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Angular Distributions in the Decay* $\psi' \rightarrow \psi\pi\pi$

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ABSTRACT

A general analysis of the amplitudes for $\psi' \rightarrow \psi\pi\pi$ is presented. Angular distributions are calculated for the case $\psi' \rightarrow \psi\pi\pi \rightarrow \mu^+\mu^-\pi\pi$ in terms of partial wave amplitudes. In principle, this decay provides a laboratory for studying $\pi\pi$ scattering. The analysis applies, mutatis mutandis, for $\psi \rightarrow \omega\pi\pi$ and similar decays of the form $\psi \rightarrow$ VPP. The determination of the partial wave amplitudes for $\psi' \rightarrow \psi\pi\pi$ will determine the phase shift difference $\delta_0^0 - \delta_2^0$ if there is sufficient π-π d-wave to produce measurable interference.

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I. INTRODUCTION

The dominant decay of the $\psi' = \psi(3.7)$ is $\psi' \to \psi\pi\pi$.[1] The apparent quantum numbers of the ψ and ψ' are $J^P = 1^-$, $I^{GC} = 0^{--}$. The $\pi\pi$ system has $I^{GC} = 0^{++}$,[2] which requires that its angular momentum, in its rest frame, be even. The decay spectrum as a function of the $\pi\pi$ invariant mass, $m_{\pi\pi}$, does not conform to naive expectations[2] (phase space for the effective Lagrangian $g\psi'_\mu \psi^\mu \underset{\sim}{\pi} \cdot \underset{\sim}{\pi}$) even when final state interactions are included. A reasonably satisfactory description is given by chiral symmetry[3] if the amplitudes which lead to anisotropic distributions are eliminated. Preliminary data indicate the anisotropies are small and are thus consistent with the chiral symmetry picture. However, chiral symmetry offers no a priori reason for the absence of anisotropies: in general they are expected to be present.

Independent of chiral symmetry, the decay $\psi' \to \psi\pi\pi$ is a remarkable source of information both for the interactions of the new particles and of "old" particles-pions. Since the hadronic interactions of the ψ' and the ψ are feeble, we expect the decay amplitudes to be real except for rescattering corrections - a situation similar to that of K_{e4}. If the π-π d-wave contribution is strong enough, it will be possible to determine $\delta_s^{I=0} - \delta_d^{I=0}$ for $\pi^+\pi^-$ scattering in the region $m_{\pi\pi} \approx 500$ MeV.

From the decays in which the ψ decays leptonically ($\sim 7\% \mu^+\mu^-$, $\sim 7\% e^+e^-$) a good deal of polarization information is available. In addition, the ψ' produced by e^+e^- annihilation is transversely polarized with respect to the beam. It is essential to exploit this polarization information to obtain the fullest understanding of the decay.

There are five invariant amplitudes for the decay $\psi' \to \psi\pi\pi$. To see this we set $q = \frac{1}{2}(q_1 - q_2)$, $Q = q_1 + q_2$ where the π^+ momentum is q_1 and the π^- momentum is q_2. Then denoting the ψ' and ψ polarizations by ϵ' and ϵ respectively we can form five invariants bilinear in $\epsilon'\epsilon^*$: $\epsilon^* \cdot \epsilon'$, $\epsilon^* \cdot q\, \epsilon' \cdot q$, $\epsilon^* \cdot Q\, \epsilon' \cdot Q$, $\epsilon^* \cdot q\, \epsilon' \cdot Q$, $\epsilon^* \cdot Q\, \epsilon' \cdot q$. Each of these can be multiplied by a function of the Lorentz invariants formed from the momenta - say Q^2 and $P' \cdot q$ where P' is the ψ' momentum.

An equally valid approach is to consider the crossed reaction $\pi\psi' \to \pi\psi$. The independent helicity amplitudes are $\langle 1 | 1 \rangle$, $\langle 0 | 1 \rangle$, $\langle -1 | 1 \rangle$, $\langle 1 | 0 \rangle$, $\langle 0 | 0 \rangle$; the others are related by parity. Again the helicity amplitudes are functions of two Lorentz invariants.

A more useful decomposition is in terms of partial waves. For fixed $m_{\pi\pi}^2$ we consider the $\pi\pi$ system as a superposition of eigenstates of angular momentum (in the $\pi\pi$ rest frame) $\ell = 0, 2, 4 \ldots$. The decay angle of the $\pi\pi$ system - the angle between one pion and some specified axis in the $\pi\pi$ rest frame plays the role of the second variable. The partial wave expansion can be truncated after a few terms substantially reducing the difficulty of the analysis. There are a variety of coupling schemes available for connecting the $\pi\pi$ system of "spin" ℓ to the

ψ of spin 1 to produce a total angular momentum 1(=spin of ψ'). We may choose to diagonalize the helicities of the ψ and the π-π system in addition to J^2 and J_3 . On the other hand, we may diagonalize L^2, the orbital angular momentum squared of the ψ-$(\pi\pi)$ system and $\mathcal{S}^2$, the spin squared of the ψ and $(\pi\pi)$ systems in addition to J^2 and J_3.

If we diagonalize the helicities, λ_ψ and $\lambda_{\pi\pi}$, there are generally five amplitudes for each value of the π-π "spin", ℓ . These correspond to $(\lambda_\psi, \lambda_{\pi\pi}) = (1,2), (1,1), (1,0), (0,1)$, and $(0,0)$. All others are related by parity or disallowed by the requirement $|\lambda_\psi - \lambda_{\pi\pi}| \le 1$ = spin of ψ' . For $\ell = 0$(π-π s-wave) only $(\lambda_\psi, \lambda_{\pi\pi}) = (1,0)$ and $(0,0)$ are permitted.

If on the other hand, we diagonalize L^2 and $\mathcal{S}^2$ where $\underset{\sim}{\mathcal{S}} = \underset{\sim}{S}_\psi + \underset{\sim}{\ell}$, for fixed ℓ we of course have five amplitudes as well. Parity conservation requires $L + \ell$ = even , while charge conjugation invariance requires ℓ = even. Thus we have the allowed values $(L, \mathcal{S}) = (\ell, \ell-1)$, (ℓ, ℓ), $(\ell, \ell+1)$, $(\ell+2, \ell+1)$, and $(\ell-2, \ell-1)$. There are only two $\pi\pi$ s-wave amplitudes $L = 0$("relative s-wave"), $\mathcal{S} = 1$ and $L = 2$ ("relative d-wave"), $\mathcal{S} = 1$.

The purposes of this paper is to relate these various amplitudes to the experimental observables. Although it is possible to deal directly with the three-body decay,[4] it is more useful here to consider sequential two-body decays $\psi' \to \psi(\pi\pi)$; $\psi \to \mu^+\mu^-$, $(\pi\pi) \to \pi\pi$. Thus we shall always describe the $\mu^+\mu^-$ in the ψ rest frame, the π's in the π-π rest frame and the ψ and the $(\pi\pi)$ in the ψ' rest frame.

The plan of the paper is as follows. In Section II, decay amplitudes are calculated in terms of partial wave amplitudes. In the following section, some of the angular distributions are presented. Implications for π-π phase shifts are discussed in Sec. IV. Section V is a summary. In Appendix A the decomposition of two-body states into partial waves is reviewed. The full angular distribution including the three lowest partial waves is presented in Appendix B.

II. DECAY AMPLITUDES

For the purposes at hand, it is simpler to deal with decay amplitudes rather than their squares. Our analysis is in terms of partial waves. We denote the π-π angular momentum in its rest frame by $\underset{\sim}{\ell}$, the spin of the ψ by $\underset{\sim}{s}$, that of the ψ' by $\underset{\sim}{s}'$. The orbital angular momentum of the ψ-$(\pi\pi)$ system is denoted by $\underset{\sim}{L}$. Then if we define the channel spin, $\underset{\sim}{\mathscr{S}}$, by

$$\underset{\sim}{\mathscr{S}} = \underset{\sim}{s} + \underset{\sim}{\ell} \quad , \tag{1}$$

we have

$$\underset{\sim}{s}' = \underset{\sim}{\mathscr{S}} + \underset{\sim}{L} \quad . \tag{2}$$

As explained in the introduction, both ℓ and L are even.

An eigenstate of $J^2 = s'^2$, L^2, $\mathscr{S}^2$, and J_z consisting of $\psi\pi^+\pi^-$ may be constructed by the techniques reviewed in Appendix A. We find

$$|s' = 1, s_z', L, \mathcal{S}, \ell\rangle = \sum_{\ell_z, s_z, \mathcal{S}_z} |L, L_z; \ell, \ell_z; s, s_z\rangle$$

$$\langle 1 s_z' | L, L_z; \mathcal{S}, \mathcal{S}_z\rangle \langle \mathcal{S}, \mathcal{S}_z | 1, s_z; \ell, \ell_z\rangle \quad , \tag{3}$$

$$|L, L_z; \ell, \ell_z; s, s_z\rangle = \prod_{0, L_z}^{L, orb} |p\hat{z}, s_z; -p\hat{z}, \ell_z\rangle\rangle \quad , \tag{4}$$

in the notation of Appendix A. Here p is the momentum of the ψ in the ψ' rest frame for some fixed value of $m_{\pi\pi}^2$, the π-π invariant mass squared. We shall suppress indicating $m_{\pi\pi}^2$ as an independent variable for notational simplicity. The $\pi\pi$ state entering Eq. (4) is given by

$$|-p\hat{z}, \ell_z\rangle\rangle = e^{i\lambda K_z} \prod_{0, \ell_z}^{\ell, orb} |q\hat{z}, -q\hat{z}\rangle \tag{5}$$

where $p = m_{\pi\pi} \sinh \lambda$ and $q = \sqrt{\frac{m_{\pi\pi}^2}{4} - m_\pi^2}$. The operator $\prod_{0, \ell_z}^{\ell, orb}$ projects out an eigenstate state of ℓ^2 and ℓ_z as explained in Appendix A. The operator K_z generates velocity transformations in the z-direction.

The invariant amplitude for ψ' to decay into $\psi\pi^+\pi^-$ with $S_z(\psi') = s_z'$ and $S_z(\psi) = s_z$ is found from Eq. (A30).

$$\langle \psi, s_z, \pi^+\pi^-(\Omega_\psi, \Omega_\pi) | \psi', s_z'\rangle$$

$$= \sum_{\substack{\ell, L, \mathcal{S} \\ \ell_z, L_z, \mathcal{S}_z}} M_{\ell, L, \mathcal{S}} \sqrt{\frac{2L+1}{4\pi}} D^L_{L_z 0}(\Omega_\psi)^*$$

(cont.)

$$\times \sqrt{\frac{2\ell+1}{4\pi}}\, D^{\ell}_{\ell_z 0}(\Omega_\pi)^* \langle 1\, s_z' \,|\, L, L_z; \mathcal{S}, \mathcal{S}_z \rangle \langle \mathcal{S}, \mathcal{S}_z \,|\, 1, s_z; \ell, \ell_z \rangle \quad . \qquad (6)$$

Here Ω_ψ describes the ψ direction in the ψ' rest frame, and Ω_π describes the $\pi^\pm$ direction in the $\pi\pi$ rest frame.

The subsequent decay of $\psi \to \ell^+\ell^-$ is most easily described by fixing the lepton helicities since the QED coupling requires $(\lambda^+, \lambda^-) = (\pm\frac{1}{2}, \mp\frac{1}{2})$. Using Eq. (A27) and absorbing certain constants we have

$$\mathcal{M}(\lambda, s_z') = \langle \mu^+\mu^-, \lambda; \pi^+\pi^-(\Omega_\psi, \Omega_\pi, \Omega_\mu) \,|\, \psi', s_z' \rangle$$

$$= \sum_{\substack{\ell, L, \mathcal{S} \\ \ell_z, L_z, \mathcal{S}_z, s_z}} M_{\ell, L, \mathcal{S}} \sqrt{\frac{2L+1}{4\pi}}\, D^{L^*}_{L_z 0}(\Omega_\psi)$$

$$\sqrt{\frac{2\ell+1}{4\pi}}\, D^{\ell^*}_{\ell_z 0}(\Omega_\pi) \sqrt{\frac{3}{4\pi}}\, D^{1^*}_{s_z, \lambda}(\Omega_\mu)$$

$$\langle 1, s_z' \,|\, L, L_z; \mathcal{S}\mathcal{S}_z \rangle \langle \mathcal{S}, \mathcal{S}_z \,|\, 1, s_z; \ell, \ell_z \rangle \quad . \qquad (7)$$

Throughout, $\lambda = \lambda^+ - \lambda^-$ is the difference of the μ^+ and μ^- helicities.

In e^+e^- annihilation, the ψ''s are produced with transverse polarization with respect to the beam and the outgoing lepton polarizations are not observed. Thus the angular distribution is

$$d\Gamma \propto \Big[|\mathcal{M}(1,1)|^2 + |\mathcal{M}(1,-1)|^2 + |\mathcal{M}(-1,1)|^2 + |\mathcal{M}(-1,-1)|^2 \Big] \, . \qquad (8)$$

Since $|\mathcal{M}(1,1)| = |\mathcal{M}(-1,-1)|$ and $|\mathcal{M}(1,-1)| = |\mathcal{M}(-1,1)|$, we can write simply the first two terms on the right-hand side of Eq. (8). Including phase space we have

$$\frac{d\Gamma}{dm_{\pi\pi}\,d\Omega_{\psi}\,d\Omega_{\pi}\,d\Omega_{\mu}} \propto qp\left[|\mathcal{M}(1,1)|^2 + |\mathcal{M}(1,-1)|^2\right] \tag{9}$$

where q is the $\pi^{\pm}$ momentum in the π-π rest frame and p is the ψ momentum in the ψ' rest frame.

In practice, the partial wave series, Eq. (7), must be terminated after a few terms. We shall, for the purpose of demonstration, and with simplicity as a criterion consider only $M_{\ell,L,\mathcal{S}} = M_{001}$, M_{201} and M_{021}. Dropping an inessential constant we have

$$\mathcal{M}(1,1) = M_{001} D^{1*}_{11}(\Omega_{\mu})$$

$$+ M_{201}\left[\sqrt{3}\,D^{2*}_{20}(\Omega_{\pi})D^{1*}_{-1,1}(\Omega_{\mu}) - \sqrt{\tfrac{3}{2}}\,D^{2*}_{10}(\Omega_{\pi})D^{1*}_{0,1}(\Omega_{\mu}) + \sqrt{\tfrac{1}{2}}\,D^{2*}_{00}(\Omega_{\pi})D^{1*}_{1,1}(\Omega_{\mu})\right]$$

$$+ M_{021}\left[\sqrt{3}\,D^{2*}_{20}(\Omega_{\psi})D^{1*}_{-1,1}(\Omega_{\mu}) - \sqrt{\tfrac{3}{2}}\,D^{2*}_{10}(\Omega_{\psi})D^{1*}_{0,1}(\Omega_{\mu}) + \sqrt{\tfrac{1}{2}}\,D^{2*}_{00}(\Omega_{\psi})D^{1*}_{1,1}(\Omega_{\mu})\right],$$

(10)

$$\mathcal{M}(1,-1) = M_{001} D^{1\,*}_{-1,1}(\Omega_\mu)$$

$$+ M_{201}\left[\sqrt{3}\,D^{2\,*}_{-20}(\Omega_\pi) D^{1\,*}_{1,1}(\Omega_\mu) - \sqrt{\tfrac{3}{2}}\; D^{2\,*}_{-10}(\Omega_\pi) D^{1\,*}_{0,1}(\Omega_\mu) + \sqrt{\tfrac{1}{2}}\, D^{2\,*}_{00}(\Omega_\pi) D^{1\,*}_{-1,1}(\Omega_\mu)\right]$$

$$+ M_{021}\left[\sqrt{3}\,D^{2\,*}_{-20}(\Omega_\psi) D^{1\,*}_{1,1}(\Omega_\mu) - \sqrt{\tfrac{3}{2}}\; D^{2\,*}_{-1,0}(\Omega_\psi) D^{1\,*}_{0,1}(\Omega_\mu) + \sqrt{\tfrac{1}{2}}\, D^{2\,*}_{00}(\Omega_\psi) D^{1\,*}_{-1,1}(\Omega_\mu)\right] . \tag{11}$$

Here $D^j_{m'm}(\Omega) = D^j_{m'm}(\phi,\theta,0)$ are the usual representation functions for the rotation group. Angular distributions, including only these three partial waves, are obtained from Eqs. (9)-(11).

III. ANGULAR DISTRIBUTIONS

The distribution as a function of Ω_ψ, Ω_π, or Ω_μ is obtained simply by integrating over the two other solid angles using the orthogonality relation, Eq. (A3). From Eqs. (9)-(11) we find[5]

$$\frac{d\Gamma}{d\Omega_\psi} \propto \Big[|M_{001}|^2 + |M_{201}|^2 + \tfrac{1}{4}|M_{021}|^2 (5-3\cos^2\theta_\psi)\Big] , \tag{12}$$

$$\frac{d\Gamma}{d\Omega_\pi} \propto \Big[|M_{001}|^2 + \tfrac{1}{4}|M_{201}|^2 (5-3\cos^2\theta_\pi) + |M_{021}|^2 \Big] , \tag{13}$$

$$\frac{d\Gamma}{d\Omega_\mu} \propto |M_{001}|^2(1+\cos^2\theta_\mu)$$

$$+\frac{1}{10}(|M_{201}|^2+|M_{021}|^2)(13+\cos^2\theta_\mu) \quad . \tag{14}$$

It is understood here that $M_{\ell, L, \mathcal{S}}$ is a function of $m_{\pi\pi}$.

If the μ's are not observed, the distribution is

$$\frac{d\Gamma}{d\Omega_\pi d\Omega_\psi} \propto |M_{001}|^2 + |M_{201}|^2\left(\frac{5}{4}-\frac{3}{4}\cos^2\theta_\pi\right)$$

$$+|M_{021}|^2\left(\frac{5}{4}-\frac{3}{4}\cos^2\theta_\psi\right)$$

$$+2\mathrm{Re}\, M_{201}M_{001}^*\left[\frac{1}{\sqrt{2}}\left(\frac{3}{2}\cos^2\theta_\pi-\frac{1}{2}\right)\right]$$

$$+2\mathrm{Re}\, M_{021}M_{001}^*\left[\frac{1}{\sqrt{2}}\left(\frac{3}{2}\cos^2\theta_\psi-\frac{1}{2}\right)\right]$$

$$+2\mathrm{Re}\, M_{201}M_{021}^*\left[\frac{9}{8}\sin^2\theta_\pi\sin^2\theta_\psi\cos 2(\phi_\mu-\phi_\psi)\right.$$

$$+\frac{9}{16}\sin 2\theta_\pi\sin 2\theta_\psi\cos(\phi_\pi-\phi_\psi)$$

$$\left.+\frac{1}{2}\left(\frac{3}{2}\cos^2\theta_\pi-\frac{1}{2}\right)\left(\frac{3}{2}\cos^2\theta_\psi-\frac{1}{2}\right)\right] \quad . \tag{15}$$

The full decay distribution, $\frac{d\Gamma}{d\Omega_\pi d\Omega_\psi d\Omega_\mu}$, is presented in Appendix B.

The single particle distributions, Eqs. (12)-(14) depend only on the magnitudes of M_{001}, M_{201} and M_{021}. Joint distributions such as Eq. (15) depend on the relative phases of the amplitudes.

IV. PI-PI PHASE SHIFTS

The angular distributions in principle determine the phases of the partial wave amplitudes - up to one overall phase. If the ψ' and ψ are regarded as inert, then the usual final state interaction argument requires $M_{\ell,L,\mathscr{S}} = e^{i\delta_\ell^0(m_{\pi\pi})} |M_{\ell,L,\mathscr{S}}|$ where δ_ℓ^0 is the $I = 0$, ℓ-wave $\pi\pi$ phase shift. Thus $\delta_0^0 - \delta_2^0$ may be obtainable from $\psi' \to \psi\pi\pi$. This is in some ways similar to the Pais-Treiman[6] method for obtaining $\pi\pi$ phase shifts from K_{e4} decays. The viability of this technique in $\psi' \to \psi\pi\pi$ depends on a number of factors:

1. Adequate data, especially for $\psi' \to \mu^+\mu^-\pi^+\pi^-$.
2. That some $\ell > 0$ contributions be significant.
3. That a few terms in the partial wave series suffice.
4. That the assumption of noninteracting ψ and ψ' be appropriate.

In principle similar techniques can be used for $\psi \to \omega\pi\pi$. The analysis is slightly different reflecting the replacement of $\psi \to \mu^+\mu^-$ by $\omega \to \pi^+\pi^-\pi^0$. However, item 4 above seems more dubious in this instance. The same is true for the SU(3) variants, e.g., $\psi \to K\overline{K}^*\pi$, $\psi \to \omega K\overline{K}$, etc.

V. SUMMARY

The results presented above and in Appendix B constitute a general treatment of the process $e^+e^- \to V' \to VP\overline{P} \to \ell^+\ell^- P\overline{P}$, although we have been primarily concerned with $\psi' \to \psi\pi\pi$. In advance of data analysis we cannot determine how many partial waves will be needed for an accurate description, but it is expected that $\psi' \to \psi\pi\pi$ will require fewer than would $\psi \to \omega\pi\pi$.

The determination of the partial wave amplitudes for $\psi' \to \psi\pi\pi$ is essential for evaluating the treatment[3] by Brown and the present author of the spectrum $d\Gamma/dm_{\pi\pi}$, which speculated that only M_{001} is significant[7] and that its dependence on $m_{\pi\pi}$ is $\propto (m_{\pi\pi}^2 - 2m_\pi^2)$. The isolation of the partial wave amplitudes would constitute some of the most refined information on the puzzling new particles.

The obverse of the investigation of new particle properties is the opportunity to measure one of the "simplest" of hadronic quantities, the elastic π-π s-wave phase shift (for I = 0) for $m_{\pi\pi} \sim 500$ MeV. If there is adequate π-π d-wave to interfere with, this will be a valuable technique, perhaps supplanting K_{e4} decays as the best clean measurement of π-π phase shifts at low values of the π-π energy.

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APPENDIX A

We review here material which is well-known[8] for the purpose of completeness and to establish our conventions. The significance of the orbital angular momentum, $\underset{\sim}{L}$, and spin angular momentum $\underset{\sim}{S}$, is stressed. Of course, the heart of the matter is the rotation group (and its covering group, SU(2)).

An element of SU(2) may be specified by three Euler angles, $0 \le \alpha \le 2\pi$, $0 \le \beta \le \pi$, $0 \le \gamma \le 4\pi$. We adopt the symbolic notation $\underset{\sim}{u}$ to indicate a set of such parameters, and $e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}}$ to indicate the associated rotation, $e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$. The group integration element is $d\underset{\sim}{u} = d\alpha\, d\cos\beta\, d\gamma/(16\pi^2)$ so $\int d\underset{\sim}{u} = 1$.

The standard representation functions have the properties

$$D^j_{m'm}(\underset{\sim}{u}) = \langle jm' |\, e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}}\, | jm\rangle = e^{-i\alpha m'} d^j_{m'm}(\beta) e^{-i\gamma m} \quad , \tag{A1}$$

$$\sum_{m'} D^j_{mm'}(\underset{\sim}{u}_1) D^j_{m'm''}(\underset{\sim}{u}_2) = D^j_{mm''}(\underset{\sim}{u}_1\underset{\sim}{u}_2) \quad , \tag{A2}$$

$$\int d\underset{\sim}{u}\; D^j_{m_2m_1}(\underset{\sim}{u})\, D^{j'}_{m_1'm_2'}(\underset{\sim}{u}^{-1}) = \frac{\delta_{jj'}\,\delta_{m_1m_1'}\,\delta_{m_2m_2'}}{(2j+1)} \quad , \tag{A3}$$

$$\delta_{\underset{\sim}{u}}(\underset{\sim}{u}^{-1}\underset{\sim}{u}_0) = \sum_{j,m,m'} (2j+1) D^j_{m'm}(\underset{\sim}{u}^{-1}) D^j_{mm'}(\underset{\sim}{u}_0) \quad , \tag{A4}$$

$$D^{j_1}_{m_1'm_1}(\underset{\sim}{u}) D^{j_2}_{m_2'm_2''}(\underset{\sim}{u}) = \sum_{j,m',m''} D^j_{m'm''}(\underset{\sim}{u}) \langle j,m' |\, j_1,m_1';j_2,m_2'\rangle \langle j,m'' |\, j_1,m_1'';j_2,m_2''\rangle \quad . \tag{A5}$$

In Eq. (A4) $\delta\, \underset{\sim}{u}(\underset{\sim}{u}^{-1}\underset{\sim}{u}_0)$ has the meaning $\int d\underset{\sim}{u}\, \delta\, \underset{\sim}{u}(\underset{\sim}{u}^{-1}\underset{\sim}{u}_0) f(\underset{\sim}{u}) = f(\underset{\sim}{u}_0)$. In Eq. (A5), $\langle j, m | j_1, m_1; j_2, m_2\rangle$ is the usual "Clebsch-Gordan" coefficient.

We suppose we have an irreducible unitary representation of the Poincaré group associated with a particle of mass m and spin s. The homogeneous transformations are generated by $\underset{\sim}{J}$ and $\underset{\sim}{K}$ ("pure boosts"). Translations are generated by P^{μ}.

We denote rest states by $|p_0, m'\rangle$ where $p_0 = (m, 0, 0, 0)$ and $J_z|p_0, m'\rangle = m'|p_0, m'\rangle$. It follows that for $\underset{\sim}{u}$, a rotation,

$$\underset{\sim}{u}\, |p_0, m'\rangle = \sum_{m''} D^{s}_{m''m'}(\underset{\sim}{u})\, |p_0, m''\rangle \quad . \tag{A6}$$

We next define two bases for our space, using improper vectors as this is most convenient. The first is the celebrated helicity representation. If θ and ϕ are the polar angles of $\underset{\sim}{p}$, we set

$$h_{\underset{\sim}{p}} = e^{-i\phi J_z}\, e^{-i\theta J_y}\, e^{-i\lambda K_z} \tag{A7}$$

where $m \cosh\lambda = p^0 = \sqrt{m^2 + \underset{\sim}{p}^2}$. For $\underset{\sim}{p} = |\underset{\sim}{p}|\,\hat{z}$, $h_{\underset{\sim}{p}} = e^{-i\lambda K_z}$; for $\underset{\sim}{p} = -|\underset{\sim}{p}|\,\hat{z}$, $h_{\underset{\sim}{p}} = e^{i\pi s}\, e^{-i\pi J_z}\, e^{-i\pi J_y}\, e^{-i\lambda K_z}$. We define the basis states

$$|\underset{\sim}{p}, \lambda\rangle = h_{\underset{\sim}{p}}\, |p_0, \lambda\rangle \quad . \tag{A8}$$

These have the well-known property

$$J\cdot\underset{\sim}{P}\, |\underset{\sim}{p}, \lambda\rangle = \lambda\, |\underset{\sim}{p}|\, |\underset{\sim}{p}, \lambda\rangle \quad . \tag{A9}$$

The second basis[9] is defined with the z-component of spin in mind, rather than helicity. We set

$$s_{\underset{\sim}{p}} = e^{-i\underset{\sim}{\lambda}\cdot\underset{\sim}{K}} \tag{A10}$$

$$|\underset{\sim}{p}, s_z \gg \; = \; s_{\underset{\sim}{p}}\, |p_0, s_z > \tag{A11}$$

where $\underset{\sim}{\lambda}$ is parallel to $\underset{\sim}{p}$ and $m \cosh|\underset{\sim}{\lambda}| = p^0$. With the conventions $J^{12} = (\underset{\sim}{J})_3$, $J^{01} = (\underset{\sim}{K})_1$, etc., $J^{\mu\nu} = -J^{\nu\mu}$, $\epsilon_{0123} = +1$ we set

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma} P^\nu J^{\lambda\sigma} \ . \tag{A12}$$

From Eq. (A12) it follows that

$$W_0 = \underset{\sim}{P}\cdot\underset{\sim}{J} \ , \tag{A13}$$

$$\underset{\sim}{W} = P^0\underset{\sim}{J} - \underset{\sim}{P}\times\underset{\sim}{K} \ , \tag{A14}$$

The vector W_μ gives rise to the Casimir operator $W_\mu W^\mu = -m^2 s(s+1)$. It is well-known that the spin operator

$$\underset{\sim}{S} = \frac{1}{m}\left(\underset{\sim}{W} - \frac{\underset{\sim}{P}W^0}{P^0+m}\right) \tag{A15}$$

satisfies SU(2) commutation relations: $\left[S_i, S_j\right] = i\,\epsilon_{ijk} S_k$. From (A15) we see $\underset{\sim}{S}$ is a vector, so $\left[J_i, S_j\right] = i\,\epsilon_{ijk} S_k$. This implies that $\underset{\sim}{L} = \underset{\sim}{J} - \underset{\sim}{S}$ commutes with $\underset{\sim}{S}$. It is easy to see that

$$\begin{aligned} \underset{\sim}{S}|\underset{\sim}{p}, s_z \gg \; &= \; s_{\underset{\sim}{p}} \frac{1}{m}\underset{\sim}{W}\,|p_0, s_z > \\ &= \; s_{\underset{\sim}{p}}\, \underset{\sim}{J}\,|\, p_0, s_z > \ . \end{aligned} \tag{A16}$$

In particular,

$$S_z \,|\, \underset{\sim}{p}, s_z \gg \; = s_z \,|\, \underset{\sim}{p}, s_z \gg \tag{A17}$$

and

$$e^{-i\underset{\sim}{u}\cdot\underset{\sim}{S}} \,|\, \underset{\sim}{p}, s_z \gg \; = \sum_{s_z'} D^s_{s_z' s_z}(\underset{\sim}{u}) \,|\, \underset{\sim}{p}, s_z' \gg \quad . \tag{A18}$$

On the other hand,

$$\begin{aligned} e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}} \,|\, \underset{\sim}{p}, s_z \gg \; &= e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}} e^{i\underset{\sim}{\lambda}\cdot\underset{\sim}{K}} \,|\, p_0, s_z > \\ &= e^{-i\underset{\sim}{\lambda}'\cdot\underset{\sim}{K}} e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}} \,|\, p_0, s_z > \\ &= \sum D^s_{s_z' s_z}(\underset{\sim}{u}) \,|\, \underset{\sim}{p}', s_z' \gg \end{aligned} \tag{A19}$$

where $\underset{\sim}{\lambda}'$ and $\underset{\sim}{p}'$ are $\underset{\sim}{\lambda}$ and $\underset{\sim}{p}$ rotated by $\underset{\sim}{u}$. Thus we infer that

$$\begin{aligned} e^{-i\underset{\sim}{u}\cdot\underset{\sim}{L}} \,|\, \underset{\sim}{p}, s_z \gg \; &= e^{+i\underset{\sim}{u}\cdot\underset{\sim}{S}} e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}} \,|\, p, s_z \gg \\ &= \sum D^s_{s_z'' s_z'}(\underset{\sim}{u}^{-1}) D^s_{s_z' s_z}(\underset{\sim}{u}) \,|\, \underset{\sim}{p}', s_z'' \gg \\ &= |\, \underset{\sim}{p}', s_z \gg \quad . \end{aligned} \tag{A20}$$

The transformations $e^{-i\underset{\sim}{u}\cdot\underset{\sim}{S}}$ and $e^{-i\underset{\sim}{u}\cdot\underset{\sim}{L}}$ are not Lorentz transformations. However their actions are well-defined through Eq. (A15). On an eigenstate of P^μ, $e^{-i\underset{\sim}{u}\cdot\underset{\sim}{S}}$ acts in the same way as a Lorentz transformation, but which particular transformation depends on both $\underset{\sim}{u}$ and $\underset{\sim}{p}$. The operators $\underset{\sim}{L}$ and $\underset{\sim}{S}$ do generate independent SU(2)'s which are realized on the single particle states.

The discussion of two particle states is facilitated by the introduction of projection operators:

$$\Pi^{j}_{m',m} = \sqrt{(2j+1)4\pi}\int d\underset{\sim}{u}\, D^{j}_{m',m}(\underset{\sim}{u}^{-1}) \quad e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}} \ . \tag{A22}$$

It follows that

$$e^{-i\underset{\sim}{u}\cdot\underset{\sim}{J}}\Pi^{j}_{m',m} = \sum_{m''} D^{j}_{m'',m}(\underset{\sim}{u})\Pi^{j}_{m',m''} \ , \tag{A23}$$

and

$$\Pi^{j}_{m',m} \quad e^{-i\alpha J_z} = e^{-i\alpha m'}\Pi^{j}_{m',m} \ . \tag{A24}$$

This means $\Pi^{j}_{m',m}$ produces states which transform as $|jm\rangle$ and annihilates $|jm''\rangle$ unless $m'' = m'$. We define

$$|jm,\lambda_1\lambda_2(W)\rangle = \Pi^{j}_{\lambda_1-\lambda_2,m}|p\hat{z},\lambda_1;-p\hat{z},\lambda_2\rangle \ , \tag{A25}$$

where $W = (\underset{\sim}{p}^2+m_1^{\ 2})^{\frac{1}{2}} + (\underset{\sim}{p}^2+m_2^{\ 2})^{\frac{1}{2}}$. The subscript $\lambda_1-\lambda_2$ is required by $J_z|p\hat{z},\lambda_1;-p\hat{z},\lambda_2\rangle = (\lambda_1-\lambda_2)|p\hat{z},\lambda_1;-p\hat{z},\lambda_2\rangle$.

Our single particle states have the covariant normalization,

$$\langle\underset{\sim}{p}',\lambda'|\underset{\sim}{p},\lambda\rangle = (2E)(2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{p}')\delta_{\lambda\lambda'} \ .$$

The matrix element between a plane wave rotated by $\underset{\sim}{u}$ from the z-orientation and a spherical wave is

$$\langle p'\hat{z},\lambda_1';-p'\hat{z},\lambda_2'| \quad e^{+i\underset{\sim}{u}\cdot\underset{\sim}{J}}\ |jm\lambda_1\lambda_2(W)\rangle$$

$$= \langle p'\hat{z},\lambda_1';-p'\hat{z},\lambda'_2|\sum_{m''} D^{j}_{m'',m}(\underset{\sim}{u}^{-1})\Pi^{j}_{\lambda_1-\lambda_2,m''}$$

$$|p\hat{z},\lambda_1;-p\hat{z},\lambda_2\rangle$$

(cont.)

$$= \sum_{m''} \int d\underset{\sim}{u}' \sqrt{(2j+1)4\pi}\, D^j_{m'',m}(\underset{\sim}{u}^{-1}) D^j_{\lambda_1-\lambda_2,m''}(\underset{\sim}{u}'^{-1})$$

$$\langle p'\hat{z}, \lambda_1'; -p'\hat{z}, \lambda_2'| \quad e^{-i\underset{\sim}{u}'\cdot \underset{\sim}{J}} |p\hat{z}, \lambda_1; -p\hat{z}, \lambda_2\rangle$$

$$= \sqrt{\frac{2j+1}{4\pi}}\; D^j_{\lambda_1-\lambda_2,m}(\underset{\sim}{u}^{-1}) \delta_{\lambda_1 \lambda_1}\delta_{\lambda_2\lambda_2}\left[\frac{2p^0(2\pi)^3 \delta(p-p)}{pp'}\right] . \quad \text{(A27)}$$

Eigenstates of L^2 and L_z are constructed with the projection operator formed from $\underset{\sim}{L} = \underset{\sim}{J} - \underset{\sim}{S}_1 - \underset{\sim}{S}_2$

$$\prod^{\ell,\mathrm{orb}}_{m',m} = \sqrt{(2\ell+1)4\pi} \int d\underset{\sim}{u}\, D^j_{m'm}(\underset{\sim}{u}^{-1}) \quad e^{-i\underset{\sim}{u}\cdot\underset{\sim}{L}} . \quad \text{(A28)}$$

We define

$$|\ell\, \ell_z s_{1z} s_{2z}(W)\rangle = \prod^{\ell,\mathrm{orb}}_{0,\ell_z} |p\hat{z}, s_{1z}; -p\hat{z}, s_{2z}\rangle\rangle . \quad \text{(A29)}$$

The matrix element between $e^{-i\underset{\sim}{u}\cdot\underset{\sim}{L}} |p'\hat{z}, s_{1z}', -p\hat{z}, s_{2z}'\rangle$ and a spherical wave is easily calculated:

$$\langle\langle p'\hat{z}, s_{1z}'; -p'\hat{z}, s_{2z}'| \quad e^{+i\underset{\sim}{u}\cdot\underset{\sim}{L}} |\ell\, \ell_z s_{1z} s_{2z}(W)\rangle\rangle$$

$$= \langle\langle p'\hat{z}, s_{1z}'; -p'\hat{z}, s_{2z}'| \sum_{m''} D^{\ell}_{m'',\ell_z}(\underset{\sim}{u}^{-1})$$

$$\sqrt{(2\ell+1)4\pi} \int du' D^{\ell}_{0,m''}(\underset{\sim}{u}'^{-1}) \quad e^{-i\underset{\sim}{u}'\cdot\underset{\sim}{L}} |pz, s_{1z}, -pz, s_{2z}\rangle\rangle$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\; D^{\ell}_{0,\ell_z}(\underset{\sim}{u}^{-1}) \delta_{s_{1z}s_{1z}'} \delta_{s_{2z}s_{2z}'}$$

$$\left[2p^0(2\pi)^3 \frac{\delta(p'-p)}{pp'}\right] . \quad \text{(A30)}$$

Since $\underset{\sim}{S} = \underset{\sim}{S}_1 + \underset{\sim}{S}_2$ and $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$, we can use the usual rules for addition of angular momentum to define

$$| jm\ell\, s(W)\gg = \sum_{\substack{\ell_z, s_z \\ s_{1z}, s_{2z}}} |\ell\, \ell_z, s_{1z}, s_{2z}(W)>$$

$$<j, m|\ell, \ell_z; s, s_z> <s, s_z| s_1, s_{1z}; s_2, s_{2z}> \quad . \tag{A31}$$

This is an eigenstate of J^2, L^2, S^2 and J_z. The transformation between $|jm\ell\, s(W)>$ and $|jm\lambda_1\lambda_2(W)>$ is given by

$$<jm'\lambda_1'\lambda_2'(W')|jm\ell\, s(W)>$$

$$= \delta_{mm'}\delta_{jj'}\sqrt{\frac{2\ell+1}{2j+1}} <s, \lambda_1'-\lambda_2'| s_1, \lambda_1'; s_2, -\lambda_2'>$$

$$<j, \lambda_1'-\lambda_2'|\ell, 0; s, \lambda_1'-\lambda_2'>\left[2p^0(2\pi)^3 \frac{\delta(p'-p)}{p'p'}\right] \quad . \tag{A32}$$

On the other hand, the orthogonality relation for $<jm\lambda_1\lambda_2(W)|$ reads

$$<jm'\lambda_1'\lambda_2'(W')|jm\lambda_1\lambda_2(W)> = \delta_{jj'}\delta_{mm'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\left[2p^0(2\pi)^3 \frac{\delta(p'-p)}{pp}\right] . \tag{A53}$$

As a final note on our conventions, the definitions of $h_{\underset{\sim}{p}}$ [Eq. (A7)] and $s_{\underset{\sim}{p}}$ [Eq. (A10)] define the coordinate systems with respect to which directions are measured. For example in Eq. (A27) the coordinate system with respect to which particle 1's helicity (and possible subsequent decay) is to be measured is obtained from the initial frame by the Lorentz transformation $h_{\underset{\sim}{p}_1}$ so the associated x-axis lies in the plane containing initial quantization axis and the direction of motion. For states $|\underset{\sim}{p}, s_z\gg$, the reference frame is obtained from the initial frame by $s_{\underset{\sim}{p}}$, the pure boost, Eq. (A10).

APPENDIX B

We display here the full spectrum obtained from Eqs. (9)-(11) which use only the first three partial waves $M_{\ell, L, \mathscr{S}}$.

$$\frac{d\Gamma}{d\Omega_\pi d\Omega_\mu d\Omega_\psi} \propto |M_{001}|^2\left[d^1_{11}(\theta_\mu)^2 + d^1_{-1,1}(\theta_\mu)^2\right]$$

$$+ |M_{201}|^2\Big[3d^2_{20}(\theta_\pi)^2\left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2\right) + 3d^2_{10}(\theta_\pi)^2 d^1_{01}(\theta_\mu)^2$$

$$+ \tfrac{1}{2} d^2_{00}(\theta_\pi)^2\left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2\right)$$

$$+ 2\sqrt{6}\, d^2_{20}(\theta_\pi) d^2_{00}(\theta_\pi) d^1_{1-1}(\theta_\mu) d^1_{11}(\theta_\mu)\cos 2(\phi_\pi - \phi_\mu)$$

$$+\left(-\sqrt{3}\, d^2_{00}(\theta_\pi) + 3\sqrt{2}\, d^2_{20}(\theta_\pi)\right) d^2_{10}(\theta_\pi) d^1_{01}(\theta_\mu)\left(d^1_{11}(\theta_\mu) - d^1_{-11}(\theta_\mu)\right)\cos(\phi_\pi - \phi_\mu)\Big]$$

$$+ |M_{021}|^2\Big[3d^2_{20}(\theta_\psi)^2\left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2\right) + 3d^2_{10}(\theta_\psi)^2 d^1_{01}(\theta_\mu)^2$$

$$+ \tfrac{1}{2} d^2_{00}(\theta_\psi)^2\left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2\right)$$

$$+ 2\sqrt{6}\, d^2_{20}(\theta_\psi) d^2_{00}(\theta_\psi) d^1_{1-1}(\theta_\mu) d^1_{11}(\theta_\mu)\cos 2(\phi_\psi - \phi_\mu)$$

$$+\left(-\sqrt{3}\, d^2_{00}(\theta_\psi) + 3\sqrt{2}\, d^2_{20}(\theta_\psi)\right) d^2_{10}(\theta_\psi) d^1_{01}(\theta_\mu)\left(d^1_{11}(\theta_\mu) - d^1_{-11}(\theta_\mu)\right)\cos(\phi_\psi - \phi_\mu)\Big]$$

$$+ 2\mathrm{Re}\, M_{201} M_{001}^*\Big[2\sqrt{3}\, d^2_{20}(\theta_\pi) d^1_{11}(\theta_\mu) d^1_{-11}(\theta_\mu) \quad \cos 2(\phi_\pi - \phi_\mu)$$

$$- \sqrt{\frac{3}{2}}\, d^2_{10}(\theta_\pi) d^1_{01}(\theta_\mu)\left(e^{i(\phi_\pi - \phi_\mu)} d^1_{11}(\theta_\mu) - e^{-(\phi_\pi - \phi_\mu)} d^1_{-11}(\theta_\mu)\right)$$

$$+ \sqrt{\tfrac{1}{2}}\, d^2_{00}(\theta_\pi)\left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2\right)\Big]$$

(cont.)

$$
\begin{aligned}
&+ 2\mathrm{Re}\, M_{021} M_{001}^{*} \Big[2\sqrt{3}\, d^2_{20}(\theta_\psi) d^1_{11}(\theta_\mu) d^1_{-11}(\theta_\mu) \cos 2(\phi_\psi-\phi_\mu) \\
&\quad - \sqrt{\tfrac{3}{2}}\, d^2_{10}(\theta_\psi) d^1_{01}(\theta_\mu) \left(e^{i(\phi_\psi-\phi_\mu)} d^1_{11}(\theta_\mu) - e^{-i(\phi_\psi-\phi_\mu)} d^1_{-11}(\theta_\mu) \right) \\
&\quad + \sqrt{\tfrac{1}{2}}\, d^2_{00}(\theta_\psi) \left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2 \right) \Big] \\
&+ 2\mathrm{Re}\, M_{201} M_{021}^{*} \Big[3 d^2_{20}(\theta_\pi) d^2_{20}(\theta_\psi) \left(e^{-2i(\phi_\pi-\phi_\psi)} d^1_{11}(\theta_\mu)^2 + e^{2i(\phi_\pi-\phi_\psi)} d^1_{-11}(\theta_\mu)^2 \right) \\
&\quad + 3\, d^2_{10}(\theta_\pi) d^2_{10}(\theta_\psi) \cos(\phi_\pi-\phi_\psi) d^1_{01}(\theta_\mu)^2 \\
&\quad + \tfrac{1}{2}\, d^2_{00}(\theta_\pi) d^2_{00}(\theta_\psi) \left(d^1_{11}(\theta_\mu)^2 + d^1_{-11}(\theta_\mu)^2 \right) \\
&\quad - \frac{3}{\sqrt{2}}\, d^2_{20}(\theta_\pi) d^2_{10}(\theta_\psi) d^1_{01}(\theta_\mu) \left(e^{i(2\phi_\pi-\phi_\psi-\phi_\mu)} d^1_{-11}(\theta_\mu) - e^{-i(2\phi_\pi-\phi_\psi-\phi_\pi)} d^1_{11}(\theta_\mu) \right) \\
&\quad - \frac{3}{\sqrt{2}}\, d^2_{10}(\theta_\pi) d^2_{20}(\theta_\psi) d^1_{01}(\theta_\mu) \left(e^{i(\phi_\pi+\phi_\mu-2\phi_\psi)} d^1_{-11}(\theta_\mu) - e^{-i(\phi_\pi+\phi_\mu-2\phi_\psi)} d^1_{11}(\theta_\mu) \right) \\
&\quad - \frac{\sqrt{3}}{2}\, d^2_{10}(\theta_\pi) d^2_{00}(\theta_\psi) d^1_{01}(\theta_\mu) \left(e^{i(\phi_\pi-\phi_\mu)} d^1_{11}(\theta_\mu) - e^{-i(\phi_\pi-\phi_\mu)} d^1_{-11}(\theta_\mu) \right) \\
&\quad - \frac{\sqrt{3}}{2}\, d^2_{00}(\theta_\pi) d^2_{10}(\theta_\psi) d^1_{01}(\theta_\mu) \left(e^{i(\phi_\mu-\phi_\psi)} d^1_{11}(\theta_\mu) - e^{-i(\phi_\mu-\phi_\psi)} d^1_{-11}(\theta_\mu) \right) \\
&\quad + \sqrt{6}\, d^2_{20}(\theta_\pi) d^2_{00}(\theta_\psi) d^1_{11}(\theta_\mu) d^1_{-11}(\theta_\mu) \cos 2(\phi_\pi-\phi_\mu) \\
&\quad + \sqrt{6}\, d^2_{00}(\theta_\pi) d^2_{20}(\theta_\psi) d^1_{11}(\theta_\mu) d^1_{-11}(\theta_\mu) \cos 2(\phi_\psi-\phi_\mu) \Big] .
\end{aligned}
\tag{B1}
$$

The frames with respect to which angles are to be measured are discussed at the end of Appendix A.

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