1+\epsilon/2} \tag{4.30}$$

The Σ_{ij} matrix is conveniently written as

$$\Sigma = \begin{pmatrix} 1 & -\frac{s}{2} \\ -\frac{s}{2} & \left(-\frac{s}{2}\right)^2 \end{pmatrix} \Sigma_{11} \tag{4.31}$$

Solving Eq. (4.14) (with $\lambda_0/2 \to \lambda_0/4$) for the T_{ij}, we find

$$T = \begin{pmatrix} 1 & -\frac{s}{2} \\ -\frac{s}{2} & \left(-\frac{s}{2}\right)^2 \end{pmatrix} T_{11}, \text{ where} \tag{4.32}$$

$$T_{11} = \frac{\Sigma_{11}}{\left(1 + \frac{\lambda_0 s}{4} \Sigma_{11}\right)} \tag{4.33}$$

The calculation of $\Gamma^{(4)}_{ijk\ell}$ proceeds as before and yields for the s-channel part the result

$$\Gamma_s^{(4)} = \frac{\lambda_0}{2}\left(2s - \Sigma p_i^{\,2}\right) - \left(\frac{\lambda_0}{4}\right)^2 \left(p_1^{\,2} + p_2^{\,2} - 2s\right) T_{11} \left(p_3^{\,2} + p_4^{\,2} - 2s\right) \tag{4.34}$$

and, for the onshell vertex,

$$\Gamma_s^{(4)}(p_i^{\,2} = 0) = \frac{\lambda_0 s}{1 - \frac{\lambda_0 N s}{2} J(\epsilon, s, 0)} \tag{4.35}$$

Renormalizing this four-point proper vertex at $s = -M^2$ according to the condition (3.22) and the definition (3.21) we find the expression for Z_λ given in Eq. (3.25). The physical onshell amplitude is then

$$A^{(s)}_{\phi\phi\to\phi\phi} = \frac{\lambda M^{-\epsilon} s}{\left[1 - \frac{\lambda}{\lambda_c} + \frac{\lambda}{\lambda_c}\left(\frac{-s}{M^2}\right)^{\epsilon/2}\right]} \qquad (4.36)$$

For $\lambda - \lambda_c \to 0_-$,

$$A^{(s)}_{\phi\phi\to\phi\phi} \to -\lambda_c(-s)^{1-\epsilon/2}$$

Now as $\lambda - \lambda_c \to 0_+$ the $\phi\phi \to \phi\phi$ scattering amplitude in the strong coupling phase is

$$\lim_{\lambda-\lambda_c\to 0_+} A^{(s)}_{\phi\phi\to\phi\phi} = -\frac{2}{NJ(\epsilon, s, 0)}$$

Using Eqs. (3.20) and (4.30) one observes that this is equal to the amplitude in the weak coupling phase in the same limit. Thus the $\phi\phi \to \phi\phi$ scattering amplitude is continuous across the phase transition.

Now we are in a position to prove the claim made in section III, that for $\lambda > \lambda_c$ the theory exists only in phase 2. This is clear since if one took Eq. (4.35), which was calculated assuming $m^2 = 0$, and let $\lambda - \lambda_c$ be greater than zero there would appear a tachyon pole at

$$s = -M^2\left(1 - \frac{\lambda_c}{\lambda}\right)^{\frac{2}{\epsilon}} \qquad (4.37)$$

Hence, as claimed, for $\lambda > \lambda_c$, the theory chooses phase 2, the strong coupling phase.

Next, we shall comment on the "progenitor" of the σ particle in the lower phase, $\lambda < \lambda_c$. The denominator of the right hand side of Eq. (4.36) is a real analytic function in the cut s-plane, $0 < \arg(s) < 2\pi$. It is a multisheeted analytic function with an infinite number of zeros at

$$s = M^2\left(\frac{\lambda_c}{\lambda} - 1\right) e^{i\pi(2/\epsilon + 1)(1 + 2k)} \tag{4.38}$$

where k runs through the integers from $-\infty$ to $+\infty$. Note that none of these zeros lie on the physical sheet. Now as $\lambda \to \lambda_c$ from below there is a confluence of zeros as they approach the origin, which is identified on all the Riemann sheets. As λ increases beyond λ_c the branch point recedes from $s = 0$ to $s = 4m^2$, and the pole of $\Gamma^{(4)}$ moves to $s = m^2$, but its residue vanishes onshell.

V. THE $\sigma\sigma \to \phi\phi$ AMPLITUDE

It is well to recall the form of the Lagrangian after the finite wave function renormalization [see Eq. (3.8)]:

$$\mathscr{L} = \frac{1}{2}:(\partial \underset{\sim}{\phi})^2:\left(1 + \frac{\lambda_0}{2} :\underset{\sim}{\phi}^2:\right)^{-2} + \frac{m^2}{2\lambda_0}\left(1 + \frac{\lambda_0}{2} :\underset{\sim}{\phi}^2:\right)^{-2} , \quad (5.1)$$

and the facts that

$$T_{11} = -i\int d^d x\, e^{ik\cdot x} \langle T(\underset{\sim}{\phi}^2(x)\underset{\sim}{\phi}^2(0))\rangle_\epsilon = \frac{4}{\lambda_0}\frac{1}{k^2 - m^2} ,$$

$$T_{12} = -i\int d^d x\, e^{ik\cdot x} \langle T(\underset{\sim}{\phi}^2(x)[\partial\underset{\sim}{\phi}(0)]^2)\rangle_\epsilon = \frac{4}{\lambda_0}\frac{m^2}{k^2 - m^2} = T_{21} ,$$

$$T_{22} = -i\int d^d x\, e^{ik\cdot x} \langle T((\partial\underset{\sim}{\phi}(x))^2(\partial\underset{\sim}{\phi}(0))^2)\rangle_\epsilon = \frac{4}{\lambda_0}\frac{m^4}{k^2 - m^2} \quad (5.2)$$

$$+ \text{ terms not singular as } k^2 \to m^2.$$

Thus, to deduce vertices involving σ's <u>on shell</u>, we can read them off from Eq. (5.1) by the mnemonic rules:

$$:\underset{\sim}{\phi}^2: \to \frac{-2}{\sqrt{\lambda_0}}\, \sigma$$

$$(5.3)$$

$$:(\partial\underset{\sim}{\phi})^2: \to \frac{-2m^2}{\sqrt{\lambda_0}}\, \sigma \quad .$$

For example, the vertex to be used for the transition: physical $\sigma \to \phi\phi$ is

$$-\frac{\lambda_0}{2} :(\partial \underset{\sim}{\phi})^2::\underset{\sim}{\phi}^2: + \frac{3}{8} \lambda_0 m^2 (:\underset{\sim}{\phi}^2:)^2$$

$$\to -\sqrt{\frac{4}{\lambda_0}}\, \sigma \lambda_0 \left[\frac{1}{4} m^2 :\underset{\sim}{\phi}:^2 - \frac{1}{2} :(\partial \underset{\sim}{\phi})^2:\right] \tag{5.4}$$

and the vertex involving two physical σ's and any number of $\underset{\sim}{\phi}$'s is given by

$$\left(\sqrt{\frac{4}{\lambda_0}}\sigma\right)^2 \left[\frac{3}{8} \lambda_0^2 :(\partial \underset{\sim}{\phi})^2: \left(1 + \frac{1}{2} \lambda_0 :\underset{\sim}{\phi}^2:\right)^{-4}\right.$$

$$\left. + \frac{3}{8} \lambda_0 m^2 \left(1 + \frac{1}{2} \lambda_0 :\underset{\sim}{\phi}^2:\right)^{-4} + \frac{1}{2} \lambda_0 m^2 \left(1 + \frac{1}{2} \lambda_0 :\underset{\sim}{\phi}^2:\right)^{-3}\right] \tag{5.5}$$

There are three classes of diagrams contributing to the on-shell process $\sigma(p_1) + \sigma(p_2) \to \underset{\sim}{\phi}(q_1) + \underset{\sim}{\phi}(q_2)$, as shown in Fig. 5.1 in the large N limit. Let us describe the computation of the diagrams of class A. The first one (with a common factor δ_{ij} removed) yields the result

$$\left(i\sqrt{\frac{4}{\lambda_0}}\right)^2 \lambda_0^2 \left[\frac{1}{2} m^2 + q_1 \cdot (p_1 - q_1)\right] \frac{i}{(p_1 - q_1)^2 - m^2} \left[\frac{1}{2} m^2 + q_2 \cdot (p_2 - q_2)\right]$$

$$= -i\, \lambda_0 (t - m^2). \tag{5.6}$$

The sum of the two diagrams of class A is

$$iA^{(A)}_{\sigma\sigma\to\phi\phi} = i\,\lambda_0(s - 2m^2) \tag{5.7}$$

The diagrams of classes B and C can be similarly computed from the interactions in Eq. (5.5). We find

$$iA^{(B)}_{\sigma\sigma\to\phi\phi} = i\,6\,\frac{1}{NJ(s)} \tag{5.8}$$

$$iA^{(C)}_{\sigma\sigma\to\phi\phi} = -i\,8\,\frac{1}{NJ(s)} - i\,\lambda_0(s - 2m^2), \tag{5.9}$$

so that the full $\sigma\sigma\to\phi\phi$ amplitude in the large N limit is

$$A_{\sigma\sigma\to\phi\phi} = -\frac{2}{NJ(s)} \tag{5.10}$$

and we see that

$$A_{\sigma\sigma\to\phi\phi}(s) = A^{(s)}_{\phi\phi\to\phi\phi}(s) \tag{5.11}$$

In order to show that the σ-particle is a physical particle of positive norm let us consider the process $\sigma\phi\phi\to\sigma\phi\phi$. Let s be the square of the total c.m. energy. The terms that contain a pole at $s = m^2$ are of the form [see Fig. 5.2]

$$i\Gamma_1(\sigma\phi\phi \to \phi\phi)\; iT_{11}(s)\, i\Gamma_1(\phi\phi \to \sigma\phi\phi)$$

$$+ i\Gamma_2(\sigma\phi\phi \to \phi\phi)\; iT_{21}(s)\, i\Gamma_2(\phi\phi \to \sigma\phi\phi)$$

$$+ i\Gamma_1(\sigma\phi\phi \to \phi\phi)\; iT_{12}(s)\, i\Gamma_2(\phi\phi \to \sigma\phi\phi)$$

$$+ i\Gamma_2(\sigma\phi\phi \to \phi\phi)\; iT_{22}(s)\, i\Gamma_2(\phi\phi \to \sigma\phi\phi) \qquad (5.12)$$

As $s \to m^2$, Eq. (5.12) reduces to

$$i\left(\Gamma_1\sqrt{\frac{4}{\lambda_0}} + \Gamma_2\sqrt{\frac{4}{\lambda_0}}\, m^2\right) \frac{i}{s - m^2}\, i\left(\sqrt{\frac{4}{\lambda_0}}\; \Gamma_1 + \sqrt{\frac{4}{\lambda_0}}\; m^2\Gamma_2\right) \qquad (5.13)$$

But according to the rules of Eq. (5.3), $\left(2/\sqrt{\lambda_0}\right)\Gamma_1 + m^2\left(2/\sqrt{\lambda_0}\right)\Gamma_2$ is just $A(\sigma\phi\phi \to \sigma)$. Thus the pole term in the process $\sigma\phi\phi \to \sigma\phi\phi$ is

$$iA(\sigma\phi\phi \to \sigma)\frac{i}{s - m^2}\; iA(\sigma \to \sigma\phi\phi)$$

which shows unequivocally that the σ-pole has a factorized residue with the correct sign for a physical particle of positive norm.

Finally, we remark that the results for $A_{\phi\phi \to \phi\phi}$, Eq. (4.26) and $A_{\sigma\sigma \to \phi\phi}$, Eq. (5.10) may be derived in the large N limit from an effective Lagrangian:

$$\mathscr{L} = \frac{1}{2} : (\partial \underset{\sim}{\Pi})^2 + (\partial\Sigma)^2 : - \frac{m^2}{2} : \underset{\sim}{\Pi}^2 + \Sigma^2 :$$

$$- \frac{1}{4N}\, a : (\underset{\sim}{\Pi}^2 + \Sigma^2)^2 : \qquad (5.12)$$

where

$$a^{-1} = J(\epsilon, p^2 = 0, m^2)$$

and the fields $\underset{\sim}{\Pi}$ and Σ are nonlinear canonical transforms of $\underset{\sim}{\phi}$ and σ, where

$$\sigma \equiv \frac{1 - \lambda_0 \phi^2}{2\sqrt{\lambda_0}} = -\frac{\sqrt{\lambda_0}}{2} : \phi^2 :$$

$$\underset{\sim}{\Pi} = \frac{\underset{\sim}{\phi}}{1 - \sqrt{\lambda_0}\,\sigma} \tag{5.13}$$

$$\Sigma = \frac{\sigma}{1 - \sqrt{\lambda_0}\,\sigma}$$

which transform as

$$\delta_i \Pi_j = \delta_{ij} \Sigma, \qquad \delta_i \Sigma = -\Pi_i \tag{5.14}$$

The effective Lagrangian (5.12) is to be supplemented by the subtraction convention:

$$\begin{aligned} &\int d^d x\, e^{ip\cdot x} \langle T\,[(\underset{\sim}{\Pi}^2(x) + \Sigma^2(x))(\underset{\sim}{\Pi}^2(0) + \Sigma^2(0))]\rangle_\epsilon \\ &\equiv -i\, N[J(\epsilon\,, p^2, m^2) - J(\epsilon, 0, m^2)] \end{aligned} \tag{5.15}$$

As an illustration, we compute the on-shell $\phi\phi \to \phi\phi$ amplitude using the effective Lagrangian (5.12) and the subtraction convention (5.15). We have, in the large N limit:

$$iA_{\phi\phi\to\phi\phi}(s) = -i\,\frac{2a}{N}\left\{1 + \sum_{n=1}^{\infty}(-a)^n\left[J(\epsilon, p^2, m^2) - J(\epsilon, 0, m^2)\right]^n\right\}$$

$$= -i\,\frac{2}{N}\,\frac{a}{1 + a\left[J(\epsilon, p^2, m^2) - J(\epsilon, 0, m^2)\right]}$$

or

$$A_{\phi\phi\to\phi\phi}(s) = -\frac{2}{NJ(\epsilon, s, m^2)} \quad .$$

The fact that Eqs. (5.12) and (5.15) describe the dynamics in the upper phase completely will be demonstrated in Appendix B.

Note that we start with a Lagrangian, Eq. (3.1), depending on one parameter, λ_0. In the massless theory of the lower phase the coupling constant renormalization introduces an arbitrary mass scale M. However the renormalized theory still depends on a single parameter; if M changes so does the renormalized coupling constant λ, as controlled by the β function. We have succeeded in constructing an effective Lagrangian for the symmetric phase which depends on only a single, (renormalization group invariant) parameter, m. Thus the fact that in this Lagrangian the mass and (quartic) coupling constant are not independent parameters as they would be for a general linear σ model should come as no surprise; it had to be the case.

The normal-ordered Lagrangian (5.12) without the subtraction prescription (5.15) is superrenormalizable. However the subtraction rule changes the theory to be just renormalizable. This is clear if one makes use of the connection between the renormalizability and the high energy behavior of a theory. Without the subtraction rule $A_{\phi\phi \to \phi\phi}(s) \to$ const $= -2a/N$ as $s \to \infty$, whereas when the rule is imposed, this amplitude grows at high energies (consistently with unitarity).

In order to compare the short and long distance behavior of the upper phase it is useful to introduce a new effective coupling constant $\bar{\lambda}$, which measures the strength of the four-point proper vertex at the distance scale M. In analogy to the renormalization of $\Gamma^{(4)}$ in the lower phase, we require

$$\Gamma_s^{(4)}(s = -M^2,\ p_i^2 = m^2) = \bar{\lambda} M^{2-\epsilon} \tag{5.16}$$

Using Eqs. (4.26) and (5.16) to compute the Callan-Symanzik function, call it $\bar{\beta}(\bar{\lambda})$, we find

$$\begin{aligned} \bar{\beta}(\bar{\lambda}) &= M \frac{\partial \bar{\lambda}}{\partial M} \\ &= \bar{\lambda}(\epsilon - 2) + \frac{\bar{\lambda}^2 N}{(4\pi)^{1+\epsilon/2}} \, \Gamma(2 - \epsilon/2) \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{\left[\frac{m^2}{M^2} + \alpha(1-\alpha)\right]^{2-\epsilon/2}} \end{aligned} \tag{5.17}$$

The fact that $\bar{\beta}$ is a nontrivial function of $\bar{\lambda}$ serves as another proof that the subtraction rule modifies the theory from being superrenormalizable to simply renormalizable. From Eq. (5.17) one can verify that $\bar{\beta}$ has an ultraviolet stable zero at $\bar{\lambda} = \lambda_c$ and furthermore that

$$\lim_{\lambda - \lambda_c \to 0_+} \bar{\beta}'(\bar{\lambda}) = \lim_{\lambda - \lambda_c \to 0_-} \beta'(\lambda) = -\epsilon$$

These properties are in accord with the general argument presented in section III based on the continuity of the short distance behavior across the phase transition. Finally, by expressing m^2 in terms of $\bar{\lambda}$ via Eqs. (5.16) and (4.26) one finds that for large $\bar{\lambda}$

$$\bar{\beta}(\bar{\lambda}) \simeq \bar{\lambda}(\epsilon - 2) \quad .$$

That is, as $M \to 0$, the effective coupling strength $\bar{\lambda}$ increases at a linearly increasing rate.

VI. CONCLUDING COMMENTS

Finally, we would like to discuss two areas for further study. The large N approximation which we use is a very powerful method that makes it possible to sum perturbation theory to all orders in λN and thereby derive nonperturbative results, such as the existence of the phase transition. It would be interesting to go further and examine the higher order corrections to our results. Already in leading order in N the theory satisfies unitarity (in particular in the strong coupling phase, where this was nontrivial and required proof) and the full O(N) symmetry is restored and realized linearly in the strong coupling phase. Higher order corrections will presumably maintain these general properties. However these corrections will presumably include derivative interaction terms of the form $\frac{1}{N^2}(\underset{\sim}{\chi}\cdot\partial\underset{\sim}{\chi})^2 f(\underset{\sim}{\chi}^2)+\frac{1}{N^2}(\partial\underset{\sim}{\chi})^2 g(\underset{\sim}{\chi}^2)$. It would be interesting to calculate these terms.

Another problem which deserves further analysis is the property of dimensional regularization whereby the positivity of operator products and associated integrals is not preserved. It is easy to see why this happens. The essence of dimensional regularization is that a divergent integral is defined by analytically continuing (in d) the function to which it is equal where it is well defined. For example the integral of Eq. (1.10) is convergent for $0 < d < 2$ (and $m^2 \neq 0$) and here it is positive, as it must be. For $d \geq 2$ it is formally defined by the continuation of the right hand side in d.

There is no reason why this analytic continuation must be positive and indeed it is not. In contrast, if one cuts off the (Euclidean) momentum integration at $k^2 = \Lambda^2$ the result is (for $\epsilon > 0$)

$$<\phi^2>_\Lambda = \frac{(m^2)^{\frac{\epsilon}{2}}}{(4\pi)^{1+\epsilon/2}} \ln \frac{\Lambda^2}{m^2} + \frac{\epsilon}{4} \ln^2 \frac{\Lambda^2}{m^2}$$

Again, this is positive as it must be since for finite Λ^2 the integral is perfectly well defined. Thus, although there is no mystery mathematically about this aspect of dimensional regulation it is inconvenient in practice. What would be desirable would be a method of regularization which is Lorentz invariant, respects the Ward-Takahashi identities, etc., but retains the naive sign of the operator products.

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APPENDIX A

We shall evaluate the tree- and one-loop-contributions to the generating functional to proper vertices Γ. We shall only present the divergent part of the one-loop contribution, mainly to verify the argument of Section II.

The method of evaluating Γ up to one loop, by applying the steepest descent method to the functional integral Z_S of Eq. (1.9), is by now well-known.[17] In the following we shall suppose this knowledge on the reader's part.

The action S is given in Eq. (1.13). We shall shift the origin of the functional integration variables $\underset{\sim}{\phi}$

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}[\underset{\sim}{J}] + \underset{\sim}{B} \tag{A.1}$$

so that $S + \int d^d x \, \underset{\sim}{J} \cdot \underset{\sim}{\phi}$ does not contain linear terms in B, which determines $\underset{\sim}{\phi}[\underset{\sim}{J}]$. Up to one loop, $\underset{\sim}{\phi}[\underset{\sim}{J}]$ is the argument of Γ.

Denoting $\underset{\sim}{\phi}[\underset{\sim}{J}]$ by $\underset{\sim}{\phi}$, and regarding $\underset{\sim}{B}(x)$ as functional integration variables, we write the Lagrange density as

$$\begin{aligned}\mathscr{L} = &\left\{\tfrac{1}{2}(\partial\underset{\sim}{\phi})^2(1+z)^{-2} - \tfrac{1}{2}\mu^2\underset{\sim}{\phi}^2\right\} \\ &+\Big\{\tfrac{1}{2}(\partial\underset{\sim}{B})^2(1+z)^{-2} - \tfrac{1}{2}\mu^2\underset{\sim}{B}^2 \\ &- \lambda_o(\partial\underset{\sim}{\phi}\cdot\partial\underset{\sim}{B})(\underset{\sim}{\phi}\cdot\underset{\sim}{B})(1+z)^{-3} - \tfrac{1}{4}\lambda_o\underset{\sim}{B}^2(\partial\underset{\sim}{\phi})^2(1+z)^{-3} \\ &+\tfrac{3}{8}\lambda_o^2(\partial\underset{\sim}{\phi})^2(\underset{\sim}{\phi}\cdot\underset{\sim}{B})^2(1+z)^{-4}\Big\} + \mathscr{O}(\underset{\sim}{B}^3)\end{aligned} \tag{A.2}$$

where we have used the abbreviation $z = \lambda_o \underset{\sim}{\phi}^2/4$. The task is to perform functional Gaussian integrations for the terms in the second bracket on the right hand side of Eq. (A.2). The actual evaluation is considerably facilitated by the change of integration variables $\underset{\sim}{B} \rightarrow \underset{\sim}{B}(1+z)$:

$$\begin{aligned}
\mathscr{L} = &\left\{\frac{1}{2}(\partial\underset{\sim}{\phi})^2(1+z)^{-1} - \frac{\mu^2}{2}\underset{\sim}{\phi}^2\right\} \\
&+ \left\{\frac{1}{2}(\partial\underset{\sim}{B})^2 - \frac{\mu^2}{2}\underset{\sim}{B}^2\right\} \\
&+ \Big\{-\lambda_o(\partial\underset{\sim}{\phi}\cdot\partial\underset{\sim}{B})(\underset{\sim}{\phi}\cdot\underset{\sim}{B})(1+z)^{-1} \\
&\quad - \frac{1}{8}\lambda_o\mu^2\underset{\sim}{B}^2\underset{\sim}{\phi}^2(2+z) \\
&\quad + \frac{1}{2}\lambda_o(\underset{\sim}{B}\cdot\partial\underset{\sim}{B})(\underset{\sim}{\phi}\cdot\partial\underset{\sim}{\phi})(1+z)^{-1} \\
&\quad + \frac{1}{8}\lambda_o^2(\underset{\sim}{\phi}\cdot\partial\underset{\sim}{\phi})^2\underset{\sim}{B}^2(1+z)^{-2} \\
&\quad - \frac{1}{2}\lambda_o^2(\underset{\sim}{\phi}\cdot\underset{\sim}{B})(\underset{\sim}{B}\cdot\partial\underset{\sim}{\phi})(\underset{\sim}{\phi}\cdot\partial\underset{\sim}{\phi})(1+z)^{-2} \\
&\quad - \frac{1}{4}\lambda_o\underset{\sim}{B}^2(\partial\underset{\sim}{\phi})^2(1+z)^{-1} \\
&\quad + \frac{3}{8}\lambda_o^2(\underset{\sim}{\phi}\cdot\underset{\sim}{B})^2(\partial\underset{\sim}{\phi})^2(1+z)^{-2}\Big\} + \mathscr{O}(B^3)
\end{aligned} \qquad (A.3)$$

The one-loop divergences in $\Gamma^{(1)}$ arise entirely from the contraction of B_iB_j in each term in the third bracket above, and the iteration of the first and third terms in the third bracket in Eq. (A.3). Thus

$$\Gamma^{(0)}[\underset{\sim}{\phi}] = \int \left[\frac{1}{2}(\partial \underset{\sim}{\phi})^2 (1+z)^{-2} - \frac{\mu^2}{2} \underset{\sim}{\phi}^2\right] d^d x \quad , \tag{A.4}$$

and

$$\left[\Gamma^{(1)}\right]_1^{\text{div}} = \langle \underset{\sim}{B}^2 \rangle_\epsilon \int d^d x \Big\{ -\frac{1}{8}\lambda_0 \mu^2 \underset{\sim}{\phi}^2 (2+z)$$

$$+ \frac{1}{8}\lambda_o^{\ 2} (\underset{\sim}{\phi} \cdot \partial \underset{\sim}{\phi})^2 (1+z)^{-2} - \frac{1}{2}\lambda_o^{\ 2} \frac{1}{N-1} (\underset{\sim}{\phi} \cdot \partial \underset{\sim}{\phi})^2 (1+z)^{-2}$$

$$- \frac{1}{4}\lambda_o (\partial \underset{\sim}{\phi})^2 (1+z)^{-1} + \frac{3}{8}\lambda_o^{\ 2} \frac{1}{N-1} (\partial \underset{\sim}{\phi})^2 \underset{\sim}{\phi}^2 (1+z)^{-2} \Big\} \tag{A.5}$$

which arises from the contraction of $B_i B_j$ within the same term, where

$$\langle \underset{\sim}{B}^2 \rangle_\epsilon = (N-1) \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - \mu^2} = I(\epsilon, \mu^2)(N-1)$$

The divergences which arise from the iteration of the first and third terms are

$$\frac{i}{2} \int d^d x \langle T\left(\partial_\mu B_i(x) B_j(x) \partial_\nu B_k(0) B_\ell(0)\right)\rangle_\epsilon \Big\{ \lambda_o^{\ 2} (\partial^\mu \phi_i)\phi_j (\partial^\nu \phi_k)\phi_\ell$$

$$- \lambda_o^{\ 2} \delta_{ij} (\underset{\sim}{\phi} \cdot \partial^\mu \underset{\sim}{\phi})(\partial^\nu \phi_k)\phi_\ell + \frac{\lambda_o^2}{4} \delta_{ij}\delta_{k\ell} (\underset{\sim}{\phi} \cdot \partial \underset{\sim}{\phi})^2 \Big\} (1+z)^{-2} \quad . \tag{A.6}$$

Now

$$i\int d^dx < T\left(\left(\partial_\mu B_i(x)\right) B_j(x)\left(\partial_\nu B_k(0)\right) B_\ell(0)\right) >$$

$$= (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})\frac{g_{\mu\nu}}{d}\, i\int\left(\frac{dp}{2\pi}\right)^d\left(\frac{i}{p^2-\mu^2}\right)^2 p^2$$

$$\simeq -\frac{g_{\mu\nu}}{d} I(\epsilon,\mu^2)(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}). \tag{A.7}$$

Combining Eqs. (A.6) and (A.7) we obtain

$$[\Gamma^{(1)}]_2^{div} = -\frac{1}{2}\frac{\lambda_o^2}{d} I(\epsilon,\mu^2)\int d^dx\left\{(\partial\underset{\sim}{\phi})^2\underset{\sim}{\phi}^2 - (\underset{\sim}{\phi}\cdot\partial\underset{\sim}{\phi})^2\right\}(1+z)^{-2} \tag{A.8}$$

Combining (A.4, 5, 6), we obtain finally

$$\begin{aligned}[\Gamma^{(0)}+\Gamma^{(1)}]^{div} = \int d^dx\Bigg[&\frac{1}{2}(\partial\underset{\sim}{\phi})^2(1+z)^{-1} - \frac{\mu^2}{2}\underset{\sim}{\phi}^2\\
&-\frac{N-1}{2\pi\epsilon}\Big\{-\frac{1}{4}\lambda_o(\partial\underset{\sim}{\phi})^2(1+z)^{-1}\\
&+\frac{\lambda_o^2}{8}\left(\frac{N-3}{N-1}\right)(\underset{\sim}{\phi}\cdot\partial\underset{\sim}{\phi})^2(1+z)^{-2}\\
&+\frac{\lambda_o^2}{8}\frac{1}{N-1}\underset{\sim}{\phi}^2(\partial\underset{\sim}{\phi})^2(1+z)^{-2}\\
&-\frac{\mu^2}{2}\lambda_o(1+z)^2\Big\}\Bigg]\end{aligned} \tag{A.9}$$

The divergent terms proportional to $(N-1)(2\pi\epsilon)^{-1}$ may be transformed away completely, except the last term proportional to $\lambda_o\mu^2$, by the following transformations:

$$\lambda_o = \lambda + \delta\lambda$$

$$= \lambda\left[1 + \frac{\lambda(N-2)}{2\pi\epsilon}\right] \quad , \tag{A.10}$$

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}_r\left[1 - \frac{1}{4}\frac{\lambda(n-1)}{2\pi\epsilon}\left(1 - \frac{N-3}{N-1}\frac{\lambda}{4}\underset{\sim}{\phi}_r{}^2\right)\right] . \tag{A.11}$$

Note further that

$$\sqrt{\frac{\lambda_o}{4}}\,\underset{\sim}{\phi} = \sqrt{\frac{\lambda}{4}}\,\underset{\sim}{\phi}_r\left[1 + \frac{1}{4}\frac{\lambda(N-3)}{2\pi\epsilon}\left(1 + \frac{\lambda}{4}\phi_r{}^2\right)\right] \tag{A.12}$$

The divergent, induced mass term is

$$g(z) = \frac{\mu^2}{2}\left(\frac{N-1}{2\pi\epsilon}\right)\lambda\left(1 + \frac{\lambda}{4}\phi_r{}^2\right) \tag{A.13}$$

APPENDIX B

The purpose of this Appendix is to present a discussion of the dynamics in the upper phase, alternative to the discussions given in sections III - V.[18] The method we shall use is, strictly speaking, justifiable for small $|\lambda_0|$. This is the case, for example, if $\lambda >> \lambda_c$ and $\lambda_c << 1$, for $\lambda_0 = -\lambda_c(1 - \lambda_c/\lambda)^{-1} \simeq -\lambda_c$. These conditions are met if λ is positive and finite, and N is large.

We begin by rewriting Z_{NL} of Eq. (1.2) in a parametric form:[19]

$$Z_{NL}[\underset{\sim}{J}] = \int \Pi d\underset{\sim}{\chi}(x) \prod_x \delta\left(\underset{\sim}{\chi}^2 - \frac{1}{\lambda_0}\right) \exp i \int d^d x \left\{\frac{1}{2}(\partial\underset{\sim}{\chi})^2 + \underset{\sim}{J}\cdot\underset{\sim}{\chi}\right\}$$

$$= C \int \prod_x d\beta(x) \int \Pi d\underset{\sim}{\chi}(x) \exp i \int d^d x \left\{\frac{1}{2}(\partial\underset{\sim}{\chi})^2 - \beta(x)\left(\underset{\sim}{\chi}^2 - \frac{1}{\lambda_0}\right) + \underset{\sim}{J}\cdot\underset{\sim}{\chi}\right\} \tag{B.1}$$

where C is a constant independent of $\underset{\sim}{J}$. We shall discard it hereafter. We are free to translate the integration variables $\beta(x)$ uniformly by a finite amount, $m^2/2$, say, where m^2 is to be chosen at our convenience. Therefore, Eq. (B.1) can be written as

$$Z_{NL}[\underset{\sim}{J}] = \int \prod_x d\beta(x)\, e^{\frac{i}{2}\int d^d x \beta(x)} Z_\beta[\underset{\sim}{J}] \quad ; \tag{B.2}$$

$$Z_\beta[\underset{\sim}{J}] = \int \Pi d\underset{\sim}{\chi}(z) \exp i \int \left[\frac{1}{2}(\partial\underset{\sim}{\chi})^2 - \frac{m^2}{2}\underset{\sim}{\chi}^2 - \frac{\lambda_0}{2}\beta(x)\underset{\sim}{\chi}^2 + \underset{\sim}{J}\cdot\underset{\sim}{\chi}\right] d^d x \tag{B.3}$$

where we have rescaled the variables β by $\beta \to \lambda_0\beta/2$, and dropped again an inessential multiplicative factor.

The functional Z_β is just the generating functional of Green's functions of a theory of N component scalar bosons interacting with an external potential $\lambda_0\beta(x)$. It can be written as

$$\ln Z_\beta[\underset{\sim}{J}] = -\frac{i}{2}\int d^dx\, d^dy\, \underset{\sim}{J}(x)\langle x|\left(-\partial^2 - m^2 - \lambda_0\beta + i\delta\right)^{-1}|y\rangle \underset{\sim}{J}(y)$$

$$-\frac{1}{2}\,\mathrm{Tr}\ln\langle x|1 - \frac{1}{-\partial^2 - m^2 + i\delta}\lambda_0\beta|y\rangle \quad , \qquad (B.4)$$

where Tr means the trace operations on N components as well as over space-time variables, and

$$\langle x|-\partial^2 - m^2 - \lambda_0\beta\,|y\rangle = \left[-\partial^2 - m^2 - \lambda_0\beta(x)\right]\delta^d(x-y)\,. \qquad (B.5)$$

It is useful to expand the right hand side of Eq. (B.4) in powers of λ_0. Symbolically

$$\underset{\sim}{J}\cdot\frac{1}{-\partial^2 - m^2 - \lambda_0\beta + i\delta}\underset{\sim}{J} = \underset{\sim}{J}\cdot\Delta_F\underset{\sim}{J} + \underset{\sim}{J}\cdot\Delta_F(\lambda_0\beta)\Delta_F\underset{\sim}{J} \qquad (B.6)$$

$$+\underset{\sim}{J}\cdot\Delta_F(\lambda_0\beta)\Delta_F(\lambda_0\beta)\Delta_F\underset{\sim}{J} + \mathcal{O}(\lambda_0{}^3)$$

and

$$-\mathrm{Tr}\ln\left(1 - \frac{1}{-\partial^2 - m^2 + i\delta}\lambda_0\beta\right) = -i\lambda_0 N I(\epsilon, m^2)\int d^dx\,\beta(x)$$

$$+\lambda_0{}^2\frac{N}{2}\int d^dx\, d^dy\,\beta(x)\left[\Delta_F(x-y)\right]^2\beta(y)$$

$$+\mathcal{O}(\lambda_0{}^3) \qquad (B.7)$$

where $\Delta_F = \langle x | (-\partial^2 - m^2 + i\delta)^{-1} | y \rangle = \Delta_F(x - y; m^2)$.

In the spirit of the steepest descent method, and buttressed by the assumption $\lambda_0 \ll 1$, we shall ignore terms of order λ_0^3 and higher in Eqs. (B.6 -7) henceforth.

We substitute Eqs. (B.6 - 7) in Eq. (B.4), and then Eq. (B.4) into Eq. (B.2). The result is

$$Z_{NL}[\underset{\sim}{J}] = e^{-\frac{i}{2}\underset{\sim}{J}\cdot\Delta_F\underset{\sim}{J}} \int \prod_x d\beta(x)\, e^{\frac{i}{2}\int d^dx\beta(x)\left[1 - \lambda_0 NI - \lambda_0 \underset{\sim}{\phi}^2(x)\right]}$$

$$\times \exp \frac{i}{2}\int d^dx\, d^dy\, \beta(x)\langle x | K | y\rangle \beta(y) \quad , \qquad \text{(B.8)}$$

where

$$\langle x | K | y \rangle = -\frac{i\lambda_0^2 N}{2} \Delta_F^2(x - y; m^2) - \lambda_0^2 \underset{\sim}{\phi}(x) \cdot \Delta_F(x - y; m^2)\underset{\sim}{\phi}(y), \qquad \text{(B.9)}$$

and, as before, $I = I(\epsilon, m^2)$ and $\underset{\sim}{\phi}(x)$ is defined as

$$\underset{\sim}{\phi}(x) = \int d^dy \Delta_F(x - y; m^2)\underset{\sim}{J}(y) \qquad \text{(B.10)}$$

We must now choose m by some criterion. We shall demand that the single particle propagation characteristic expressed by the exponent of the first factor on the right hand side of Eq. (B.8)

$$\frac{i}{2}\underset{\sim}{J}\cdot\Delta_F\underset{\sim}{J} = \frac{i}{2}\int d^dx\, d^dy \underset{\sim}{J}(x)\cdot\Delta_F(x - y; m^2)\underset{\sim}{J}(y)$$

not be modified by the terms that arise from the $\beta(x)$ integrations. As we shall see, this is achieved by choosing m such that

$$1 - \lambda_0 \, NI(\epsilon, m^2) = 0 \tag{B.11}$$

This is the mass eigenvalue equation for m^2, discussed elsewhere in the text: $m^2 \neq 0$ only if $\lambda_0 < 0$.

Equation (B.8) is now simple:

$$\begin{aligned} Z_{NL}[\underset{\sim}{J}] &= \exp\left(-\frac{i}{2}\underset{\sim}{J} \cdot \Delta_F \underset{\sim}{J}\right) \\ &\times \int \prod_x d\beta(x) \exp i \int \left[\frac{1}{2}\beta K \beta - \left(\frac{\lambda_0}{2} \underset{\sim}{\phi}^2\right)\beta\right] , \end{aligned} \tag{B.12}$$

in a symbolic notation. The functional Gaussian integrations implied can be performed, with the result that

$$\begin{aligned} \frac{1}{i} \ln Z_{NL}[\underset{\sim}{J}] &= \frac{1}{2}\int d^d x \, d^d y \Big\{ - \underset{\sim}{J}(x) \cdot \Delta_F(x - y; m^2) \underset{\sim}{J}(y) \\ &\quad - \left(\frac{\lambda_0}{2}\right)^2 \underset{\sim}{\phi}^2(x) \langle x | K^{-1} | y \rangle \underset{\sim}{\phi}^2(y) \Big\} \\ &\quad + \frac{i}{2} \operatorname{tr} \ln \langle x | K | y \rangle \end{aligned} \tag{B.13}$$

where tr denotes the trace operation over space-time variables (and not over N components). It turns out that the contribution of the last term is of order N^{-1} compared to the rest (we leave this statement for the reader's verification), and we shall ignore it henceforth.

It is necessary for us to examine the structure of $<x|K^{-1}|y>$ more closely. We write

$$<x|K|y> = \frac{\lambda_0{}^2 N}{2} a^{-1} [\delta^d(x - y) - a <x|L|y>] \qquad (B.14)$$

where $a^{-1} = J(\epsilon, 0, m^2)$

$$J(\epsilon, p^2, m^2) = i\int\left(\frac{dk}{2\pi}\right)^d \frac{i}{k^2 - m^2} \frac{i}{(p-k)^2 - m^2} \qquad (B.15)$$

as before, and

$$<x|L|y> = \left[i\Delta_F^2(x-y) + J\delta^d(x-y)\right] + \frac{2}{N} \underset{\sim}{\phi}(x) \cdot \Delta_F(x-y; m^2)\underset{\sim}{\phi}(y). \quad (B.16)$$

We can now invert K:

$$\begin{aligned} <x|K^{-1}|y> &= \frac{2}{\lambda_0{}^2 N} a <x|[1 - aL]^{-1}|y> \\ &= \frac{2}{\lambda_0{}^2 N} a \sum_{m=0} a^m <x|L^m|y>, \; L^{(0)} = 1 \end{aligned} \qquad (B.17)$$

Substituting Eq. (B. 17) into Eq. (B. 13) we obtain finally

$$\frac{1}{i}\ln Z_{NL}[\underset{\sim}{J}] = \int d^dx\, d^dy \Big\{\frac{1}{2}\underset{\sim}{\phi}(x)<x|-\partial^2 - m^2|y>\underset{\sim}{\phi}(y) - \underset{\sim}{J}(x)\cdot \delta^d(x-y)\underset{\sim}{\phi}(x)$$

$$- \frac{1}{4N}a\sum_{m=0} a^m \underset{\sim}{\phi}^2(x)<x|L^m|y>\underset{\sim}{\phi}^2(y)\Big\} \quad , \qquad (B.18)$$

where $\underset{\sim}{\phi}$ is given by Eq. (B.10). It is of interest to note the Fourier transform of <x| L |y>:

$$\langle p|L|q\rangle = \int d^dx\, d^dy\, e^{i(p\cdot x - q\cdot y)}\langle x|L|y\rangle$$

$$= -(2\pi)^d\delta^d(p-q)[J(\epsilon, p^2, m^2) - J(\epsilon, 0, m^2)] \quad \text{(B.19)}$$

$$+\frac{2}{N}\int\left(\frac{dk}{2\pi}\right)^d \underset{\sim}{\tilde{\phi}}(p-k)\cdot\frac{1}{k^2-m^2+i\delta}\underset{\sim}{\tilde{\phi}}(k-q)$$

where

$$\tilde{\phi}(k) = \int d^dx\, e^{ik\cdot x}\phi(x).$$

As noted in Section V, the above result (B.18) may be derived from an effective Lagrangian

$$\mathscr{L}_{\text{eff}} = :\frac{1}{2}(\partial\underset{\sim}{\chi})^2 - \frac{1}{2}m^2\underset{\sim}{\chi}^2 - \frac{a}{4N}(\underset{\sim}{\chi}^2)^2 : \quad \text{(B.20)}$$

in the large N-limit, with the proviso that in the construction of the S-matrix, the following rule be used:

$$i\int d^dx\, e^{ip\cdot x}\langle T\left(\underset{\sim}{\chi}^2(x)\underset{\sim}{\chi}^2(0)\right)\rangle_\epsilon = N[J(\epsilon, p^2, m^2) - J(\epsilon, 0, m^2)] \quad \text{(B.21)}$$

Finally let us evaluate the effective coupling constant a:

$$a^{-1} = i\int\left(\frac{dk}{2\pi}\right)^d\left(\frac{i}{k^2-m^2}\right)^2 = \Gamma\left(1-\frac{\epsilon}{2}\right)\left(\frac{m^2}{4\pi}\right)^{\frac{\epsilon}{2}}(4\pi m^2)^{-1} \quad \text{(B.22)}$$

so that

$$a = 4\pi m^2 \tag{B.23}$$

as $\epsilon \to 0_+$. In the same limit the effective Lagrangian is

$$\mathscr{L}_{\text{eff}} = :\frac{1}{2}(\partial \underset{\sim}{\chi})^2 - \frac{1}{2} m^2 \underset{\sim}{\chi}^2 - \frac{\pi m^2}{N} (\underset{\sim}{\chi}^2)^2 : \tag{B.24}$$

APPENDIX C

The following identities are useful in understanding Section II:

(1)
$$\int d^d x\, X_{ij}^{(0)} \frac{\delta \Gamma}{\delta \xi_j(x)} = (\partial \underset{\sim}{\xi})^2 \xi_i \left[2f_1 + (1+z)f_1'\right] + (\xi \cdot \partial_\mu \xi)\partial^\mu \xi_i (1+z) f_2 + (\xi \cdot \partial \xi)^2 \xi_i \left[2f_2 + (1+z)f_2'\right] + 2\xi_i (1+z) g' \qquad (C.1)$$

where $X_{ij}(0)$ is given in Eq. (2.14a) and the form of Γ is given in Eq. (2.22).

(2)
$$\int d^d x\, X_{ij} \frac{\delta \Gamma^{(0)}}{\delta \xi_j(x)} = -\frac{2}{\lambda_0}\Big\{(\partial \underset{\sim}{\xi})^2 \xi_i (1+z)^{-3}\Big[2\big(K_2(z) - K_1(z)\big) - (1+z)K_1(z)\Big] - (\xi \cdot \partial_\mu \xi)\partial^\mu \xi_i (1+z)^{-2}\Big[2K_1'(z) + K_2(z)\Big] - (\xi \cdot \partial \xi)^2 \xi_i\, 2(1+z)K_2'\Big\} \qquad (C.2)$$

where X_{ij} has the form given in Eq. (2.33), and $\Gamma^{(0)}$ is given in Eq. (2.19)

FOOTNOTES AND REFERENCES

[1]K. Wilson, Phys. Rev. D10, 2445 (1974); Erice Lectures (unpublished).

[2]R. Balian, J. Drouffe, and C. Itzykson, Phys. Rev. D10, 3376 (1974); D11, 2098, 2104 (1975); C. Korthals-Altes, talk at the Marseilles Colloquium on Lagrangian Field Theory, 1974; J. Kogut and L. Susskind, Phys. Rev. D11, 395 (1975); T. Banks, L. Susskind, and J. Kogut, Phys. Rev. D13, 1043 (1976).

[3]A. Polyakov, Phys. Lett. 59B, 79 (1975); A. Migdal, Landau Institute of Theoretical Physics preprint, 1975.

[4]E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976); CEN Saclay preprint. We did not have the benefit of the latter preprint until after the completion of this paper.

[5]W. Bardeen and R. Pearson, Fermilab preprint 76/24-THY.

[6]The large N expansion in field theory has been considered previously by a number of authors. See, among others, L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974); H. Schnitzer, Phys. Rev. D10, 1800 (1974) D10, 2042 (1974); S. Coleman, R. Jackiw, and H. Politzer, Phys. Rev. D10, 2491; D. Gross and A. Neveu, Phys. Rev. D10, 3235 (1975); G. 't Hooft, Nucl. Phys. B72, 461 (1974), B75, 461 (1974); G. 't Hooft and B. deWit, private communication; D. Gross, N. Coote, and C. Callan, Princeton preprint; M. Einhorn, Fermilab preprint; L. Abbot, J. Kang,

and H. Schnitzer, Brandeis preprint; P. Townsend, Phys. Rev. D12, 2269 (1975); Brandeis preprints. The large N expansion has also been used in statistical mechanics; see, e.g. H. Stanley, Phase Transitions and Critical Phenomena (Cambridge Univ. Press, 1971). Dynamical mass generation has been considered before by Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961), and more recently by other authors.

[7] We are being somewhat cavalier about renormalization here. See sections II, III and Appendices A, B, and C for a careful discussion of renormalization.

[8] This cancellation was shown by I. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, Phys. Rev. D3, 2486 (1971).

[9] G. 't Hooft and M. Veltman, Nucl. Phys. B44, 185 (1973); G. Liebbrandt

We have enclosed a corrected version of the page (p. 36) on which these formulas appear. These changes have no effect on any of the rest of the paper.

Secondly, in footnote 6, the second last sentence, "The large N expansion has also been used in statistical mechanics..." should be expanded to read "The large N expansion ... (Cambridge Univ. Press, 1971); R. Abe and S. Hikami, Prog. Theor. Phys. 49, 1851 (1973); S. K. Ma, Rev. Mod. Phys. 45, 589 (1973) and references therein."

op cit; lectures at Les Houches, 1975 (to be published).

[13]By "second order" phase transition we mean only that the order parameter is continuous across the transition.

[14]The N which appears in Eq. (3.21) for λ_c is really N - 2, reflecting the fact that the nonlinear O(2) σ model is a free field theory. In this case there is of course no phase transition, no renormalization, and no β function.

[15]The critical exponent for an order parameter p, called β, is defined by

$$\beta = \lim_{\Delta \to 0_+} \frac{\partial \log p}{\partial \log \Delta}$$

where $\Delta = \lambda - \lambda_c$. There should be no confusion of this constant with the Callan-Symanzik β function.

[16]S. Coleman, Commun, Math. Phys. 31, 259 (1973).

[17]See, B. de Witt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967); B. W. Lee and J. Zinn-Justin, Phys. Rev. D5, 3121, 3137, 3155 (1972); E. Abers and B. W. Lee, Phys. Repts. 9C, 1 (1973).

[18]Our approach here is in spirit similar to, but in emphasis different from, that of E. Brézin and J. Zinn-Justin, Ref. 4. See also D. Bessis and J. Zinn-Justin, Phys. Rev. D5, 1313 (1972).

[19]Here $\underset{\sim}{\chi}$ and $\underset{\sim}{\phi}$ (see Eq. (B.10)) are N-component vectors.

FIGURE CAPTIONS

Fig. 3.1: Dominant graphs which contribute to $\Delta_{F\,ij}$ for the Lagrangian (1.9).

Fig. 3.2: Dominant classes of graphs which contribute to $\Delta_{F\,ij}$ for the Lagrangian (1.3).

Fig. 3.3: m as a function of λ/λ_c for (a)ϵ = .1; (b)ϵ = .01.

Fig. 3.4: m as a function of λN for $\epsilon = 0$.

Fig. 3.5: The Callan-Symanzik function $\beta(\lambda)$.

Fig. 4.1: The leading graphs which contribute to the T_{ij} functions. The graphs for T_{21} are simply reflections of those for T_{12} and hence are not shown.

Fig. 4.2: The bare four-point vertex $i\Gamma^{(4)}_{ijk\ell}$ in the upper phase. The vertex includes an infinite sum of daisy loop corrections.

Fig. 4.3: Graphical representation of the matrix Σ.

Fig. 4.4: Graphical and analytic form of the equation (4.14) for T_{ij}. The figure illustrates the ij = 11 and 12 components of the equation.

Fig. 4.5: The leading classes of graphs which contribute to $\Gamma_s^{(4)}$ and $A^{(s)}_{\phi\phi\to\phi\phi}$.

Fig. 5.1: The classes of graphs which give the dominant contribution to $A_{\sigma\sigma\to\phi\phi}$. The dotted lines denote σ particles and the solid lines denote ϕ particles.

Fig. 5.2: The classes of graphs which give the dominant contribution to the $\sigma\phi\phi \to \sigma\phi\phi$ scattering amplitude. The dotted lines denote σ particles and the solid lines denote ϕ particles.

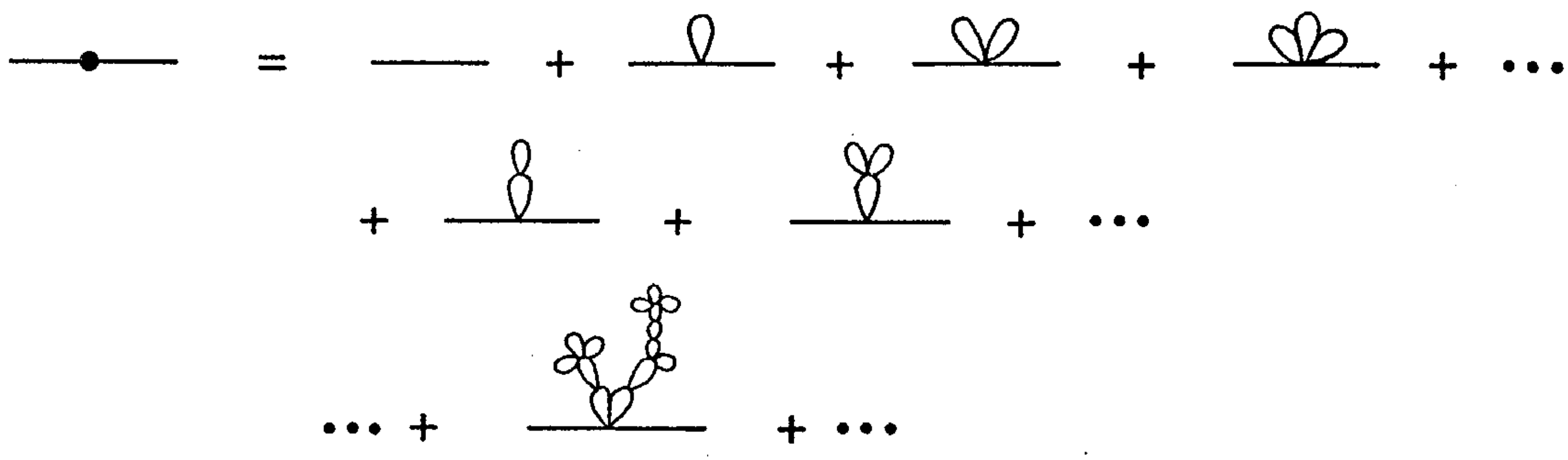


Fig. 3.1

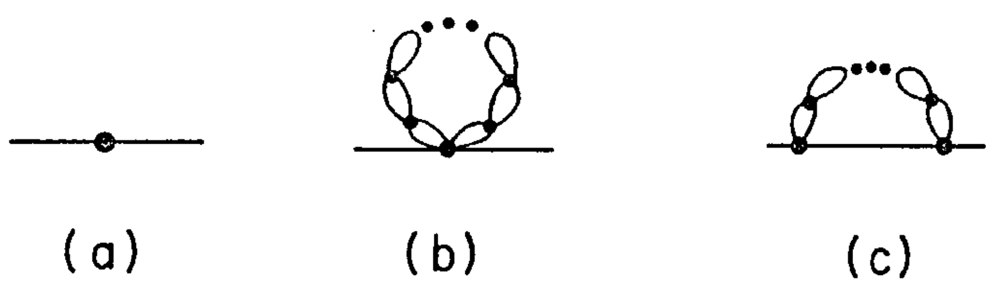


Fig. 3.2

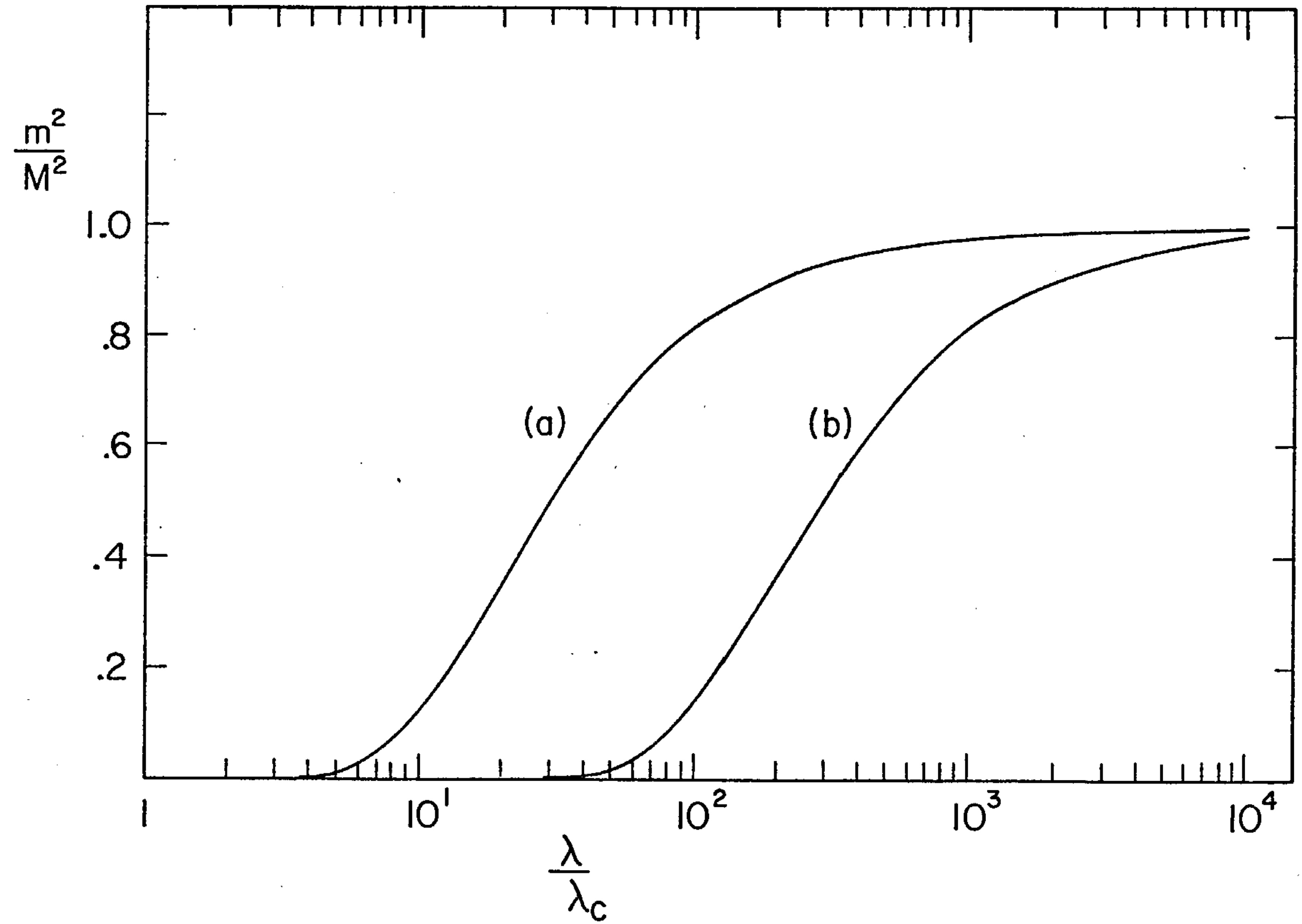


Fig. 3.3

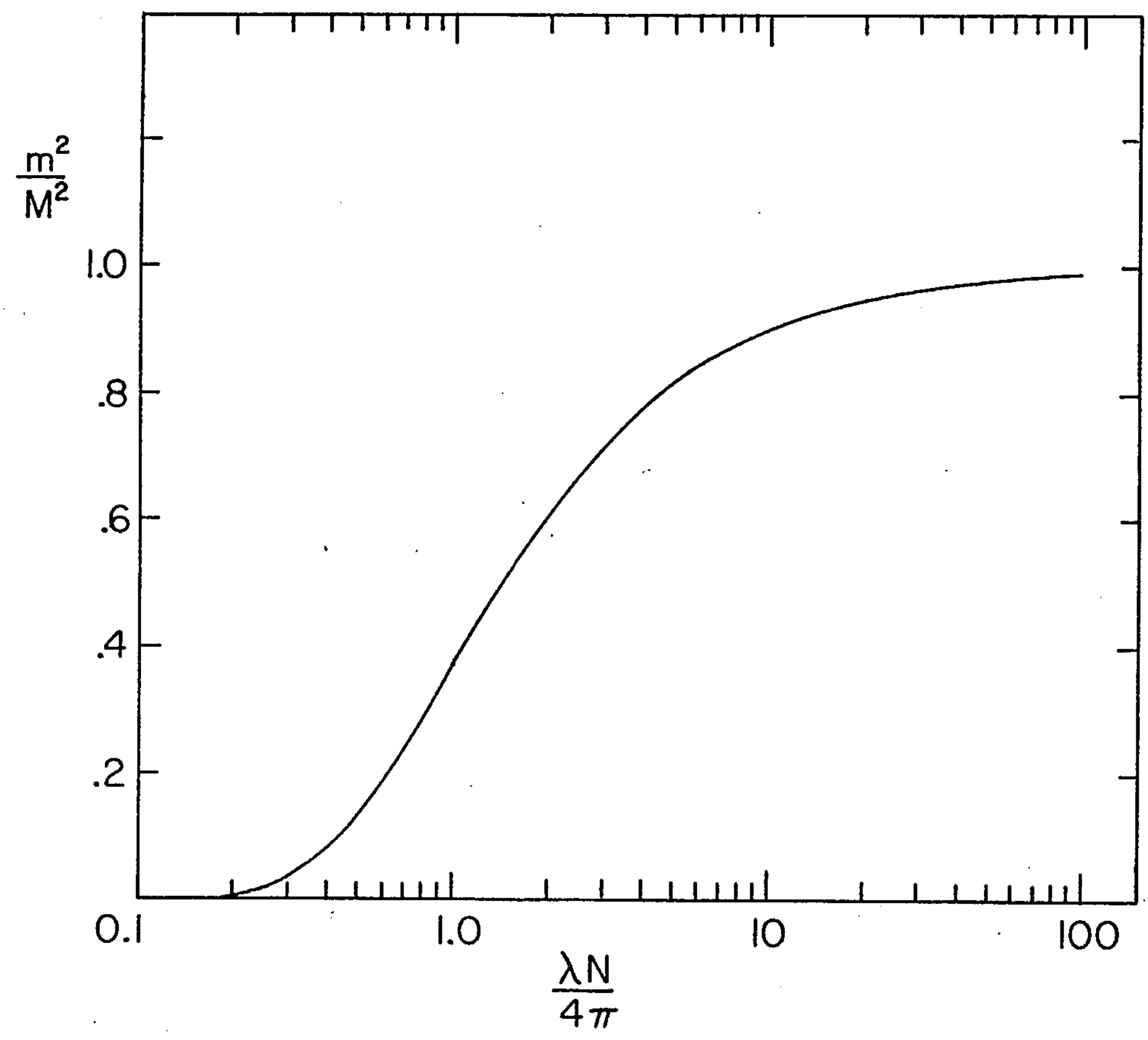


Fig. 3.4

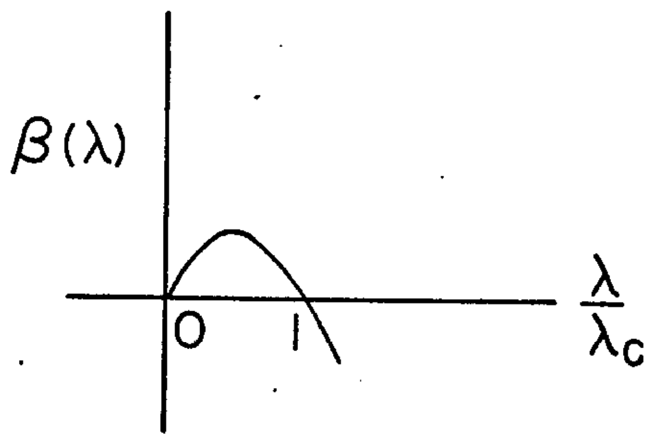


Fig. 3.5

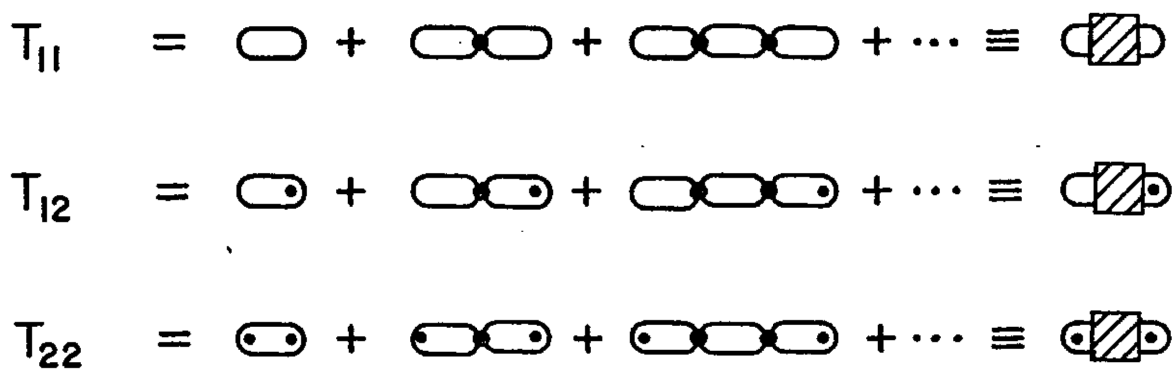
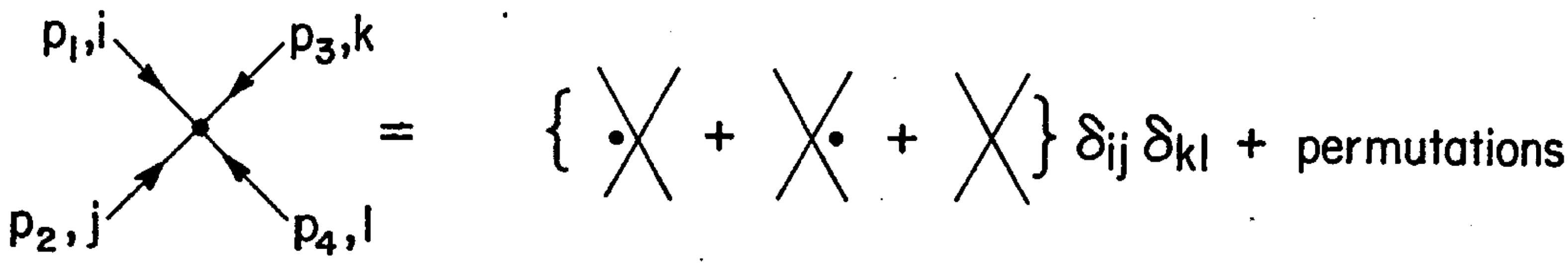
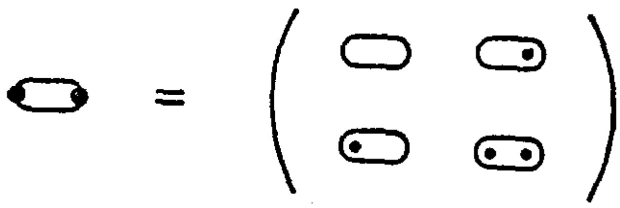


Fig. 4.1



$$i\,\Gamma^{(4)}_{ijkl} = i\lambda_0\left[\left\{2p_1\cdot p_2 + 2p_3\cdot p_4 + 3m^2\right\}\delta_{ij}\delta_{kl} + \text{permutations}\right]$$

Fig. 4.2



$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Fig. 4.3

$$T_{11} = \Sigma_{11} + \frac{\lambda_0}{2}\left[\left(\Sigma_{12} - \frac{3}{2} m^2 \Sigma_{11}\right) T_{11} + \Sigma_{11}\, T_{21}\right]$$

$$T_{12} = \Sigma_{12} + \frac{\lambda_0}{2}\left[\left(\Sigma_{12} - \frac{3}{2} m^2 \Sigma_{11}\right) T_{12} + \Sigma_{11}\, T_{22}\right]$$

Fig. 4.4

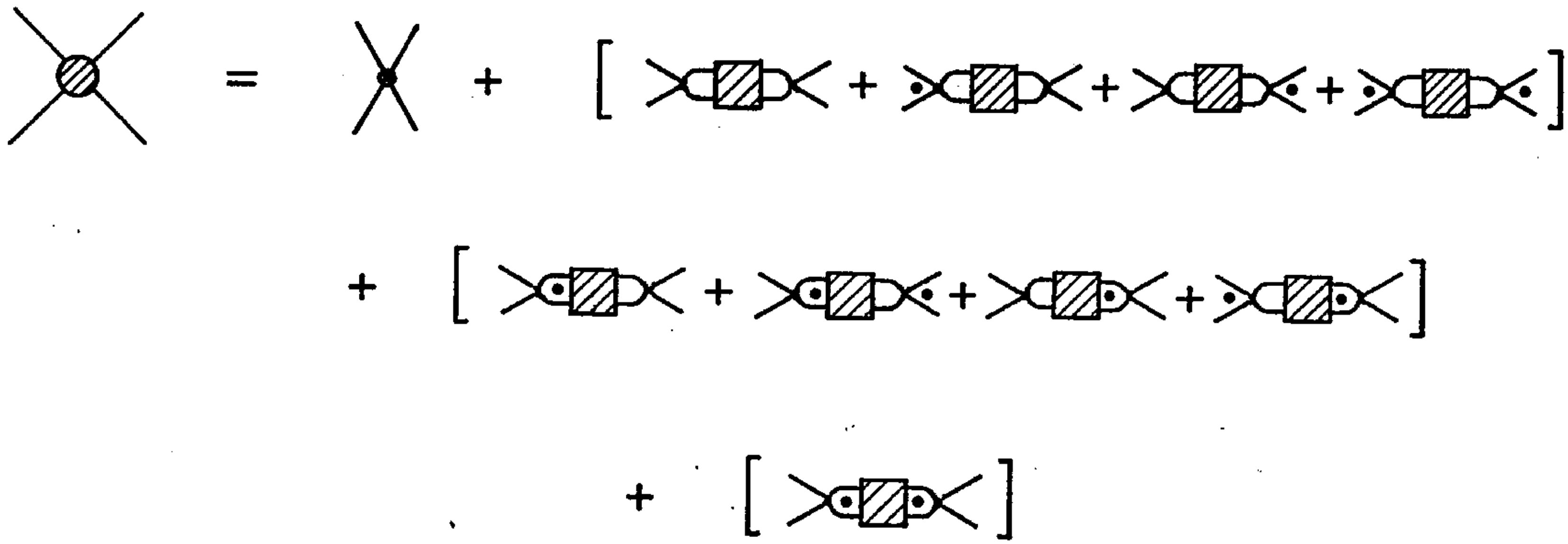


Fig. 4.5

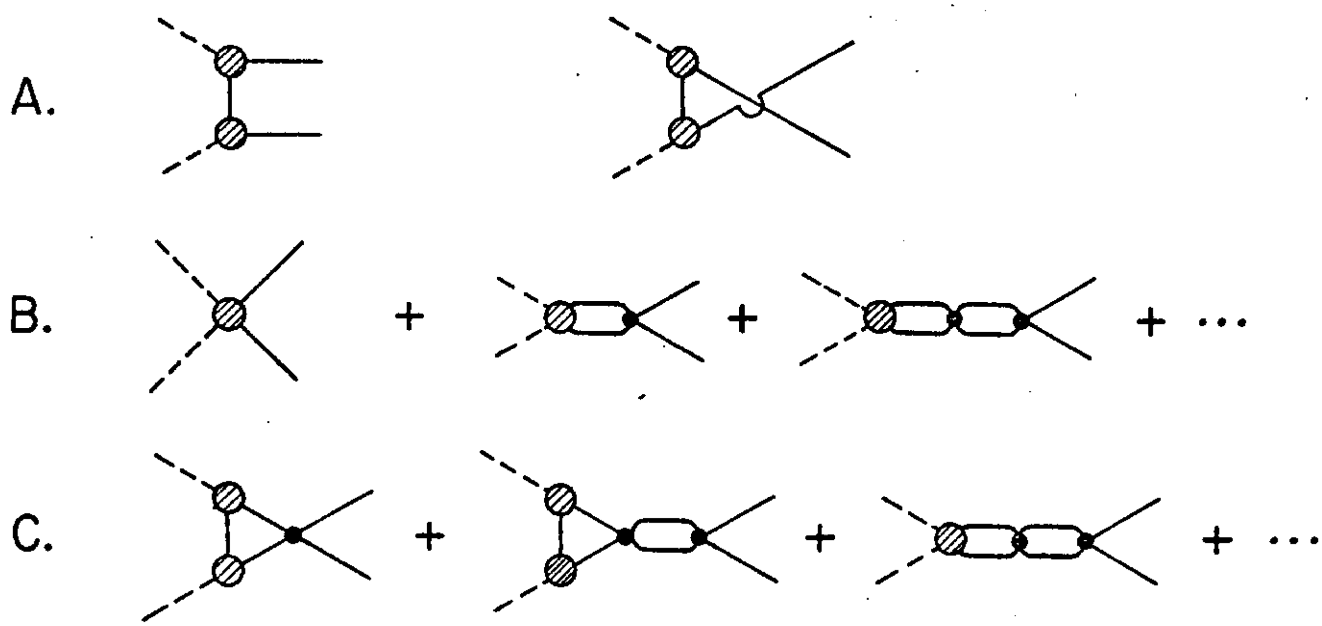


Fig. 5.1

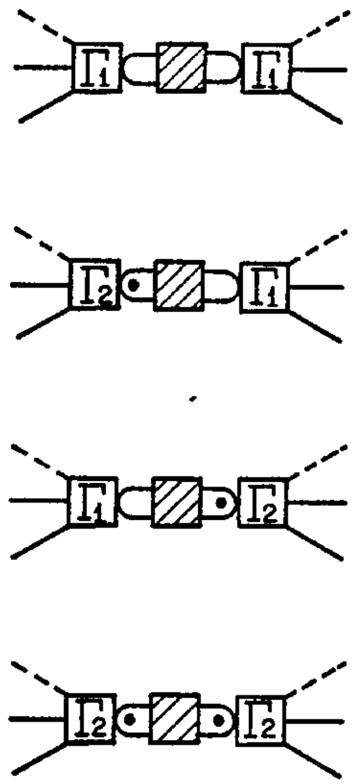


Fig. 5.2