



Renormalization of Spinor and Scalar Electrodynamics
With Bilinear Gauge Conditions

SATISH D. JOGLEKAR

Fermi National Accelerator Laboratory, Batavia, Illinois 60510
and
State University of New York at Stony Brook, N. Y. 11790

ABSTRACT

We apply the recently developed technique of Ward Takahashi identities for proper vertices in gauge theories to the problem of renormalization of electrodynamics - as a simple example of a gauge theory - when the gauge condition chosen is bilinear in fields. We show that spinor electrodynamics is renormalizable when the gauge condition is $f(A) \equiv \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2) = 0$ where ξ and α are real and arbitrary, and parameter ξ is renormalized independently. We also show that scalar electrodynamics is renormalizable with the gauge condition $f[A] \equiv \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2 - \frac{1}{2} \eta \phi^* \phi) = 0$ where ξ and η are real and arbitrary. ξ and η must be renormalized independently.



I. INTRODUCTION

Renormalization of gauge theories (broken and spontaneously broken) has been discussed at length over past few years. The earlier discussions on the renormalization of gauge theories have been based on the Ward-Takahashi (WT) identities for Green's functions.¹ Recently, renormalization of gauge theories has been discussed using the Ward-Takahashi identity for $\Gamma[\Phi]$, the generating functional of the one particle irreducible (proper) vertices.² Since renormalization procedure is stated in terms of proper vertices, use of the Ward-Takahashi identity for $\Gamma[\Phi]$ simplifies the discussion of renormalizability greatly. Above discussions on renormalizability of gauge theories have been carried out in which the gauge conditions chosen to quantize the theory are linear in fields. It is of some interest to see whether the proof of renormalizability goes through when the gauge condition chosen is bilinear in fields.⁵ (That is how far one can go if the gauge term is not to exceed dimensions four.)

Here, we apply the method of Ref. 2, viz. the Ward-Takahashi identity for $\Gamma[\Phi]$ in order to carry out the renormalization of the simplest possible gauge theory. We work out the renormalization of electromagnetic field interacting with a Dirac field or a complex scalar field. It is hoped that this exercise will help understand the renormalization of more complicated (e.g., non-abelian) gauge theories in bilinear gauge conditions.

In Sec. II, we begin considering the Lagrangian for free

electromagnetic field with the gauge condition $f[A] \equiv \frac{1}{\sqrt{\alpha}} \left[\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A_{\mu} A^{\mu} \right] = 0$, with ξ and α as free parameters. Though the theory is trivial from the point of view its physical content (S-matrix), it is non-trivial from the point of view of renormalization. In fact the renormalization of free electromagnetic field makes it considerably simpler to treat the interacting cases in this type of gauge. We note that, in this gauge ($\xi \neq 0$), there are $(A_{\mu})^3$, $(A_{\mu})^4$ and $(\bar{c}c A_{\mu})$ vertices. (\bar{c}, c are the Faddeev-Popov ghost fields.³⁾ We obtain the Ward-Takahashi identity for proper vertices. We use the dimensional regularization. We analyze the divergences in $G^{-1}[\Phi]$ (the generating functional of proper vertices with two external ghosts.) and in $\mathcal{F}_{\alpha}[\Phi]$ which is essentially the expectation value of the gauge functional in presence of external sources. Using WT identity for $\Gamma[\Phi]$, we obtain relations among the divergences in $\Gamma[\Phi]$, $G^{-1}[\Phi]$ and $\mathcal{F}_{\alpha}[\Phi]$ and show by an inductive proof that they can be removed by multiplicative renormalization on fields and parameters α and ξ . (We shall not state any specific renormalization conditions which determine the finite parts of renormalization constants.)

In Sec. III, we give results of one loop calculation to carry out the renormalization program of Sec. II and verify the relations among the divergences obtained there.

In Sec. IV, we show that the 4-photon S-matrix amplitude vanishes in this gauge, as it should.

In Sec. V, we consider spinor electrodynamics. The extension from non-interacting case is, more or less, straightforward. We prove the renormalizability of spinor electrodynamics and obtain the usual Ward identity between the renormalizations of electron photon vertex and electron propagator.

In Sec. VI, we consider scalar electrodynamics. We find that in the gauge $f[A] \equiv \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2) = 0$, we cannot make proper vertices finite by multiplicative renormalizations on α and ξ (and fields, etc.). This is essentially because, in this case, $\mathcal{F}_{\alpha}[\Phi]$ is such that its derivatives cannot be made finite to all orders by multiplicative renormalizations on α and ξ (and fields, etc.). We however find that if we choose the gauge condition $f[A] \equiv \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2 - \frac{1}{2} \eta \phi^{*} \phi) = 0$, and renormalize α , ξ and η independently, the renormalization program goes through. This is explained in Sec. VI.

It is found that renormalization of ξ (or ξ and η) is (are) independent of those of other parameters and fields. From practical point of view, such gauges would have been more useful were the renormalizations on ξ and η dependent on other renormalization constants, for then, certain simplifications in the effective action could be made and maintained to all orders.

(II-4) Preliminary

In the following, we consider the Lagrangian for the electromagnetic field $A_\mu(x)$,

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu .$$

\mathcal{L}_0 is invariant under a local gauge transformation,

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \omega(x) . \tag{1}$$

We shall choose the nonlinear gauge function,

$$f[A] = +\frac{1}{\sqrt{\alpha}} (\partial_\mu A^\mu - \frac{1}{2} \xi A_\mu A^\mu) . \tag{2}$$

Then the gauge term, to be added to \mathcal{L}_0 , is given by,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2} \{ f[A] \}^2 .$$

Henceforth, we shall use a summation - integration convention (used, for example in Ref. 2). Thus, the gauge functional of Eq.(2) is,

$$f_\alpha[A] = +\frac{1}{\sqrt{\alpha}} (\partial_i^\alpha A_i - \frac{1}{2} \xi_{ij}^\alpha A_i A_j) \tag{3}$$

$$[\partial_i^\alpha \equiv \partial_\mu \delta^4(x_\alpha - x_i) \quad \xi_{ij}^\alpha \equiv \xi \cdot \delta^4(x_\alpha - x_i) \delta^4(x_i - x_j) \text{ etc.}] .$$

As shown by Faddeev and Popov,³ the Feynman rules for constructing Green's functions can be deduced from the effective Lagrangian,

$$\mathcal{L}_{\text{eff}}[A, c, \bar{c}] = \mathcal{L}_0 + \mathcal{L}_{\text{gauge}} + \bar{c}_\alpha M_{\alpha\beta} c_\beta . \tag{4}$$

Where, \bar{c}_α and c_β are fictitious, anticommuting complex scalar fields which generate the Faddeev-Popov ghost loops, and $M_{\alpha\beta}$ is given by,

$$M_{\alpha\beta} = \frac{\delta f_\alpha[A]}{\delta A_i} \left(\frac{1}{e} \partial_i^\beta \right) e\sqrt{\alpha}$$

$$= \partial_i^\beta \left(\partial_i^\alpha - \xi_{ij}^\alpha A_j \right) . \quad (5)$$

We note that there are $(A_\mu)^3$ and $(A_\mu)^4$ vertices arising out of the gauge term and a $\bar{c}c A_\mu$ vertex from the ghost term. These Feynman rules are given in Fig. 1. (The dotted lines denote the ghost lines, the wiggly lines denote photons.)

(II-2) Ward-Takahashi Identity for Proper Vertices

We shall deal with unrenormalized, but dimensionally regularized quantities (in dimensions $4-\epsilon$). We shall use the notation of Ref. 2.

The generating functional of Green's functions is given by,

$$W_F[J] = \int [dA dc d\bar{c}] \exp i \left\{ \mathcal{L}_{\text{eff}}[A, c, \bar{c}] + J_i A_i \right\} . \quad (6)$$

As a result of gauge invariance, $W_F[J]$ satisfies the WT identity²; which to our specific case, reads

$$\left\{ - \frac{1}{\sqrt{\alpha}} f_\alpha \left[\frac{1}{i} \frac{\delta}{\delta J} \right] + J_i \partial_i^\beta M_{\beta\alpha}^{-1} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] \right\} W_F[J] = 0 . \quad (7)$$

$Z[J]$, the generating functional of connected Green's functions is defined by,

$$W_F[J] = \exp iZ[J] .$$

We define,

$$\Phi_i = \frac{\delta Z[J]}{\delta J_i} \quad (8)$$

Then the functional $\Gamma[\Phi]$ defined by,

$$\Gamma[\Phi] = Z[J] - J_i \Phi_i \quad (9)$$

generates the proper vertices.

It follows from Eq. (9),

$$J_i = - \frac{\delta \Gamma}{\delta \Phi_i} \quad (10)$$

We go back to the WT identity of Eq. (7) and use the operator identity:

$$B\left[\frac{1}{i} \frac{\delta}{\delta J}\right] e^{iZ[J]} = e^{iZ[J]} B\left[\Phi + \frac{1}{i} \frac{\delta}{\delta J}\right] \quad (11)$$

and thus obtain,

$$- \frac{1}{\sqrt{\alpha}} f_\alpha \left[\Phi + \frac{1}{i} \frac{\delta}{\delta J} \right] \cdot 1 + \partial_i^\beta J_i M_{\beta\alpha}^{-1} \left[\Phi + \frac{1}{i} \frac{\delta}{\delta J} \right] \cdot 1 = 0 \quad (11)$$

Now,

$$\frac{\delta}{\delta J_i} = \frac{\delta \Phi_j}{\delta J_i} \frac{\delta}{\delta \Phi_j} \equiv - \Delta_{ij}[\Phi] \frac{\delta}{\delta \Phi_j} \quad .$$

It can be shown that $\Delta_{ij}[\Phi]$ is the propagator when fields A_i are constrained to have expectation values Φ_i . Then using Eq. (10), Eq. (11) becomes,

$$- \frac{1}{\sqrt{\alpha}} f_\alpha \left[\Phi + i\Delta \frac{\delta}{\delta \Phi} \right] \cdot 1 - \partial_i^\beta \frac{\delta \Gamma}{\delta \Phi_i} G_{\beta\alpha}[\Phi] = 0 \quad (12)$$

where,

$$G_{\beta\alpha}[\Phi] \equiv M_{\beta\alpha}^{-1} \left[\Phi + i\Delta \frac{\delta}{\delta \Phi} \right] \cdot 1 \quad , \text{ and can be shown to be the}$$

generating functional of proper vertices with two external ghost lines.

Now,

$$\begin{aligned} & \sqrt{\alpha} f_{\alpha} \left[\Phi + i\Delta \frac{\delta}{\delta \Phi} \right] \cdot 1 \\ &= \left\{ \partial_i^{\alpha} \left[\Phi_i + i\Delta_{ij} \frac{\delta}{\delta \Phi_j} \right] - \frac{1}{2} \xi_{ij}^{\alpha} \left[\Phi_i + i\Delta_{ik} \frac{\delta}{\delta \Phi_k} \right] \left[\Phi_j + i\Delta_{j\ell} \frac{\delta}{\delta \Phi_{\ell}} \right] \right\} \cdot 1 \\ &= \left\{ \partial_i^{\alpha} \Phi_i - \frac{1}{2} \xi_{ij}^{\alpha} (\Phi_i \Phi_j + i\Delta_{ij}) \right\} . \end{aligned}$$

Thus we obtain the WT identity for proper vertices:

$$G_{\beta\alpha}[\Phi] \partial_i^{\beta} \frac{\delta \Gamma[\Phi]}{\delta \Phi_i} = -\frac{1}{\alpha} \left\{ \partial_i^{\alpha} \Phi_i - \frac{1}{2} \xi_{ij}^{\alpha} (\Phi_i \Phi_j + i\Delta_{ij}[\Phi]) \right\} . \quad (13)$$

(II-3) Expression for $G_{\alpha\gamma}^{-1}[\Phi]$

$G_{\alpha\gamma}^{-1}[\Phi]$ is the generating functional of the proper vertices with two ghost fields at α and γ . In order to carry through the renormalization program, we need to show that the renormalized $G^{-1}[\Phi]$ is a finite functional. Hence we need, first, to obtain an expression for $G_{\alpha\gamma}^{-1}[\Phi]$.

We have the identity:

$$M_{\alpha\gamma} \left[\Phi + i\Delta \frac{\delta}{\delta \Phi} \right] M_{\gamma\beta}^{-1} \left[\Phi + i\Delta \frac{\delta}{\delta \Phi} \right] \cdot 1 = \delta_{\alpha\beta} .$$

Using definitions of $M_{\alpha\gamma}$ [Eq. (3)] and $G_{\gamma\beta}$ [Eq. (12)];

$$\left[\partial_i^{\alpha} - \xi_{ij}^{\alpha} \left(\Phi_j + i\Delta_{jk} \frac{\delta}{\delta \Phi_k} \right) \right] \partial_i^{\gamma} G_{\gamma\beta}[\Phi] = \delta_{\alpha\beta} .$$

Using,

$$\frac{\delta}{\delta \Phi_k} G_{\gamma\beta}[\Phi] = -G_{\gamma\xi} \frac{\delta G_{\xi\eta}^{-1}}{\delta \Phi_k} G_{\eta\beta} ,$$

we obtain,

$$\left[\partial_i^\alpha \partial_i^\gamma - \xi_{ij}^\alpha \Phi_j \partial_i^\eta \Delta_{jk} G_{\eta\zeta} \frac{\delta G_{\zeta\gamma}^{-1}}{\delta \Phi_k} \right] G_{\gamma\beta} = \delta_{\alpha\beta} \quad .$$

Hence,

$$G_{\alpha\gamma}^{-1}[\Phi] = \partial_i^\alpha \partial_i^\gamma - \xi_{ij}^\alpha \Phi_j \partial_i^\eta \Delta_{jk} G_{\eta\zeta} \frac{\delta G_{\zeta\gamma}^{-1}}{\delta \Phi_k} \quad (14)$$

A diagrammatic representation for the last term is given in Fig. 2.

(II-4) Renormalization Transformations

To prove renormalizability of the theory, we have to show that the derivatives of $\Gamma[\Phi]$ about its minimum can be rendered finite as $\epsilon \rightarrow 0$, by rescaling fields and parameters appearing in the Lagrangian $\mathcal{L}_{\text{eff}}[\Phi, c, \bar{c}]$. We therefore define renormalized parameters and fields by the following renormalization transformations:

$$\begin{aligned} c &= \tilde{Z}^{\frac{1}{2}} c^{(r)} & \Phi &= \tilde{Z}^{\frac{1}{2}} \Phi^{(r)} \\ \frac{1}{\alpha} &= \frac{W}{\tilde{Z}} \frac{1}{\alpha^{(r)}} & \xi &= Y \tilde{Z}^{-1} \tilde{Z}^{\frac{1}{2}} \xi^{(r)} \end{aligned}$$

We also define:

$$G_{\alpha\beta}[\Phi] = \tilde{Z} G_{\alpha\beta}^{(r)}[\Phi^{(r)}], \quad \Gamma[\Phi, \alpha, \xi] = \Gamma^{(r)}[\Phi^{(r)}, \alpha^{(r)}, \xi^{(r)}] \quad .$$

In the following, we shall always express everything in terms of renormalized quantities and drop the superscript (r). Thus the expression for (renormalized) $G_{\alpha\gamma}^{-1}[\Phi]$ becomes, (from Eq. (14)):

$$G_{\alpha\gamma}^{-1}[\Phi] = \left\{ \tilde{Z} \partial_i^\alpha \partial_i^\gamma - Y \xi_{ij}^\alpha \Phi_j \partial_i^\eta \Delta_{jk} G_{\eta\zeta} \frac{\delta G_{\zeta\gamma}^{-1}}{\delta \Phi_k} \right\} \quad (15)$$

While the WT identity of Eq. (13) becomes,

$$G_{\beta\alpha} \partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} = -\frac{1}{\alpha} \frac{W}{\tilde{Z}} \left\{ \partial_i^\alpha \Phi_i - \frac{Y}{2\tilde{Z}} \xi_{ij}^\alpha \Phi_i \Phi_j - \frac{iY}{2\tilde{Z}} \xi_{ij}^\alpha \Delta_{ij} \right\} \\ \equiv -\frac{1}{\alpha} \mathcal{F}_\alpha[\Phi] \quad . \quad (16)$$

To carry through the renormalization program, we start with the unperturbed Lagrangian expressed in renormalized fields,

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2$$

and expand the proper vertices in terms of the loops the Feynman diagram contains. In each loop approximation we must determine the renormalization constants by a given prescription.

In the following, it will be assumed that renormalization constants Z , \tilde{Z} , W , Y are determined up to $(n-1)$ loop approximation and that these make $G_{\alpha\gamma}^{-1}[\Phi]$, $\Gamma[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$ (defined in Eq. (16)) finite to each order, up to $(n-1)$ loop approximation, in perturbation theory. We write, up to $(n-1)$ loop approximation:

$$Z(\epsilon) = 1 + z_1(\epsilon) + z_2(\epsilon) + \dots + z_{n-1}(\epsilon), \quad \text{etc., etc.}$$

Then we shall show that an appropriate choice of z_n , \tilde{z}_n , y_n , w_n can be made as to make $G_{\alpha\gamma}^{-1}[\Phi]$, $\Gamma[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$ finite up to n loop approximation.

(II-5) Analysis of Divergences in $G_{\alpha\gamma}^{-1}[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$

In order to show that $G_{\alpha\gamma}^{-1}[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$ can be made finite by

appropriate choice of the renormalization constants in n-loop approximation, we must show that various derivatives of $G_{\alpha\gamma}^{-1}[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$ at $\Phi = 0$ (the minimum) in the n-loop approximation have received all the internal subtractions (the meaning of this statement will be clear soon.), so that the divergences in these (those which are renormalization parts) are polynomials in external momenta and therefore that these can be removed by the local counter terms provided by the appropriate choices of the renormalization parameters.

We note from Fig. 1 that the Feynman rule at the $\bar{c}cA_\mu$ vertex is proportional to the momentum of the incoming ghost; so that in any proper vertex with two ghost lines, there is a factor of p_μ for the incoming ghost of momentum p . This effectively decreases the degree of divergence (D) by one. Therefore, $G_{\alpha\gamma}^{-1}[\Phi] \Big|_{\Phi=0}$ and $\frac{\delta G_{\alpha\gamma}^{-1}[\Phi]}{\delta \Phi_k} \Big|_{\Phi=0}$ are renormalization parts but higher derivatives of $G_{\alpha\gamma}^{-1}[\Phi]$ are not renormalization parts. Also only the first two derivatives of $\mathcal{F}_\alpha[\Phi]$ at $\Phi = 0$ are renormalization parts.

(A) We begin by considering $G_{\alpha\gamma}^{-1}[\Phi] \Big|_{\Phi=0}$ in n-loop approximation.

We write this down in momentum space as Feynman Diagrams. [See Fig. (3)].

Consider $[\Sigma]_n$ i.e., Σ in n-loop approximation. The shaded blobs in $[\Sigma]_n$ contain at most (n-1) loops and the counter terms introduced up to (n-1) loop approximation provide the necessary subtractions for

the subdiagrams in the blobs making them finite by our hypothesis.

However $[\Sigma]_n$ needs further subtractions for the renormalization parts which are subdiagrams of $[\Sigma]_n$ and contain the rightmost vertex in $[\Sigma]_n$.

From the remark made earlier about the $(\bar{c}cA_\mu)$ vertex, such renormalization parts arise only from two particle cuts in $\frac{\delta G_{\xi Y}^{-1}}{\delta \Phi_k}$, the leftmost blob in $[\Sigma]_n$. See Fig. 4.

Thus these renormalization parts needing overall subtractions consist of the three point proper vertex $\delta G^{-1}/\delta \Phi$ to various loop approximations [up to (n-1) loops]. We shall show that the overall subtraction for such subdiagram is provided by y_r . Hence the additional internal subtractions needed by $[\Sigma]_n$ consist of $\sum_{r=1}^{y-1} y_r(\Sigma)_{n-r}$. Therefore,

$$(Y\Sigma)_n = \sum_{r=1}^{n-1} y_r(\Sigma)_{n-r}$$

has its divergence a polynomial in momentum. Due to Lorentz transformation property and dimensions of $G^{-1}(p)$,

$$(Y\Sigma)_n^{\text{div}} = p^2 K(\epsilon) .$$

(Here we note that there are no dimensional parameters in \mathcal{L}_{eff} .)

Therefore by choosing $\tilde{z}_n^{\text{div}} = K(\epsilon)$ we can make $[G^{-1}(p)]_n$ finite.

(B) Next, consider $\left. \frac{\delta G_{\alpha Y}^{-1}[\Phi]}{\delta \Phi_k} \right|_{\Phi=0}$. From Eq. (15),

$$\left. \frac{\delta G_{\alpha\gamma}^{-1}}{\delta \Phi_k} \right|_{\Phi=0} = -Y \partial_i^\gamma \xi_{ik}^\alpha + iY \xi_{ij}^\alpha \partial_i^\eta \frac{\delta}{\delta \Phi_k} \Delta_{jm} G_{\eta\zeta} \left. \frac{\delta G_{\zeta\gamma}^{-1}}{\delta \Phi_m} \right|_{\Phi=0} . \quad (17)$$

We express the Fourier transform of Eq. (17) diagrammatically in Fig. 5.

As before, we need consider the internal subtractions needed to $[\gamma_1 + \gamma_2 + \gamma_3]_n$ for their subdiagrams containing the rightmost vertex in each only. It is easy to see that $(\gamma_2)_n$ and $(\gamma_3)_n$ does not have subdiagrams which are renormalization parts. $(\gamma_1)_n$ has, however, such subdiagrams which arise out of two particle cut in the proper $(\bar{c}c A_\mu A_\nu)$ vertex on the left. These fall into three categories shown in Fig. 6.

As before, it is clear that the internal subtractions to $(\gamma_1 + \gamma_2 + \gamma_3)_n$ are provided by $\sum_{\gamma=1}^{n-1} y_\gamma (\gamma_1 + \gamma_2 + \gamma_3)_{n-\gamma}$. Hence,

$$[Y(\gamma_1 + \gamma_2 + \gamma_3)]_n^{\text{div}} = p_\mu J(\epsilon)$$

(Here one must remember that each diagram is proportional to p_μ .)

Therefore,

$$[G^{(3)}(p, q, r, \mu)]_n^{\text{div}} = p_\mu [\xi y_n(\epsilon) + J(\epsilon)]$$

and can be made finite by appropriate choice of $y_n(\epsilon)$.

(C) Consider divergences in $\mathcal{F}_\alpha[\Phi]$.

We note,

$$\mathcal{F}_\alpha[\Phi] \Big|_{\Phi=0} = \frac{-iY}{2W\tilde{Z}^2} \xi_{ij}^\alpha \Delta_{ij}$$

$$\text{F. T.} \left\{ \mathcal{F}_\alpha[\Phi] \right\}_{\Phi=0} = \frac{-iY\xi}{WZ^2} \int [\Delta_{\mu\mu}(q^2)]_{n-1} d^4q \quad (19)$$

where $\Delta_{\mu\nu}(q^2)$ is the photon propagator. Since the right hand side must have dimensions (momentum)² and since there are no dimensional quantities in the integral that it can depend on, it must be zero in dimensional regularization.

Next, we consider the divergences in $\left. \frac{\delta \mathcal{F}_\alpha}{\delta \Phi_k} \right|_{\Phi=0}$ and $\left. \frac{\delta^2 \mathcal{F}_\alpha}{\delta \Phi_k \delta \Phi_l} \right|_{\Phi=0}$

$$\left. \frac{\delta \mathcal{F}_\alpha}{\delta \Phi_k} \right|_{\Phi=0} = \frac{W}{Z} \left[\partial_k^\alpha - \frac{Y}{2\tilde{Z}} \xi_{ij}^\alpha \frac{\delta}{\delta \Phi_k} (i\Delta_{ij}) \right] \Bigg|_{\Phi=0} . \quad (20)$$

$$\left. \frac{\delta^2 \mathcal{F}_\alpha}{\delta \Phi_k \delta \Phi_l} \right|_{\Phi=0} = -\frac{WY}{2\tilde{Z}^2} \left\{ 2\xi_{kl}^\alpha + \xi_{ij}^\alpha \frac{\delta^2}{\delta \Phi_k \delta \Phi_l} (i\Delta_{ij}) \right\} \Bigg|_{\Phi=0} . \quad (21)$$

We tabulate these in Fig. 7.

The constants, A, B, C are defined in each order in perturbation to be the overall divergences left in Δ , $\square_{\mu\nu}$ and $\Pi_{\mu\nu}$ (They are defined in Fig. 7) respectively when all subtractions are performed on their subdiagrams which are renormalization parts. Note that we have not yet specified how the finite parts of A, B, C (alternately of Δ , $\square_{\mu\nu}$, and $\Pi_{\mu\nu}$) are to be defined. [For clarity, we note that Δ , $\square_{\mu\nu}$, and $\Pi_{\mu\nu}$ are proper diagrams. The distinction between $\square_{\mu\nu}$ and $\Pi_{\mu\nu}$ is that when they are opened at the vertex denoted by cross ($\sim \times \sim$), $\Pi_{\mu\nu}$ gives rise to a 4 photon proper vertex while $\square_{\mu\nu}$ gives rise to a 4 photon improper vertex.]

The subtractions needed for subdiagrams in the shaded blobs in $(\Delta)_n$, $[\square_{\mu\nu}]_n$ and $[\Pi_{\mu\nu}]_n$ (See Fig. 7) are provided by the counter terms already introduced in the Lagrangian up to $(n-1)$ loop approximation. Thus apart from overall subtraction these need subtractions for the renormalization parts which are subdiagrams containing the leftmost vertex denoted by a cross (\times). We tabulate these subdiagrams and subtractions needed to them in Fig. 8.

Thus, from Fig. 7 and Fig. 8, it follows that,

$$(i) \quad [\Delta_\mu(p)]_n - \sum_{r=1}^{n-1} B_r [\Delta_\mu(p)]_{n-r} - \frac{1}{2} \sum_{r=1}^{n-1} C_r [\Delta_\mu(p)]_{n-r} - A_n p_\mu = \text{finite}$$

which can be written in a condensed form:

$$[\Delta_\mu(p)(1-B-\frac{C}{2}) - A p_\mu]_n = \text{finite} \quad . \quad (22)$$

$$(ii) \quad 2[\square_{\mu\nu}]_n - 2B_n g_{\mu\nu} = \text{finite} \quad .$$

$$(iii) \quad [\Pi_{\mu\nu}]_n - \frac{1}{2} \sum_{r=1}^{n-1} C_r [\Pi_{\mu\nu}]_{n-r} - \sum_{r=1}^{n-1} C_r [\square_{\mu\nu}]_{n-r} - 2 \sum_{r=1}^{n-1} B_r [\square_{\mu\nu}]_{n-r} - \sum_{r=1}^{n-1} B_r [\Pi_{\mu\nu}]_{n-r} = \text{finite} \quad .$$

Adding (ii) and (iii) and writing in a condensed form:

$$[(2g_{\mu\nu} + 2\square_{\mu\nu} + \Pi_{\mu\nu})(1-B-\frac{C}{2})]_n = \text{finite} \quad . \quad (23)$$

Now,

$$\text{F. T.} \left\{ \frac{\delta^2 \mathcal{F}_\alpha}{\delta \Phi_k} \Big|_{\Phi=0} \right\} = i \left\{ \frac{W}{\tilde{Z}} p_\mu + \frac{\xi Y W}{2 \tilde{Z}^2} \Delta_\mu(p) \right\} \quad (24)$$

and,

$$\text{F. T.} \left\{ \frac{\delta^2 \mathcal{F}_\alpha}{\delta \Phi_k \delta \Phi_\ell} \Big|_{\Phi=0} \right\} = - \frac{W Y \xi}{2 \tilde{Z}^2} \left\{ 2g_{\mu\nu} + 2\Omega_{\mu\nu} + \Pi_{\mu\nu} \right\}. \quad (25)$$

Comparing Eq. (22) with Eq. (24) and Eq. (23) with Eq. (25), it is clear that the two derivatives of $\mathcal{F}_\alpha[\Phi]$ can be made finite simultaneously if we choose the factors $\frac{\xi Y W}{2 \tilde{Z}^2}$ and $\frac{W}{\tilde{Z}}$ appropriately [i. e., equal to $(1-B-\frac{C}{2})$ and $-\frac{\xi A}{2}$ respectively.] However, since we would like to (though it is not necessary) determine the finite parts of W and Y by renormalization conditions on derivatives of $\Gamma[\Phi]$ rather than of $\mathcal{F}_\alpha[\Phi]$, we will state it differently. Suppose, we have chosen w_r and y_r in rth loop ($r < n$) approximation by appropriate renormalization conditions and if they satisfy,

$$\begin{aligned} \text{(i)} \quad \left(\frac{Y W}{\tilde{Z}^2} \right)_r &= (1-B-\frac{C}{2})_r & 0 \leq r \leq n-1 \\ \text{(ii)} \quad \left(\frac{2}{\xi} \frac{W}{\tilde{Z}} \right)_r^{\text{div}} &= -A_r^{\text{div}} & 0 \leq r \leq n-1 \end{aligned} \quad (26)$$

then we have to show that w_n and y_n chosen in nth loop approximation will satisfy:

$$\text{(i)} \quad \left(\frac{Y W}{\tilde{Z}^2} \right)_n = (1-B-\frac{C}{2})_n$$

$$(ii) \left(\frac{2}{\xi} \frac{W}{\bar{Z}} \right)_r^{\text{div}} = -A_n^{\text{div}}$$

and hence will make $\left. \frac{\delta \mathcal{F}_\alpha}{\delta \Phi_i} \right|_{\Phi=0}$ and $\left. \frac{\delta^2 \mathcal{F}_\alpha}{\delta \Phi_k \delta \Phi_\ell} \right|_{\Phi=0}$ finite to n loop approximation.

Here we note the convention to define the finite parts of Δ_μ and $(2\Box_{\mu\nu} + \Pi_{\mu\nu})$. Once the finite parts of Y, W and Z are chosen by a given set of renormalization prescriptions in r loop approximation, Eq. (26) then defines the finite part of $(1-B-\frac{C}{2})_r$ and hence of $(2\Box_{\mu\nu} + \Pi_{\mu\nu})_r$. Finite part of $\left[\Delta_\mu(p) \right]_r$ is so defined in rth loop approximation that WT identity of Eq. (30) below is satisfied by the finite parts.

The higher derivatives of $\mathcal{F}_\alpha[\Phi]$ are not renormalization parts. The proof that they become finite in n-loop approximation once Eq. (26) are satisfied proceeds similarly.

(II-6) Proof of Renormalizability:

Consider the inverse photon propagator $\Gamma_{\mu\nu}(p)$. Because of the Lorentz transformation property and the fact that there are no dimensional parameters in the theory, it follows that,

$$\left[\Gamma_{\mu\nu}(p) \right]_n^{\text{div}} = \left(g_{\mu\nu} p^2 - p_\mu p_\nu \right) N^t(\epsilon) + p_\mu p_\nu N^\ell(\epsilon) \quad .$$

Define $\Gamma^{(t)}$ and $\Gamma^{(\ell)}$ by:

$$\Gamma_{\mu\nu}(p) = \left(g_{\mu\nu} p^2 - p_\mu p_\nu \right) \Gamma^{(t)}(p) + p_\mu p_\nu \Gamma^{(\ell)}(p) \quad (28)$$

Then,

$$\left[\Gamma^{(t)}(p) \right]_n^{\text{div}} = N^t(\epsilon); \left[\Gamma^\ell(p) \right]_n^{\text{div}} = N^\ell(\epsilon) \quad .$$

We shall choose z_n and w_n such that $N^t(\epsilon)$ and $N^\ell(\epsilon)$ respectively become finite. As shown earlier, we can choose \tilde{z}_n such that $\left[G^{-1}(p) \right]_n$ is finite. Then we have to show that proper choice of y_n can be made so that the $(\bar{c}cA_\mu)$, $(A_\mu)^3$ and $(A_\mu)^4$ vertices become finite and that relations of Eq. (27) are satisfied.

To this end, we consider the WT identity of Eq. (16). We consider successive derivatives of this identity at $\Phi=0$ and equate the quantities on both sides in the n-loop approximation.

(A) Differentiate the WT identity with respect to Φ_k and set $\Phi=0$. We obtain,

$$G_{\beta\alpha} \partial_i^\beta \frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Phi_k} = - \frac{1}{\alpha} \frac{W}{\tilde{Z}} \left\{ \partial_k^\alpha - \frac{iY}{2Z} \xi_{ij}^\alpha \frac{\delta(\Delta_{ij})}{\delta \Phi_k} \right\} \Bigg|_{\Phi=0} \quad . \quad (29)$$

Writing this in momentum space, using the Fourier transforms defined earlier [See Eqs. (24) and (28)].

$$G(p^2) p_\nu p^2 \Gamma^{(\ell)}(p^2) = - \frac{1}{\alpha} \left\{ \frac{W}{\tilde{Z}} p_\nu + \xi \frac{YW}{2\tilde{Z}^2} \Delta_\nu(p) \right\} \quad . \quad (30)$$

Left hand side of Eq. (30) is finite in n-loop approximation (with already chosen \tilde{z}_n and w_n), so that

$$\text{finite} = - \left\{ \frac{W}{\tilde{Z}} p_\nu + \xi \frac{YW}{2\tilde{Z}^2} \Delta_\nu(p) \right\}_n \quad ,$$

$$\begin{aligned}
 &= - \left(\frac{W}{\tilde{Z}} \right)_n p_\nu - \frac{\xi}{2} \sum_{r=0}^{n-1} \left(\frac{YW}{\tilde{Z}^2} \right)_r \left[\Delta_\nu(p) \right]_{n-r} \\
 &= - \left(\frac{W}{\tilde{Z}} \right)_n p_\nu - \frac{\xi}{2} \sum_{r=0}^{n-1} \left(1 - B - \frac{C}{2} \right) \left[\Delta_\nu(p) \right]_{n-r} \quad \text{by Eq. (26)} \\
 &= - \left(\frac{W}{\tilde{Z}} \right)_n p_\nu - \frac{\xi}{2} A_n p_\nu + \text{finite} \quad \text{by Eq. (22)} \\
 &\therefore \left[A \right]_n^{\text{div}} = - \frac{2}{\xi} \left(\frac{W}{\tilde{Z}} \right)_n^{\text{div}} . \quad (31)
 \end{aligned}$$

(B) Next, differentiate the WT identity with respect to Φ_k and Φ_ℓ and set $\Phi=0$. We obtain,

$$\begin{aligned}
 &G_{\beta\alpha} \partial_i^\beta \frac{\delta^3 \Gamma}{\delta \Phi_i \delta \Phi_k \delta \Phi_\ell} + \frac{\delta}{\delta \Phi_k} (G_{\beta\alpha}) \partial_i^\beta \frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Phi_\ell} + (k \leftrightarrow \ell) \\
 &= \frac{1}{\alpha} \frac{WY}{2\tilde{Z}^2} \left\{ 2\xi_{\ell k}^\alpha + \xi_{ij}^\alpha \frac{\delta^2 (i\Delta_{ij})}{\delta \Phi_\ell \delta \Phi_k} \right\} . \quad (32)
 \end{aligned}$$

Let us define:

$$\text{F. T.} \left\{ \frac{\delta^3 \Gamma}{\delta \Phi_i \delta \Phi_k \delta \Phi_\ell} \right\} = -i \Gamma_{\alpha\mu\nu}^{(3)}(p, q, r) . \quad (33)$$

Then, the left hand side of Eq. (32) has the diagrammatic representation shown in Fig. 9.

Now, $\left\{ \Gamma_{\alpha\mu\nu}^{(3)}(p, q, r) \right\}_n^{\text{div}}$ must be a polynomial linear in external momenta ($p+q+r=0$) and a Bose symmetric Lorentz tensor. This implies that,

$$\left\{ \Gamma_{\alpha\mu\nu}^{(3)}(p, q, r) \right\}_n^{\text{div}} = D(\epsilon) \left\{ g_{\alpha\mu} r_\nu + g_{\alpha\nu} q_\mu + g_{\mu\nu} p_\alpha \right\} . \quad (34)$$

As shown in Eq. (18),

$$\left[G^{(3)}(p, q; r, \mu) \right]_n^{\text{div}} = p_\mu \left[\xi y_n + J(\epsilon) \right] \equiv a(\epsilon) p_\mu . \quad (35)$$

Similarly,

$$\left[G^{(3)}(p, q; r, \nu) \right]_n^{\text{div}} = a(\epsilon) p_\nu . \quad (36)$$

Hence Eq. (32), in momentum space, becomes:

$$\begin{aligned} & \frac{D}{p} p_\alpha \left\{ g_{\alpha\mu} r_\nu + g_{\alpha\nu} q_\mu + g_{\mu\nu} p_\alpha \right\} + \frac{a}{\alpha} \frac{p_\mu r_\nu}{p^2} + \frac{a}{\alpha} \frac{p_\nu q_\mu}{p^2} \\ &= \left\{ \frac{YW\xi}{2\alpha\tilde{Z}^2} \left[2g_{\mu\nu} + 2\Box_{\mu\nu}(q, r) + \Pi_{\mu\nu}(q, r) \right] \right\}_n^{\text{div}} + \text{finite} \\ &= \left(\frac{\xi W Y}{\alpha \tilde{Z}^2} \right)_n^{\text{div}} g_{\mu\nu} + \frac{\xi}{2\alpha} \sum_{r=0}^{n-1} \left(\frac{YW}{\tilde{Z}^2} \right)_r \left(2g_{\mu\nu} + 2\Box_{\mu\nu} + \Pi_{\mu\nu} \right)_{n-r} + \text{finite} \end{aligned}$$

using Eq. (26),

$$= \left(\frac{\xi W Y}{\alpha \tilde{Z}^2} \right)_n^{\text{div}} g_{\mu\nu} + \frac{\xi}{2\alpha} \sum_{r=0}^{n-1} \left(1 - B - \frac{C}{2} \right)_r \left(2g_{\mu\nu} + 2\Box_{\mu\nu} + \Pi_{\mu\nu} \right)_{n-r} + \text{finite}$$

using Eq. (23),

$$= \left(\frac{\xi W Y}{\alpha \tilde{Z}^2} \right)_n^{\text{div}} g_{\mu\nu} - \frac{\xi}{\alpha} \left(1 - B - \frac{C}{2} \right)_n g_{\mu\nu} + \text{finite} . \quad (37)$$

Expressing Eq. (37) as a function of p and q ,

$$\begin{aligned}
 D \left[-(p+q)_\nu p_\mu + p_\nu q_\mu + g_{\mu\nu} p^2 \right] - \frac{a}{\alpha} p_\mu (p+q)_\nu + \frac{a}{\alpha} p_\nu q_\mu \\
 = \frac{\xi}{\alpha} p^2 \left[\left(\frac{WY}{\tilde{Z}^2} \right)_n^{\text{div}} - \left(1-B-\frac{C}{2} \right)_n \right] g_{\mu\nu} + \text{finite}.
 \end{aligned}$$

Comparing coefficients of $p_\mu q_\nu$ and $g_{\mu\nu} p^2$, we obtain,

$$D = -\frac{a}{\alpha} + \text{finite} = \frac{\xi}{\alpha} \left[\left(\frac{WY}{\tilde{Z}^2} \right)_n^{\text{div}} - \left(1-B-\frac{C}{2} \right)_n \right] + \text{finite} . \quad (38)$$

But from Eq. (18), $a \equiv \xi y_n(\epsilon) + K_n(\epsilon)$ can be made finite with appropriate choice of $y_n(\epsilon)$. Then $D(\epsilon)$ becomes finite and

$$\left(1-B-\frac{C}{2} \right)_n^{\text{div}} = \left(\frac{YW}{\tilde{Z}^2} \right)_n^{\text{div}} . \quad (39)$$

As remarked earlier, finite part of $\left(1-B-\frac{C}{2} \right)$ will be defined such that

$$\left(1-B-\frac{C}{2} \right)_n = \left(\frac{YW}{\tilde{Z}^2} \right)_n . \quad (40)$$

(C) We shall consider, finally, the WT identity differentiated thrice with respect to Φ_k , Φ_l and Φ_m and set $\Phi = 0$. With choices of renormalization constants in n -loop approximation already made, all vertices entering the equation are made finite except (possibly) the $(A_\mu)^4$ vertex. From this equation it trivially follows that the $(A_\mu)^4$ vertex is also finite. Thus all renormalization parts of $\Gamma[\Phi]$ are shown to become finite.

Thus we have shown that if renormalization constants are chosen up to (n-1) loop approximation such that $\Gamma[\Phi]$, $G_{\alpha\beta}^{-1}[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$ are finite in the limit $\epsilon \rightarrow 0$ in loop approximation, then renormalization constants z_n , \tilde{z}_n , w_n , y_n can be chosen to make $\Gamma[\Phi]$, $G_{\alpha\beta}^{-1}[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$ finite up to n loop approximation. For n=1, this is trivially true if we choose $z_0 = \tilde{z}_0 = w_0 = y_0 = 1$. Hence the proof by induction is complete.

We shall present the results of one loop calculation in Sec. III.

III. RESULTS OF ONE LOOP CALCULATION

In this section, we state the results of one loop calculation to verify the relations between divergences [See Eq. (27)] in $G_{\alpha\beta}^{-1}[\Phi]$, $\Gamma[\Phi]$ and $\mathcal{F}_\alpha[\Phi]$.

(A) The inverse photon propagator⁴:

The diagrams of Fig. 10 contribute to the inverse photon propagator in one loop approximation.

We use dimensional regularization to compute these and state the divergences in units of

$$I \equiv \frac{-i}{(2\pi)^4} \left[\int \frac{d^4 q}{q^4} \right]_{\text{pole}} = \frac{\pi^2}{(2\pi)^4} \frac{2}{\epsilon} . \quad (41)$$

Let $m(a)$ denote the diagram of Fig. 10(a) evaluated with usual Feynman rules; let $\text{Div} \{ m(a) \}$ denote the terms in $m(a)$ which have

pole in ϵ .

We find:

$$\text{Div}\{m(a)\} = \frac{i\xi^2 [2\alpha^2 - 3\alpha + 3]}{2\alpha^2} I p_\mu p_\nu - i\xi^2 I \left\{ \frac{1}{3} (g_{\mu\nu} p^2 - p_\mu p_\nu) - \frac{1}{4} g_{\mu\nu} p^2 \right\}$$

$$\text{Div}\{m(b)\} = i\xi I \left\{ \frac{1}{3} (g_{\mu\nu} p^2 - p_\mu p_\nu) - \frac{1}{4} g_{\mu\nu} p^2 \right\}$$

$$m(c) = m(d) = 0 \quad .$$

Therefore,

$$\text{Div}\{i \Gamma_{\mu\nu}(p)\} = \frac{i\xi^2 [2\alpha^2 - 3\alpha + 3]}{2\alpha^2} I p_\mu p_\nu \quad . \quad (42)$$

Writing $Z = 1+z$, $W = 1+w$ etc. to one loop approximation, the counter term is:

$$-iz (g_{\mu\nu} p^2 - p_\mu p_\nu) - \frac{i}{\alpha} (w-z) p_\mu p_\nu \quad . \quad (43)$$

Hence, we find that the following choice will make renormalized inverse propagator finite:

$$\text{Div}(z) = 0 \quad , \quad \text{Div}(w) = \frac{\xi^2}{2\alpha} [2\alpha^2 - 3\alpha + 3] I \quad . \quad (44)$$

Here, we see that the transverse part is unrenormalized to one loop approximation while the longitudinal part is renormalized.

(B) The Inverse Ghost Propagator:

Diagrams of Fig. 11 contribute to the inverse ghost propagator.

We find,

$$\begin{aligned} \text{Div}\{m(a)\} &= -\frac{i}{4} \xi^2 (3-\alpha) I p^2 \\ m(b) &= 0 \quad . \end{aligned} \quad (45)$$

Writing $\tilde{Z} = 1+z$; the divergence in the inverse ghost propagator will be cancelled by the counter term $-ip \frac{2}{\tilde{Z}}$ if we choose,

$$\text{Div}(\tilde{Z}) = -\frac{(3-\alpha)\xi^2 I}{4} \quad . \quad (46)$$

(C) The Ghost-Ghost-Photon ($\tilde{c}cA_\mu$) Vertex: Diagrams of Fig. 12 contribute to the proper vertex. It is found that,

$$\begin{aligned} \text{Div}\{m(a)\} &= \frac{3}{4} \xi^3 I p_\alpha \\ \text{Div}\{m(b)\} &= -\frac{1}{4} \xi^3 I p_\alpha \quad . \end{aligned}$$

Thus,

$$\text{Div}\{m(a) + m(b)\} = \frac{(3-\alpha)}{4} \xi^3 I p_\alpha \quad . \quad (47)$$

The counter term is $y\xi p_\alpha$. Hence we choose,

$$\text{Div}(y) = -\frac{(3-\alpha)\xi^2 I}{4} \quad . \quad (48)$$

Thus far, we have determined the divergent parts of the renormalization constants. Now we shall verify the relations between divergent parts of

$$\Delta_\mu, \square_{\mu\nu}, \Pi_{\mu\nu}, \Gamma_{\alpha\mu\nu}^{(3)}, \Gamma_{\alpha\beta\mu\nu}^{(4)} \quad .$$

(D) $\Delta_\mu(p)$: The diagram for $\Delta_\mu(p)$ is shown in Fig. 7. It is found that,

$$\text{Div}(A) = \frac{3\xi}{2\alpha} (\alpha^2 - \alpha + 2) I \quad . \quad (49)$$

Then from Eqs. (49), (46) and (44) we easily verify Eq. [27(ii)] viz:

$$\text{Div}(A) = -\frac{2}{\xi} \text{Div}(w-\tilde{Z}) \quad . \quad (50)$$

(E) $\square_{\mu\nu}(q, r)$ and $\Pi_{\mu\nu}(q, r)$: These are defined in Fig. 7. Here, we obtain:

$$\begin{aligned} \text{Div}(B) &= \frac{3\xi^2(1+\alpha)}{2\alpha} I \\ \text{Div}(C) &= \frac{3\xi^2(3+\alpha^2)}{2\alpha} I \quad . \end{aligned} \quad (51)$$

Therefore, we can varify Eq. [27(i)], viz:

$$-\text{Div}(B + \frac{C}{2}) = \text{Div}(y+w-2\tilde{Z}) \quad . \quad (52)$$

(F) The 4-photon vertex:

Diagrams of Fig. 13 contribute to the 4-photon vertex $\Gamma_{\alpha\beta\mu\nu}^{(4)}$.

The results are,

$$\begin{aligned} \text{Div}\{m(a)\} &= \frac{5i\xi^4}{4\alpha} I A_{\mu\nu\alpha\beta} \\ \text{Div}\{m(b)\} &= \frac{-2i\xi^4(\alpha+2)}{\alpha^2} I A_{\mu\nu\alpha\beta} \\ \text{Div}\{m(c)\} &= \frac{-i\xi^4(17+2\alpha+5\alpha^2)}{4\alpha^2} I A_{\mu\nu\alpha\beta} \\ \text{Div}\{m(d)\} &= \frac{-i\xi^4}{4} I A_{\mu\nu\alpha\beta} \end{aligned}$$

where,

$$A_{\mu\nu\alpha\beta} = g_{\mu\nu} g_{\alpha\beta} + g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} \quad . \quad (53)$$

Hence, we can verify that the counter term $\left[-i(2y+w-2\tilde{z}) \frac{\xi^2}{\alpha} A_{\mu\nu\alpha\beta} \right]$ cancels the divergence in the 4-point vertex.

(G) 3-photon proper vertex:

Diagrams of Fig. 14 contribute to the three-photon proper vertex.

The results are,

$$\begin{aligned} \text{Div}\{m(a)\} &= -\frac{\xi^3}{\alpha^2} \left(\frac{3}{4} + \alpha \right) I F_{\mu\nu\sigma} \\ \text{Div}\{m(b)\} &= \frac{\xi^3}{\alpha^2} \left(\frac{9}{4} - \frac{\alpha}{2} + \frac{5}{4} \alpha^2 \right) I F_{\mu\nu\sigma} \\ \text{Div}\{m(c)+m(d)\} &= -\frac{\xi^3}{4} I F_{\mu\nu\sigma} \end{aligned} \quad (54)$$

where,

$$F_{\mu\nu\sigma} = p_\mu g_{\nu\sigma} + q_\nu g_{\mu\sigma} + r_\sigma g_{\mu\nu} \quad .$$

Then it is easy to verify that the total divergence in the $(A_\mu)^3$ vertex is cancelled by the counterterm:

$$-\frac{\xi}{2} (y+w-\tilde{z}) F_{\mu\nu\sigma} \quad .$$

IV. S-MATRIX

In this section, we shall show that the renormalized three and four photon S-matrix elements vanish.

First let us note that the polarization vector $\epsilon_\mu(p)$ of a physical photon ($p^2=0$) of momentum p satisfied $p \cdot \epsilon = 0$. $\left\{ \epsilon_\mu(p) \equiv \langle p | A_\mu(0) | 0 \rangle \right.$.

With the linear gauge condition, $\partial_\mu A^\mu = 0$, it immediately follows that $p \cdot \epsilon = 0$. In our case the gauge condition $f_\alpha[A] = \frac{1}{\sqrt{\alpha}} (\partial_\mu A^\mu - \xi A_\mu A^\mu) = 0$ means that matrix elements of $f_\alpha[A]$ between physical states vanish: $\langle p | f_\alpha[A] | 0 \rangle = 0$. This translates into $\epsilon^\mu [p_\mu - a \Delta_\mu(p)] = 0$ where a is some constant. Since $\Delta_\mu(p) = p_\mu \Delta(p)$; it follows that $p \cdot \epsilon = 0$.

It is easy to see that the three photon amplitude vanishes on mass shell. Three photons of momenta p, q, r (with $p+q+r=0$) can be on mass shell only when $p=\alpha q=\beta r$ for some α and β . Thus there is only one independent 4-vector. Any tensor with three Lorentz indices constructed out of it vanishes when dotted with polarization vectors.

Finally, we wish to show that the 4-photon amplitude vanishes on mass shell. Since the amplitude is a truncated Green's function, it is easier to use the WT identity for $Z[J]$, the generating functional of the connected Green's functions. Referring back to Eq. (16), we can write the WT identity for $Z^{(r)}[J^{(r)}]$; in terms of renormalized quantities dropping the superscript (r):

$$G_{\beta\alpha}[J] \partial_i^\beta J_i - \frac{1}{\alpha} \frac{W}{\tilde{Z}} \left\{ \partial_i^\alpha \frac{\delta Z}{\delta J_i} - \frac{Y}{2\tilde{Z}} \xi_{ij}^\alpha \left(\frac{\delta Z}{\delta J_i} \frac{\delta Z}{\delta J_j} - \frac{i \delta^2 Z}{\delta J_i \delta J_j} \right) \right\}. \quad (55)$$

Differentiating with respect to J_k, J_ℓ, J_m and setting $J=0$,

$$\frac{\delta^2 G_{\beta\alpha}}{\delta J_k \delta J_m} \partial_\ell^\beta + (2 \text{ permutations of } k, \ell, m) - \frac{1}{\alpha} \frac{W}{\tilde{Z}} \partial_i^\alpha \frac{\delta^4 Z}{\delta J_i \delta J_k \delta J_m \delta J_\ell}$$

$$\begin{aligned}
 &= -\frac{1}{\alpha} \frac{WY}{2\tilde{Z}^2} \xi_{ij}^\alpha \left\{ \frac{\delta^3 Z}{\delta J_i \delta J_k \delta J_\ell} \frac{\delta^2 Z}{\delta J_j \delta J_m} + \text{permutations} \right\} \\
 &\quad + \frac{iYW}{2\tilde{Z}^2} \xi_{ij}^\alpha \frac{\delta^5 Z}{\delta J_i \delta J_j \delta J_k \delta J_\ell \delta J_m} . \tag{56}
 \end{aligned}$$

We show the Fourier transform of the Eq(56) in Fig. 15. (A shaded box stands for connected truncated Green's function.)

The first term (and its permutations) does not contribute when dotted with polarization vectors since it is proportional to $s_{\lambda\mu}^{\nu}$. The first term on the right hand side does not contribute because it does not have a pole at $p^2=0$. One can verify that (at least) in one loop calculation the last term does not have a pole at $p^2=0$ that would contribute with on mass shell photons. Therefore, from Eq. (56) it follows that

$$\begin{aligned}
 &\lim_{p^2, q^2, r^2, s^2 \rightarrow 0} p^2 q^2 r^2 s^2 \epsilon^\mu(q) \epsilon^\nu(r) \epsilon^\lambda(s) p^\alpha G_{\alpha\mu\nu\lambda}^{(4)}(p, q, r, s) \\
 &\quad \equiv p^\alpha T_{\alpha\mu\nu\lambda}^{(4)} \epsilon^\mu(q) \epsilon^\nu(r) \epsilon^\lambda(s) = 0 . \tag{57}
 \end{aligned}$$

Here $G_{\alpha\mu\nu\lambda}^{(4)}(p, q, r, s)$ is a connected 4-photon Green's function .

Equation (57) is just the statement of gauge invariance of the T-matrix under an arbitrary gauge transformation $A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \omega(x)$. Since an $\omega(x)$ exists which can change $f[\alpha, \xi, A(x)] \rightarrow f[\alpha, \xi + d\xi, A(x)]$; it follows, in particular, that

$$\frac{\partial}{\partial \xi} \left[T_{\alpha\mu\nu\lambda}^{(4)}(p, q, r, s) \epsilon^\alpha(p) \epsilon^\mu(q) \epsilon^\nu(r) \epsilon^\lambda(s) \right] = 0 \quad . \quad (58)$$

Since we know that $T_{\alpha\mu\nu\lambda}^{(4)} = 0$ at $\xi = 0$; Eq. (58) tells us that

$$T_{\alpha\mu\nu\lambda}^{(4)}(p, q, r, s) \epsilon^\alpha(p) \epsilon^\mu(q) \epsilon^\nu(r) \epsilon^\lambda(s) = 0 \quad (59)$$

for any ξ .

V. RENORMALIZATION OF SPINOR ELECTRODYNAMICS

(V-1) In this section we shall consider a Dirac field (electron) interacting with the electromagnetic field quantized with the same gauge condition (of Eq. (3)). We shall show that we can remove the divergences in all the proper vertices by multiplicative renormalizations on the electron field and electric charge e , in addition to the renormalizations done in Sec. II, and by choosing a mass counterterm δm . We shall be brief.

The Lagrangian (in terms of unrenormalized fields and parameters) is,

$$\mathcal{L}(\psi, \bar{\psi}, A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi \quad (60)$$

\mathcal{L} is invariant under the local gauge transformations,

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \omega(x);$$

$$\psi(x) \rightarrow e^{-i\omega(x)} \psi(x) \quad , \quad \bar{\psi}(x) \rightarrow e^{i\omega(x)} \bar{\psi}(x) \quad . \quad (61)$$

We note that $M_{\alpha\beta}[A]$ of Eq. (5) is still unchanged and hence $\mathcal{L}_{\text{eff}}[\psi, \bar{\psi}; A_\mu; c, \bar{c}]$ is given by

$$\mathcal{L}_{\text{eff}}[\psi, \bar{\psi}; A_\mu; c, \bar{c}] = \left[\psi, \bar{\psi}, A_\mu \right] + \mathcal{L}_{\text{gauge}} + \bar{c}_\alpha M_{\alpha\beta} c_\beta . \quad (62)$$

We note that there are no basic ghost-electron vertices.

The generating functional of the Green's functions is now constructed by introducing sources (corresponding to fermion fields) η_α and $\bar{\eta}_\beta$. They anticommute among themselves and with the electron field.

We have,

$$W_F[J, \eta, \bar{\eta}] = \int \left[dA d\psi d\bar{\psi} dc d\bar{c} \right] \exp i \left\{ \mathcal{L}_{\text{eff}}(A; c, \bar{c}, \psi, \bar{\psi}) + J_i A_i + \bar{\psi}_i \eta_i + \bar{\eta}_i \psi_i \right\} . \quad (63)$$

We define fields $\bar{\chi}$ and χ , the expectation values of electron fields $\bar{\psi}$ and ψ by,

$$\bar{\chi}_\alpha = - \frac{\delta Z}{\delta \eta_\alpha} \quad \chi_\beta = \frac{\delta Z}{\delta \bar{\eta}_\beta} . \quad (64)$$

Define:

$$Z[J, \bar{\eta}, \eta] = -i \ln W[J, \eta, \bar{\eta}] .$$

We define the generating functional of proper vertices by,

$$\Gamma[\Phi; \bar{\chi}, \chi] = Z[J, \eta, \bar{\eta}] - \bar{\chi}_i \eta_i - \bar{\eta}_i \chi_i - J_i \Phi_i . \quad (65)$$

The inverse propagator for the electron field in presence of external sources is,

$$s_{ij}^{-1} = \frac{\delta^2 \Gamma}{\delta \chi_i \delta \bar{\chi}_j} \quad (66)$$

while, the propagator s_{ij} is given by,

$$s_{ij} = - \frac{\delta^2 Z}{\delta \bar{\eta}_i \delta \eta_j} \equiv - \bar{s}_{ji} \quad (67)$$

(V-2) WT Identities:

Let us obtain the WT identity for $W_F [J, \eta, \bar{\eta}]$. Following the procedure of Ref. 2 and noting the transformation properties of fields (Eq. 61), we obtain the following identity:

$$\left\{ -\frac{1}{\sqrt{\alpha}} f_{\alpha} \left[\frac{1}{i} \frac{\delta}{\delta J} \right] + \left[J_i \partial_i^{\beta} + ie \zeta_{ij}^{\beta} \left(\bar{\eta}_i \frac{\delta}{\delta \bar{\eta}_j} + \eta_i \frac{\delta}{\delta \eta_j} \right) \right] M_{\beta\alpha}^{-1} \left[\frac{1}{i} \frac{\delta}{\delta J_i}, -\frac{1}{i} \frac{\delta}{\delta \eta}, \frac{1}{i} \frac{\delta}{\delta \bar{\eta}} \right] \right\} \times W_F [J, \eta, \bar{\eta}] = 0$$

$$\left[\text{where } \zeta_{ij}^{\beta} = \delta^4(x_{\beta} - x_i) \delta^4(x_{\beta} - x_j) \delta_{ij} \right] \quad .$$

Going through the steps analogous to those of Sec. (II-2), we obtain:

$$-\frac{1}{\alpha} f_{\alpha} \left[\Phi + i \Delta \frac{\delta}{\delta \Phi} \right] + \left\{ \frac{\delta \Gamma}{\delta \Phi_i} \partial_i^{\beta} + e \zeta_{ij}^{\beta} \left[\frac{\delta \Gamma}{\delta \chi_i} \left(\chi_j + i s_{jk} \frac{\delta}{\delta \bar{\chi}_k} \right) - \frac{\delta \Gamma}{\delta \bar{\chi}_i} \left(\bar{\chi}_j + i \bar{s}_{jk} \frac{\delta}{\delta \chi_k} \right) \right] \right\} G_{\beta\alpha} [\Phi, \bar{\chi}, \chi] = 0 \quad .$$

Thus, the WT identity for generating functional of proper vertices

$\Gamma[\Phi, \bar{\chi}, \chi]$ is,

$$\begin{aligned} & \left[\delta_{\eta\beta} \frac{\delta\Gamma}{\delta\Phi_i} \delta_i^\beta + e \zeta_{ij}^\beta \left(\chi_j \frac{\delta\Gamma}{\delta\chi_i} - \bar{\chi}_j \frac{\delta}{\delta\bar{\chi}_i} \right) \delta_{\eta\beta} \right. \\ & \left. + ie \zeta_{ij}^\beta \left\{ s_{jk} \frac{\delta G_{\xi\eta}^{-1}}{\delta\bar{\chi}_k} G_{\beta\xi} \frac{\delta\Gamma}{\delta\bar{\chi}_i} - \bar{s}_{jk} \frac{\delta G_{\xi\eta}^{-1}}{\delta\chi_k} G_{\beta\xi} \frac{\delta\Gamma}{\delta\chi_i} \right\} \right] G_{\eta\alpha}[\Phi, \bar{\chi}, \chi] \\ & = -\frac{1}{\alpha} \left\{ \partial_i^\alpha \Phi_i - \frac{1}{2} \xi_{ij}^\alpha (\Phi_i \Phi_j + i \Delta_{ij}) \right\} . \end{aligned} \quad (69)$$

(V-3) Renormalization Transformations:

In addition to the renormalization transformations defined in Sec. (II-4), we define the following renormalizations on fields χ , $\bar{\chi}$ and on electric charge e ,

$$\begin{aligned} \chi &= Z^{\frac{1}{2}} \chi^{(r)} , & \bar{\chi} &= Z^{\frac{1}{2}} \bar{\chi}^{(r)} ; \\ e &= e^{(r)} X Z_{\chi}^{-1} Z^{-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} G_{\alpha\beta}[\Phi, \chi, \bar{\chi}; \alpha, \xi, e] &= \tilde{Z} G_{\alpha\beta}^{(r)}[\Phi^{(r)}, \chi^{(r)}, \bar{\chi}^{(r)}; \alpha^{(r)}, \xi^{(r)}, e^{(r)}] \\ \Gamma[\Phi, \chi, \bar{\chi}; \alpha, \xi, e] &= \Gamma^{(r)}[\Phi^{(r)}, \chi^{(r)}, \bar{\chi}^{(r)}; \alpha^{(r)}, \xi^{(r)}, e^{(r)}] . \end{aligned}$$

In the following we shall express everything in terms of renormalized quantities and drop the superscript (r) .

The WT identity of Eq. (69) becomes,

$$\left[\delta_{\eta\beta} \partial_i^\beta \frac{\partial \Gamma}{\delta \Phi_i} + \frac{eX}{Z_\chi} \zeta_{ij}^\beta \left(\chi_i \frac{\delta \Gamma}{\delta \chi_i} - \bar{\chi}_j \frac{\delta \Gamma}{\delta \bar{\chi}_j} \right) \delta_{\eta\beta} \right. \\ \left. + \frac{X}{Z_\chi} \left(\bar{K}_{\eta i} \frac{\delta \Gamma}{\delta \bar{\chi}_i} - K_{\eta i} \frac{\delta \Gamma}{\delta \chi_i} \right) \right] G_{\eta\alpha} = -\frac{1}{\alpha} \mathcal{F}_\alpha \quad (70)$$

where,

$$K_{\eta i} \equiv +ie \zeta_{ij}^\beta \bar{s}_{jk} \frac{\delta G_{\xi\eta}^{-1}}{\delta \chi_k} G_{\beta\xi} \\ \bar{K}_{\eta i} = ie \zeta_{ij}^\beta s_{jk} \frac{\delta G_{\xi\eta}}{\delta \bar{\chi}_k} G_{\beta\xi} \\ \mathcal{F}_\alpha = +\frac{W}{Z} \left\{ \partial_i^\alpha \Phi_i - \frac{Y}{2Z} \xi_{ij}^\alpha (\Phi_i \Phi_j + i\Delta_{ij}) \right\} \quad (71)$$

Diagrammatic representation for $\bar{K}_{\eta i}$ is shown in Fig. 17.

(V-4) Analysis of Divergences in $G^{-1}[\Phi, \chi, \bar{\chi}]$, $K_{\eta i}[\Phi, \chi, \bar{\chi}]$, $\mathcal{F}_\alpha[\Phi, \bar{\chi}, \chi]$:

(A) $G^{-1}[\Phi, \chi, \bar{\chi}]$:

Referring back to the discussion of Sec. II-5(A), (See Fig. 3 and Fig. 4), we need only worry about the internal subtractions for renormalization parts containing the rightmost vertex in Fig. 3. In introducing the fermion fields we do not introduce any such additional renormalization parts; since any diagram with two ghost lines and two or more fermion lines have superficial degree of divergence $D \leq -1$. Thus discussion of Secs. II-5(A) and II-5(B) goes through

(B) $\mathcal{F}_\alpha[\Phi, \chi, \bar{\chi}]$:

Here, too, we do not introduce any new renormalization parts in derivatives of $\mathcal{F}_\alpha[\Phi, \chi, \bar{\chi}]$ which contain the leftmost vertex denoted by

a cross. (See Figs. 7 and 8.) Here, too, any subdiagram containing this vertex and two or more fermion lines has $D \leq -1$. Hence, the discussion of Sec. II-5(C) goes through unchanged. Similar discussion,

as applied to $\frac{\delta^2 \mathcal{F}_\alpha}{\delta \bar{\chi}_i \delta \chi_j} \Big|_{\Phi=\chi=\bar{\chi}=0}$ shows that it becomes finite to n-loop

approximation once the appropriate choices of renormalization constants up to (n-1) loop approximation is made according to Eq. (26).

$$(C) K_{\eta i} [\Phi, \chi, \bar{\chi}] :$$

We shall show that $K_{\eta i}$ and $\bar{K}_{\eta i}$ become finite to n-loop approximation once the proper vertices to (n-1) loop approximation are made finite. It is clear that the lowest derivative of $K_{\eta i}$ which is nonzero at $\phi=\chi=\bar{\chi}=0$

is $\frac{\delta K_{\eta i}}{\delta \chi_j} \Big|_{\Phi=\chi=\bar{\chi}=0}$; since $K_{\eta i} [\phi=0=\chi=\bar{\chi}] = 0$. The first derivative

is shown in Fig. (18).

The blobs in Fig. (18) are made finite by renormalization counter-terms introduced up to (n-1) loop approximation and it needs subtractions for renormalization parts containing the rightmost vertex. But there are no such renormalization parts. A suspect renormalization part shown in Fig. 19 is not a renormalization part because the leftmost vertex on the ghost line within this subdiagram must be a $\bar{c}cA_\mu$ vertex and it contains a factor of external momentum q_μ (external to this subdiagram).

Thus this subdiagram has $D=-1$. Furthermore $\frac{\delta K_{\eta i}}{\delta \chi_j} \Big|_{\Phi=\chi=\bar{\chi}=0}$ is

it self not a renormalization part $\left[D=-1 \right]$ and hence it becomes finite in n-loop approximation once the counterterms are chosen up to (n-1) loop approximation. Similar discussion goes through for higher derivatives of $K_{\eta i}$ and also for $\bar{K}_{\eta i} \left[\Phi, \chi, \bar{\chi} \right]$.

(V-5) Proof of Renormalizability:

Here we shall deal only with the new renormalization part, the $\bar{\psi} \psi A_{\mu}$ vertex. The discussion for the remaining renormalization parts proceeds parallel to the discussion in Sec. II-6 and will not be repeated here.

We assume that renormalization constants and mass counterterm have been chosen up to (n-1) loop approximation making all the proper vertices finite up to (n-1) loop approximation. We assume that proper choice of $z_n, w_n, \bar{z}_n, z_{\chi(n)}, (\delta m)_n$ has been made making the photon, the ghost and the electron propagators finite to n-loop approximation. We shall show that it is possible to make $\bar{c} c A_{\mu}$ vertex finite with appropriate choices of x_n and that we may choose $x_n = (z_{\chi})_n$ if we have chosen $x_r = (z_{\chi})_r; 0 \leq r \leq n-1$; yielding the usual Ward identity.

Differentiate the WT identity of Eq. (70) with respect to χ_m and $\bar{\chi}_n$ and set $\Phi = \chi = \bar{\chi} = 0$. We get:

$$\begin{aligned} & \partial_i^{\beta} \frac{\delta^3 \Gamma}{\delta \Phi_i \delta \chi_m \delta \bar{\chi}_n} G_{\beta\alpha} + e \frac{X}{Z_{\chi}} \zeta_{im}^{\beta} \left[\frac{\delta^2 \Gamma}{\delta \bar{\chi}_n \delta \chi_i} - \zeta_{in}^{\beta} \frac{\delta^2 \Gamma}{\delta \chi_m \delta \bar{\chi}_i} \right] G_{\beta\alpha} \\ & + \frac{X}{Z_{\chi}} G_{\eta i} \left[\frac{\delta \bar{K}_{\eta i}}{\delta \bar{\chi}_m} \frac{\delta^2 \Gamma}{\delta \chi_n \delta \bar{\chi}_i} - \frac{\delta K_{\eta i}}{\delta \chi_n} \frac{\delta^2 \Gamma}{\delta \bar{\chi}_m \delta \chi_i} \right] = -\frac{1}{\alpha} \frac{\delta^2 \mathcal{F}_{\alpha}}{\delta \chi_m \delta \bar{\chi}_n} \quad (72) \end{aligned}$$

Equating the n-loop divergence on both sides,

$$\partial_i^\beta \left[\frac{\delta^3 \Gamma}{\delta \Phi_i \delta \chi_m \delta \bar{\chi}_n} \right]_n^{\text{div}} [G_{\beta\alpha}]_0 - e \left(\frac{X}{Z_\chi} \right)_n^{\text{div}} \left[+ \zeta_{im}^\beta S_{in}^{-1} - \zeta_{in}^\beta S_{mi}^{-1} \right]_0 [G_{\beta\alpha}]_0$$

$$= \text{finite.} \tag{73}$$

$$\left\{ \text{Since, } \left[\frac{\delta K_{\eta i}}{\delta \bar{\chi}_m} \right]_{\Phi=0=\chi=\bar{\chi}}^{\text{div}} = 0 = \left[\frac{\delta^2 \mathcal{F}_\alpha}{\delta \chi_m \delta \bar{\chi}_n} \right]_{\Phi=0=\chi=\bar{\chi}}^{\text{div}} \text{ etc.} \right\} .$$

It is clear from Eq. (73) that if we choose x_n such that

$$\left[\frac{X}{Z_\chi} \right]_n^{\text{div}} = \text{finite} \tag{74}$$

we will have

$$\left[\frac{\delta^3 \Gamma}{\delta \Phi_i \delta \chi_m \delta \bar{\chi}_n} \right]_n^{\text{div}} = \text{finite} .$$

Further, if we have chosen $x_r = (z_\chi)_r$, $0 \leq r \leq n-1$, then Eq. (74)

gives:

$$\text{Div}(x_n) = \text{Div} \left[(z_\chi)_n \right] .$$

and we may choose the finite part of x_n such that $x_n = (z_\chi)_n$.

VI: A COMPLEX SCALAR FIELD INTERACTING WITH ELECTROMAGNETIC FIELD:

In this section we shall discuss the renormalization of a complex scalar field interacting with electromagnetic field when the gauge

condition chosen is bilinear. We shall consider only the unbroken version of the theory ($\mu_{\text{ren}}^2 > 0$)

The Lagrangian in terms of unrenormalized fields is:

$$\mathcal{L} = |(\partial_{\mu} - ie A_{\mu})\phi|^2 - \mu^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (75)$$

\mathcal{L} is invariant under the electromagnetic gauge transformation:

$$A_{\mu}(x) \rightarrow A_{\mu}(x) - \frac{1}{e} \partial_{\mu} \omega(x) ; \quad \phi(x) \rightarrow e^{-i\omega(x)} \phi(x) , \quad \phi^*(x) \rightarrow e^{i\omega(x)} \phi^*(x) .$$

We shall show the following:

(i) With a simple counter example, we shall show that if we choose the previous gauge function $f[A] = \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2)$, it is not possible to make proper vertices finite by renormalization on fields and parameters ξ , e , α , μ^2 and λ .

(ii) However, that, if we choose the gauge function $f' = \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2 - \frac{1}{2} \eta \phi^* \phi)$ and renormalize parameter η independently, all the proper vertices can be made finite.

$$\text{(VI-1) The Gauge Function } f = \frac{1}{\sqrt{\alpha}} (\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2)$$

Since this discussion is similar to that in Sec. V, we shall be brief.

Let us introduce two sources K_a ($a=1$ or 2 ; x_a) corresponding to fields ϕ_a [$\phi_1 \equiv \phi^*(x)$, $\phi_2 \equiv \phi(x)$]. The generating functional of the Green's functions is given by:

$$W[J, K] = \int [dA_i d\phi_a d\bar{c} d\bar{c}] \text{Exp } i \left\{ \mathcal{L}_{\text{eff}}[A_i, \phi_a, c, \bar{c}] + J_i A_i + K_a \phi_a \right\}$$

$$Z[J, K] \equiv -i \ln W[J, K]. \quad (76)$$

As before, we define expectation values Ψ_a for field ϕ_a in presence of external sources by,

$$\Psi_a = \delta Z[J, K] / \delta K_a .$$

We note that under an infinitesimal gauge transformation,

$$A_i \rightarrow A_i + \frac{1}{e} \partial_i^\beta \omega_\beta ; \quad \phi_a \rightarrow \phi_a + i \zeta_{ab}^\beta \phi_b \omega_\beta \quad (77)$$

$$\left[\zeta_{ab}^\alpha \equiv \delta^4(x_a - x_b) \delta^4(x_\alpha - x_a) \zeta_{ab} ; \quad \zeta_{11} = -\zeta_{22} = 1 ; \quad \zeta_{12} = \zeta_{21} = 0 \right].$$

Following the derivation of WT identity, we obtain the following WT identity for $\Gamma[\Phi, \Psi]$:

$$G_{\beta\alpha} \left[\partial_i^\beta \frac{\delta}{\delta \Phi_i} - e \zeta_{dc}^\sigma \left(\Psi_c^\delta \delta_{\beta\sigma} - i P_{ca} G_{\sigma\xi} \frac{\delta G_{\xi\beta}^{-1}}{\delta \Psi_a} \right) \frac{\delta \Gamma}{\delta \Psi_\alpha} \right]$$

$$= -\frac{1}{\alpha} \left\{ \partial_i^\alpha \Phi_i - \frac{1}{2} \xi_{ij}^\alpha (\Phi_i \Phi_j + i \Delta_{ij}) \right\} \quad (78)$$

$$\left[\text{where, } P_{ca} \equiv -\frac{\delta^2 Z}{\delta K_c \delta K_a} = \text{propagator of } \phi \text{ field.} \right].$$

To exhibit the difficulty, let us consider Eq. (78) up to one loop approximation. Since there are no basic $(\bar{c}c\phi)$ vertices, the term $i e P_{ca} G_{\sigma\xi} \frac{\delta G_{\xi\beta}^{-1}}{\delta \Psi_a} \zeta_{dc}^\sigma$ contains at least two loops and hence will be dropped in this consideration.

Introduce the renormalization transformation identical to those of Sec. (II-4); in addition to the renormalization of e and the scalar field,

$$\Psi = Z_{\psi}^{\frac{1}{2}} \Psi^{(r)} \quad ; \quad e = e_r X Z_{\psi}^{-1} Z^{-\frac{1}{2}} \quad . \quad (79)$$

Then expressing Eq. (80) to one loop approximation, [in terms of renormalized quantities, dropping superscript (r)] ,

$$\begin{aligned} G_{\beta\alpha} & \left[\partial_i^{\beta} \frac{\delta \Gamma}{\delta \Phi_i} - e \frac{X}{Z_{\psi}} \zeta_{dc}^{\beta} \frac{\Psi_c}{\Psi_d} \frac{\delta \Gamma}{\delta \Psi_d} \right] \\ & = - \frac{1}{\alpha} \frac{W}{\tilde{Z}} \left\{ \partial_i^{\alpha} \Phi_i - \frac{Y}{2\tilde{Z}} \xi_{ij}^{\alpha} (\Phi_i \Phi_j + i\Delta_{ij}) \right\} . \end{aligned} \quad (80)$$

Let us write $Z = 1+z$, $W = 1+w$, etc., etc.

Suppose, we have chosen a , w , $\delta\mu^2$, z_{ψ} , \tilde{Z} to make the photon, scalar and ghost propagators finite to one loop approximation. Then we shall show that charge renormalization alone cannot remove the divergence in $(\phi^* \phi A_{\mu})$ vertex. Essentially, this happens because the $(\phi^* \phi A_{\mu})$ vertex in one loop approximation has a divergence proportional to photon momentum in addition to the divergence of the form of the bare vertex.

Differentiating Eq. (80) with respect to Ψ_a and Ψ_e and setting $\Phi = 0 = \Psi$ (the vacuum expectation values), we obtain:

$$\begin{aligned} G_{\beta\alpha} \partial_i^{\beta} \frac{\delta^3 \Gamma}{\delta \Phi_i \delta \Psi_a \delta \Psi_e} - e(x-z_{\psi}) G_{\beta\alpha} & \left[\zeta_{da}^{\beta} \frac{\delta^2 \Gamma}{\delta \Psi_e \delta \Psi_d} + \zeta_{de}^{\beta} \frac{\delta^2 \Gamma}{\delta \Psi_a \delta \Psi_d} \right] \\ & = \frac{i}{2\alpha} \left(\frac{WY}{\tilde{Z}^2} \right) \xi_{ij}^{\alpha} \frac{\delta^2 \Delta_{ij}}{\delta \Psi_a \delta \Psi_e} . \end{aligned} \quad (81)$$

Let us choose $a = 1, e = 2$ in Eq. (81).

Remembering that the propagators are made finite to one loop approximation, we may equate the divergence on both sides of Eq. (81) in one loop approximation:

$$\begin{aligned} [G_{\beta\alpha}]_0 \partial_i^\beta \left[\frac{\delta^3 \Gamma}{\delta \Phi_i \delta \Psi_a \delta \Psi_e} \right]_1^{\text{div}} - e(x-z)_\psi [G_{\beta\alpha}]_0 \left[\zeta_{da}^\beta \frac{\delta^2 \Gamma}{\delta \Psi_e \delta \Psi_d} + (a \leftrightarrow e) \right]_0 \\ = \frac{i}{2\alpha} \left[\xi_{ij}^\alpha \frac{\delta^2 \Delta_{ij}}{\delta \Psi_a \delta \Psi_e} \right]_1^{\text{div}} + \text{finite} . \end{aligned} \quad (82)$$

Let us define:

$$\text{F. T.} \left\{ \frac{\delta^3 \Gamma}{\delta \Phi_i \delta \Psi_a \delta \Psi_e} \right\} = i \Gamma_\mu^{(3)}(p; q, r)$$

$$\text{Then} \quad \left[\Gamma_\mu^{(3)}(p; q, r) \right]_1^{\text{div}} = (q+r)_\mu b(\epsilon) + (q-r)_\mu c(\epsilon)$$

$$\text{F. T.} \left\{ \frac{\delta^2 \Gamma}{\delta \Psi_d \delta \Psi_e} \right\}_0 = r^2 - \mu^2 ;$$

$$\text{F. T.} \left\{ \frac{\delta^2 \Gamma}{\delta \Psi_a \delta \Psi_d} \right\} = q^2 - \mu^2 .$$

$$\text{F. T.} \left\{ G_{\beta\alpha} \right\}_0 = \frac{1}{p^2} ; \quad \text{F. T.} \left\{ \xi_{ij}^\alpha \frac{\delta^2 \Delta_{ij}}{\delta \Psi_a \delta \Psi_e} \right\}_1^{\text{div}} \equiv \Sigma(\epsilon) .$$

Then Eq. (82) becomes,

$$b(\epsilon) + \frac{q^2 - r^2}{(q+r)^2} c(\epsilon) - e(x-z)_\psi \frac{(r^2 - q^2)}{(r+q)^2} = \frac{1}{2\alpha} \Sigma(\epsilon) + \text{finite} . \quad (83)$$

Thus, we can make $c(\epsilon)$ finite by choosing $x = z_\chi + \text{finite}$. Also,

$$b(\epsilon) = \frac{1}{2\alpha} \Sigma(\epsilon) + \text{finite} \quad .$$

An explicit calculation of $\Sigma(\epsilon)$ shows that it is divergent. Hence, it follows that the $\phi^* \phi A_\mu$ vertex will necessarily contain divergence.

[We show the graphs contributing to $\Sigma(\epsilon)$ in Fig. 20.]

We may express this in another way. We saw in Eq. (83) that a derivative of $\mathcal{F}_\alpha[\Phi, \Psi]$ viz. $\left. \frac{\delta^2 \mathcal{F}_\alpha}{\delta \Psi_a \delta \Psi_e} \right|_{\substack{\Phi=0 \\ \Psi=0}}$ contained (nonrenormalizable)

divergence and by virtue of WT identity, there must be divergence in its left hand side which must come from a proper vertex. The reason why

$\mathcal{F}_\alpha[\Phi, \Psi]$ cannot be made finite as against the previous two cases is that

the derivatives of $\mathcal{F}_\alpha[\Phi, \Psi]$ at $\Phi = \Psi = 0$ need additional internal

subtractions, (in addition to those shown in Fig. 8. See discussion of

Sec. II-5). For example, we may consider $\Delta_\mu(p)$ defined in Fig. 7.

The additional subtractions needed are shown in Fig. 21 (a,b). Clearly

these subtractions cannot be expressed as (divergent constant) $X \Delta_\mu(p)$;

unlike the subtractions in Fig. 8. These are rather generated out of

derivatives of a loop consisting of a scalar propagator. See Fig. 21(c).

This suggests that we modify the gauge functional to $f' = \frac{1}{\sqrt{\alpha}} \left[\partial_\mu A^\mu - \frac{1}{2} \xi A^2 - \frac{1}{2} \eta \phi^* \phi \right]$. Then we may be able to make $f_\alpha \left[\Phi + i\Delta \frac{\delta}{\delta \Phi}, \Psi + iP \frac{\delta}{\delta \Psi} \right] \cdot 1$ finite.

(VI-2) The Gauge Condition $f' = \frac{1}{\sqrt{\alpha}} \left[\partial_{\mu} A^{\mu} - \frac{1}{2} \xi A^2 - \frac{1}{2} \eta \phi^* \phi \right]$.

Let us choose the gauge functional

$$f'_{\alpha}[A, \Phi] = \frac{1}{\sqrt{\alpha}} \left\{ \partial_i^{\alpha} A_i - \frac{1}{2} \xi_{ij}^{\alpha} A_i A_j - \frac{1}{2} \eta_{ab}^{\alpha} \phi_a \phi_b \right\} . \quad (84)$$

In this section, we shall use hermitian fields.

$$\phi_1 = \frac{\phi + \phi^*}{2} \quad \phi_2 = \frac{\phi - \phi^*}{2i}$$

[so that $\eta_{ab}^{\alpha} = \delta^4(x_{\alpha} - x_a) \delta^4(x_a - x_b) \eta_{ab}$; $\eta_{11} = \eta_{22} = 1$; $\eta_{12} = \eta_{21} = 0$]

Since the last term is gauge invariant, $M_{\alpha\beta}$ of Eq. (5) is unchanged; i. e., the ghost Feynman rules are unchanged. The Feynman rules for $\phi^* \phi A_{\mu}$, $(\phi^* \phi)^2$ and $(\phi^* \phi A^2)$ vertices are changed. $G^{-1}[\Phi, \Psi]$ is still given by the same formal expression of Eq. (14). The new WT identity of $\Gamma[\Phi, \Psi]$ is,

$$\begin{aligned} G_{\beta\alpha} & \left[\partial_i^{\beta} \frac{\delta \Gamma}{\delta \Phi_i} - e \xi_{ac}^{\sigma} \left(\delta_{\sigma\beta} \Psi_c - i P_{ce} G_{\sigma\xi} \frac{\delta G_{\xi\beta}^{-1}}{\delta \Psi_e} \right) \frac{\delta \Gamma}{\delta \Psi_a} \right] \\ & = - \frac{1}{\alpha} f'_{\alpha} \left[\Phi + i\Delta \frac{\delta}{\delta \Phi}, \Psi + i P \frac{\delta}{\delta \Psi} \right] \cdot 1 \\ & = - \frac{1}{\alpha} \left\{ \partial_i^{\alpha} \Phi_i - \frac{1}{2} \xi_{ij}^{\alpha} (\Phi_i \Phi_j + i\Delta_{ij}) - \frac{1}{2} \eta_{bc}^{\alpha} (\Psi_b \Psi_c + i P_{bc}) \right\} \quad (85) \end{aligned}$$

[$\xi_{12} = -\xi_{21} = 1$, $\xi_{11} = \xi_{22} = 0$] .

We define renormalized fields, parameters and renormalization constants by

$$\begin{aligned}\Psi_a &= Z^{\frac{1}{2}} \Psi_a^{(r)} & \eta &= \eta^{(r)} V & e &= e^{(r)} X Z_{\psi}^{-1} Z^{-\frac{1}{2}} \\ \mu^2 &= \mu_a^2(r) + \delta \mu^2\end{aligned}\quad (86)$$

in addition to those defined in Sec. (II-4).

Expressing WT identity in terms of renormalized quantities (and dropping the suffix (r)),

$$\begin{aligned}G_{\beta\alpha} \left[\partial_i^{\beta} \frac{\delta \Gamma}{\delta \Phi_i} + e \frac{X}{Z_{\psi}} \zeta_{ac}^{\sigma} \left(\delta_{\sigma\beta} \Psi_c - i P_{ce} G_{\sigma\xi} \frac{\delta G_{\xi\beta}^{-1}}{\delta \Psi_e} \right) \frac{\delta \Gamma}{\delta \Psi_a} \right] \\ = - \frac{1}{\alpha} \mathcal{F}_{\alpha} [\Phi, \Psi]\end{aligned}\quad (87)$$

where,

$$\begin{aligned}\mathcal{F}_{\alpha} [\Phi, \Psi] &= \frac{W}{\tilde{Z}} \left\{ \partial_i^{\alpha} \Phi_i - \frac{Y}{2\tilde{Z}} \xi_{ij}^{\alpha} (\Phi_i \Phi_j + i \Delta_{ij}) \right. \\ &\quad \left. - \frac{1}{2} W \eta_{bc}^{\alpha} (\Psi_b \Psi_c + i P_{bc}) \right\} .\end{aligned}\quad (88)$$

We define:

$$L_a^{\beta} = e \frac{X}{Z_{\psi}} \left[\zeta_{ab}^{\beta} \Psi_b - i \zeta_{ac}^{\sigma} P_{ce} G_{\sigma\xi} \frac{\delta G_{\xi\beta}^{-1}}{\delta \Psi_e} \right] .\quad (89)$$

Then the WT identity reads:

$$G_{\beta\alpha} \left[\partial_i^{\beta} \frac{\delta \Gamma}{\delta \Phi_i} + L_a^{\beta} \frac{\delta \Gamma}{\delta \Psi_a} \right] = - \frac{1}{\alpha} \mathcal{F}_{\alpha} [\Phi, \Psi]\quad (90a)$$

$$\text{i. e. , } \left[\partial_i^{\beta} \frac{\delta \Gamma}{\delta \Phi_i} + L_a^{\beta} \frac{\delta \Gamma}{\delta \Psi_a} \right] = - \frac{1}{\alpha} \mathcal{F}_{\alpha} [\Phi, \Psi] G_{\alpha\beta}^{-1} .\quad (90b)$$

The second term in L_a^{β} [Eq. (89)] is identical in form to $K[\Phi, \chi, \bar{\chi}]$ of Sec. (V-3) and has the same diagrammatic representation (See Fig. 17).

We shall see that, like the general linear gauge (discussed in Ref. 2), the scalar field develops a vacuum expectation value even though $\mu_{\text{ren}}^2 > 0$.

Consider the WT identity of Eq. (90a) for $\Phi_i = 0$, $\Psi_a = u_a$, where u_a are real constants. Let v_a be the vacuum expectation value of ϕ_a . In vacuum,

$$\left. \frac{\delta \Gamma}{\delta \Psi_a} \right|_{\text{vac.}} \equiv 0 \quad ; \quad \frac{\delta \Gamma}{\delta \Phi_i} \equiv 0 \quad . \quad (91)$$

[Here, the meaning of $\left. \frac{\delta \Gamma}{\delta \Phi_i} \right|_{\text{vac.}} \equiv 0$ should be carefully noted.

For $\Phi = 0$, $\Psi_a = u_a$; $\frac{\delta \Gamma}{\delta \Phi_i}$ has, in momentum space, the form,

$$\text{F. T.} \left\{ \left. \frac{\delta \Gamma}{\delta \Phi_i} \right|_{\Phi=0, \Psi_a=u_a} \right\} = p_\mu J(p^2, \sqrt{u_1^2 + u_2^2}) \Big|_{p \rightarrow 0}$$

and thus it is zero for any

u_a . However, it is not true that $J(p^2, \sqrt{u_1^2 + u_2^2}) \Big|_{p \rightarrow 0}$ is zero for any

u_a . In vacuum, $J(p^2, \sqrt{v_1^2 + v_2^2}) = 0$; so that

$$\begin{aligned} \text{F. T.} \left\{ G_{\beta\alpha} \partial_i^\beta \left. \frac{\delta \Gamma}{\delta \Phi_i} \right|_{\substack{\Phi=0 \\ \Psi_a=v_a}} \right\} &\sim \frac{1}{p} p_\mu p^\mu J(p^2, \sqrt{v_1^2 + v_2^2}) \Big|_{p \rightarrow 0} \\ &= J(p^2, \sqrt{v_1^2 + v_2^2}) \Big|_{p \rightarrow 0} \text{ is zero too.} \end{aligned}$$

Hence from Eq. (90a) we obtain that v_a satisfies:

$$\mathcal{F}_\alpha \left[\Phi = 0, \Psi_a = v_a \right] = 0 \quad . \quad (92)$$

v_a is, in general, nonzero and ϵ -dependent. (i.e., infinite in the limit $\epsilon \rightarrow 0$). v_a 's are to be determined from solutions of

$$\left. \frac{\delta \Gamma}{\delta \Psi_a} \right|_{\Psi_a = v_a, \Phi = 0} = 0 . \quad (93)$$

We, then, define the shifted fields Ψ'_a by,

$$\Psi_a = v_a + \Psi'_a$$

and make this substitution in the effective action. The proper vertices of the theory are obtained by expanding $\Gamma[\Phi_i, \Psi'_a]$ around $\Phi_i = 0$, $\Psi'_a = 0$.

We note that as a result of the substitution $\Psi_a = \Psi'_a + v_a$ in the effective Lagrangian, there are new vertices $(\phi' A_\mu, \phi' A^2, \phi'^3, \phi')$ created in higher orders. All these vertices have dimensions three or lower.

We note that the presence of these vertices do not create any new renormalization subdiagrams in $G^{-1}[\Phi, \Psi']$, $L[\Phi, \Psi']$ and $\mathcal{F}'_\alpha[\Phi, \Psi']$, of the kind that would need further internal subtractions (i.e., subtractions not taken care of by renormalization counter-terms \rightarrow See the discussion of Sec. II-5 and Sec. V-4). This follows because, as mentioned earlier such renormalization parts have $D = 0$ at most, and inclusion of any of the new vertices lower D by one.

The presence of these vertices creates new renormalization parts in derivatives of $\Gamma[\Phi, \Psi']$ and $\mathcal{F}'_\alpha[\Phi, \Psi']$, they are: $\frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Psi'_a}$, $\frac{\delta^3 \Gamma}{\delta \Psi'_a \delta \Phi_i \delta \Phi_j}$,

$$\frac{\delta^3 \Gamma}{\delta \Psi_a' \delta \Psi_b' \delta \Psi_c'} ; \frac{\delta \mathcal{F}'_\alpha}{\delta \Psi_a'}$$

We need to show that these become finite with others.

$$\left. \frac{\delta^2 \mathcal{F}'_\alpha}{\delta \Phi_i \delta \Psi_a'} \right|_{\Phi=0=\Psi_a'} \quad \text{and} \quad \left. \frac{\delta G_{\alpha\beta}^{-1}}{\delta \Psi_a'} \right|_{\Phi=0=\Psi_a'}$$

are not renormalization parts.

Taking these facts into account, we can carry out the analysis of divergences in $G_{\alpha\beta}^{-1}[\Phi, \Psi_a']$, $L_a^\alpha[\Phi, \Psi_a']$ and $\mathcal{F}'_\alpha[\Phi, \Psi_a']$ analogous to that in sections (II-5) and (V-4). In the discussion for $\mathcal{F}'_\alpha[\Phi, \Psi_a']$, we only need to remember the presence of additional subtractions needed which are shown in Fig. 21. Qualitatively, the result is the same, namely, with appropriate choices of W, Y and V in each loop

approximation, $\left. \frac{\delta \mathcal{F}'_\alpha}{\delta \Phi_i} \right|_{\Phi=0=\Psi_a'}$, $\left. \frac{\delta^2 \mathcal{F}'_\alpha}{\delta \Psi_i' \delta \Psi_j'} \right|_{\Phi=0=\Psi_a'}$ and higher derivatives of \mathcal{F}'_α can be made finite. $\left. \frac{\delta^2 \mathcal{F}'_\alpha}{\delta \Phi_i \delta \Psi_a'} \right|_{\Phi=0=\Psi_a'}$ also becomes finite, since

it is not a renormalization part. This, however, does not apply to

$$\left. \frac{\delta \mathcal{F}'_\alpha}{\delta \Psi_a'} \right|_{\Phi=0=\Psi_a'}$$

. Also, derivatives of $L_a^\alpha[\Phi, \Psi_a']$ become finite in n-loop

approximation once the counter terms up to (n-1) loop approximation

are chosen to make $\Gamma[\Phi, \Psi_a']$ and $G[\Phi, \Psi_a']$ finite. $L_a^\alpha[\Phi=0=\Psi_a']$ may

contain divergence, which in momentum space, is independent of external momentum.

To prove renormalizability, let us assume that the counter terms

chosen up to (n-1) loop approximation, and the choice of v up to (n-1) loop approximation make derivatives of $\Gamma[\Phi, \Psi'_a]$, $G_{\alpha\beta}^{-1}[\Phi, \Psi'_a]$, $L_a^\alpha[\Phi, \Psi'_a]$ and $\mathcal{F}'_\alpha[\Phi, \Psi'_a]$ around $\Phi = \Psi'_a = 0$ finite. Then we have to show that we can choose the counter terms to n loop approximation (and determine v to n-loop approximation) which will make the derivatives of $\Gamma[\Phi, \Psi'_a]$, $G_{\alpha\beta}^{-1}[\Phi, \Psi'_a]$, $L_a^\alpha[\Phi, \Psi'_a]$ and $\mathcal{F}'_\alpha[\Phi, \Psi'_a]$ finite in n-loop approximation.

Let us choose $z_{(n)}$, \tilde{z}_n so as to make the transverse part of the photon propagator and ghost propagator finite in n-loop approximation.

Let us further choose divergent parts of $w_{(n)}$, $y_{(n)}$ and $v_{(n)}$ so as to

make
$$\left. \frac{\delta \mathcal{F}'_\alpha}{\delta \Phi_i} \right|_{\Phi=0=\Psi'_a}, \quad \left. \frac{\delta^2 \mathcal{F}'_\alpha}{\delta \Phi_i \delta \Phi_j} \right|_{\Phi=0=\Psi'_a} \quad \text{and} \quad \left. \frac{\delta^2 \mathcal{F}'_\alpha}{\delta \Psi'_a \delta \Psi'_b} \right|_{\Phi=0=\Psi'_a}$$

finite in n-loop approximation.

Differentiate Eq. (90b) with respect to Ψ'_b and set $\Phi_i = 0 = \Psi'_a$.

$$\partial_i^\beta \frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Psi'_b} + L_a^\beta \frac{\delta^2 \Gamma}{\delta \Psi'_a \delta \Psi'_b} = -\frac{1}{\alpha} G_{\alpha\beta}^{-1} \frac{\delta \mathcal{F}'_\alpha}{\delta \Psi'_b} \quad . \quad (94)$$

In momentum space, the right hand side and the first term on the left hand side are proportional to p^2 . Since $\frac{\delta^2 \Gamma}{\delta \Psi'_a \delta \Psi'_b}$ does not have a zero in p^2 (for any a and b) it follows that $L_a^\beta(p^2)$ [for both a=1, 2] and in particular $[L_a^\beta(p^2)]_n^{\text{div}}$ are proportional to p^2 . [This can be seen more easily if one performs a global U(1) transformation on

ϕ_1, ϕ_2 such that only one of them has a vacuum expectation value = $\sqrt{\frac{v_1^2 + v_2^2}{2}}$.] But since $[L_a^\beta(p^2)]_n^{\text{div}}$ must be a constant independent of p^2 , it must be finite.

Differentiate Eq. (90b) with respect to Φ_j and set $\Phi=0=\Psi'_a$

$$\partial_i^\beta \frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Phi_j} + L_a^\beta \frac{\delta^2 \Gamma}{\delta \Psi'_a \delta \Phi_j} = - \frac{1}{\alpha} \frac{\delta \mathcal{F}'_\alpha}{\delta \Phi_j} G_{\alpha\beta}^{-1} . \quad (95)$$

We equate the n-loop divergence on both sides of Eq. (95) in momentum space, noting

$$\left[\frac{\delta^2 \Gamma}{\delta \Psi'_a \delta \Phi_j} \right]_0 = 0 , \quad [G_{\alpha\beta}^{-1}]_n = \text{finite} , \quad \left[\frac{\delta \mathcal{F}'_\alpha}{\delta \Phi_j} \right]_n = \text{finite} ,$$

we obtain:

$$p^\mu [\Gamma_{\mu\nu}(p)]_n^{\text{div}} = \text{finite}.$$

Hence the longitudinal part of the photon propagator is also finite. Further, since $L_a^\beta(p^2)$ and $G^{-1}(p^2)$ are proportional to p^2 , we find, from Eq. (95),

$$p^\mu \Gamma_{\mu\nu} \propto p^2 p_\nu .$$

Hence, the photon mass is zero.

Henceforth, let us use a compact notation for derivatives of Γ , G , L , \mathcal{F}'_α , for convenience. The letters a, b, c, \dots will be used for the scalar fields, letters i, j, \dots will be used for the photon field. Thus

$$\Gamma_{aj} \equiv \frac{\delta^2 \Gamma}{\delta \Phi_j \delta \Psi'_a} \Bigg|_{\Phi=0=\Psi'_a} \quad L_{a,j}^\alpha \equiv \frac{\delta L_a^\alpha}{\delta \Phi_j} \Bigg|_{\Phi=0=\Psi'_a} \quad \text{etc, etc.}$$

Differentiate the WT identity of Eq. (90b) with respect to Ψ'_b and Φ_j and set $\Psi' = 0 = \Phi$.

$$\begin{aligned} & \partial_i^\beta \Gamma_{bij} + L_a^\beta \Gamma_{abj} + L_{a,j}^\beta \Gamma_{ab} + L_{a,b}^\beta \Gamma_{aj} \\ &= -G_{\alpha\beta}^{-1} \frac{1}{\alpha} \mathcal{F}'_{\alpha,bj} - \frac{1}{\alpha} \mathcal{F}'_{\alpha,b} G_{\alpha\beta,j}^{-1} - \frac{1}{\alpha} \mathcal{F}'_{\alpha,j} G_{\alpha\beta,b}^{-1} . \end{aligned} \quad (96)$$

Now, $L_{a,j}^\beta$, $L_{a,b}^\beta$, $\mathcal{F}'_{\alpha,bj}$ and $G_{\alpha\beta,b}^{-1}$ are not renormalization parts and $[L_a^\beta]_0 = [L_{a,j}^\beta]_0 = [\mathcal{F}'_{\alpha,b}]_0 = [\Gamma_{aj}]_0 = 0$. Hence, equating the n-loop divergence on both sides of Eq. (96), we get

$$\partial_i^\beta [\Gamma_{bij}]_n^{\text{div}} + [L_{ab}^\beta]_0 [\Gamma_{aj}]_n^{\text{div}} = -\frac{1}{\alpha} [\mathcal{F}'_{\alpha,b}]_n^{\text{div}} [G_{\alpha\beta,j}^{-1}]_0 . \quad (97)$$

We define:

$$\text{F.T.} \{ \Gamma_{bij} \} = -i \Gamma_{\mu\nu}^b(p, q, r) \quad \text{F.T.} \{ \Gamma_{aj} \} = r_\nu \Gamma^{(a)}(r) .$$

Then, using $[L_{ab}^\beta]_0 = e \zeta_{ab}$, Eq. (97) yields:

$$p^\mu [\Gamma_{\mu\nu}^b(p, q, r)]_n^{\text{div}} + e \zeta_{ab} r_\nu [\Gamma^a(r)]_n^{\text{div}} = -\frac{\xi}{\alpha} r_\nu [\mathcal{F}'_{ab}]_n^{\text{div}} . \quad (98)$$

Since,

$$[\Gamma_{\mu\nu}^b(p, q, r)]_n^{\text{div}} = g_{\mu\nu} \times (\text{divergent constant}) ,$$

It follows from Eq. (98) that

$$[\Gamma_{\mu\nu}^b(p, q, r)]_n^{\text{div}} = \text{finite}$$

and,

$$e \zeta_{ab} [\Gamma^a(r)]_n^{\text{div}} = -\frac{1}{\alpha} \xi [\mathcal{F}'_{\alpha,b}]_n^{\text{div}} + \text{finite} \quad (99)$$

while Eq. (94) gives:

$$r^2 \left[\Gamma^b(r) \right]_n^{\text{div}} = -\frac{1}{\alpha} \xi \left[\mathcal{F}'_{\alpha, b} \right]_n^{\text{div}_2} r^2 + \text{finite} . \quad (100)$$

Equation (99) and Eq. (100) imply:

$$\left[\Gamma^b(r) \right]_n^{\text{div}} = \text{finite} ; \quad \left[\mathcal{F}'_{\alpha, b} \right]_n^{\text{div}} = \text{finite} . \quad (101)$$

Differentiate Eq. (98) with respect to Ψ'_b , Ψ'_c and set $\Phi=0=\Psi'$:

$$\begin{aligned} & \partial_i^\beta \Gamma_{bci} + L_a^\beta \Gamma_{abc} + L_{a,b}^\beta \Gamma_{ac} + L_{a,c}^\beta \Gamma_{ab} \\ & = -\frac{1}{\alpha} G_{\alpha\beta}^{-1} \mathcal{F}'_{\alpha, bc} - \frac{1}{\alpha} G_{\alpha\beta, b}^{-1} \mathcal{F}'_{\alpha, c} - \frac{1}{\alpha} \mathcal{F}'_{\alpha, b} G_{\alpha\beta, c}^{-1} \end{aligned} \quad (102)$$

Equating the n-loop divergence on both sides of Eq. (102),

$$\begin{aligned} & \partial_i^\beta \left[\Gamma_{bci} \right]_n^{\text{div}} + e \zeta_{ab} \left[\Gamma_{ac} \right]_n^{\text{div}} + e \zeta_{ac} \left[\Gamma_{ab} \right]_n^{\text{div}} \\ & + e \left(\frac{X}{Z_\psi} \right) \left\{ \zeta_{ab} \left[\Gamma_{ac} \right]_0 + \zeta_{ac} \left[\Gamma_{ab} \right]_0 \right\} = \text{finite} . \end{aligned} \quad (103)$$

Choose $b = 1$, $c = 2$ $\left[\zeta_{12} = -\zeta_{21} = 1 \right]$. In this case, we express

Eq. (103) in momentum space using:

$$i\Gamma_\mu^{bc}(p; q, r) \equiv \text{F. T.} \left\{ \Gamma_{bci} \right\} = +i \left[A(q+r)_\mu + B(q-r)_\mu \right]$$

$$\text{F. T.} \left[\Gamma_{ac} \right]_n^{\text{div}} = r^2 \left(c_2 + z_{\psi(n)} \right) + \left(D_2 - \delta_{\mu n}^2 \right)$$

$$\text{F. T.} \left[\Gamma_{ab} \right]_n^{\text{div}} = q^2 \left(c_1 + z_{\psi(n)} \right) + \left(D_1 - \delta_{\mu n}^2 \right)$$

$$\text{F. T.} \left[\Gamma_{ab} \right]_0 = (q^2 - \mu^2)$$

$$\text{F. T.} \left[\Gamma_{ac} \right]_0 = r^2 - \mu^2 \quad .$$

Then we obtain:

$$\begin{aligned} & A(q+r)^2 + B(q^2 - r^2) + e \left[q^2 (c_1 + z_{\psi(n)}) - r^2 (c_2 + z_{\psi(n)}) \right] \\ & + e \left(\frac{X}{Z_{\psi_n}} \right) (q^2 - r^2) + (D_2 - D_1) = \text{finite} \quad . \end{aligned} \quad (104)$$

Hence, it follows that:

$$A = \text{finite}$$

$$D_1 = D_2 + \text{finite} \quad (105)$$

$$C_1 = C_2 + \text{finite} \quad . \quad (106)$$

From Eq. (105) and Eq. (106), it follows that a mass renormalization term and a wave function renormalization term of the form $-\delta_{\mu_n}^2 (\phi_1^2 + \phi_2^2)$ and $+z_{\psi(n)} (\partial_{\mu} \phi_1 \partial^{\mu} \phi_1 + \partial_{\mu} \phi_2 \partial^{\mu} \phi_2)$ respectively, will remove divergences in the propagators for ϕ_1' and ϕ_2' fields with the choices

$$\delta_{\mu(n)}^2 = D_1 + \text{finite} \quad , \quad c_1 = -z_n + \text{finite} \quad .$$

Once this is done, Eq. (104) yields:

$$B = e \left(X/Z_{\psi_n} \right) + \text{finite} \quad .$$

Hence the choice of x_n such that $(X/Z_{\psi_n}) = \text{finite}$ will make

$$\left[\Gamma_{\mu}^{cb}(p, q, r) \right]_n \text{ finite. In particular, if we have chosen } x_r = z_{\psi(r)}$$

($0 \leq r \leq n-1$), then we may choose $x_n = z_{\psi(n)}$.

Now, choose $b=c=1$ in Eq. (103). We write:

$$\left[\Gamma_{\mu}^{bc}(p, q, r) \right]_n^{\text{div}} = E(q+r)_{\mu} + F(q-r)_{\mu}$$

$$\text{F. T. } \left[\Gamma_{ac} \right]_n^{\text{div}} = r^2 G + H \quad [a=2, c=1] \text{ etc.}$$

Then, we obtain:

$$E(q+r)^2 + F(q^2-r^2) = eG(r^2+q^2) + 2He + \text{finite}$$

Therefore, it is clear that E, F, G and H are finite.

Finally, we differentiate Eq. (90b) with respect to Ψ'_b, Ψ'_c and Ψ'_d and set $\Phi=0=\Psi'$. We equate the n -loop divergence on both sides.

We obtain:

$$\zeta_{ab} \left[\Gamma_{acd} \right]_n^{\text{div}} + \text{permutations} = \text{finite} \quad (107)$$

choosing $b = c = d = 1$ in Eq. (107), we obtain,

$$\left[\Gamma_{211} \right]_n^{\text{div}} = \text{finite} \quad (108)$$

Choosing $b = c = d = 2$ in Eq. (107), we obtain,

$$\left[\Gamma_{122} \right]_n^{\text{div}} = \text{finite} \quad (109)$$

Choosing $b = 2, c = d = 1$ in Eq. (107) and using Eqs. (108) and (109),

we find

$$\left[\Gamma_{111} \right]_n^{\text{div}} = \text{finite} \quad (110)$$

Choosing $b = 1, c = d = 2$, we get

$$\left[\Gamma_{222} \right]_n^{\text{div}} = \text{finite} \quad (111)$$

Thus we have shown that symmetric mass and wavefunction renormalization counter terms remove divergences in the propagators of ϕ_1^{\prime} and ϕ_2^{\prime} . We have also shown that all the newly introduced renormalization parts become finite in n-loop approximation. The rest of the proof (4-point functions and etc.) is trivial and hence will not be given here.

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FIGURE CAPTIONS

- Fig. 1 Feynman rules.
- Fig. 2 Diagrammatic representation for the last term in Eq. (14).
- Fig. 3 Diagrammatic representation for $G^{-1}(p)$.
- Fig. 4 Subdiagrams of $\Sigma(p)$ needing subtraction.
- Fig. 5 Diagrammatic representation for $G^{(3)}(p, q; r \mu)$.
- Fig. 6 Subdiagrams of γ_1 needing subtraction.
- Fig. 7 Derivatives of $\mathcal{F}_\alpha[\Phi]$ which are renormalization parts.
- Fig. 8 Subtractions needed for $\Delta_\mu(p)$, $\square_{\mu\nu}(q, r)$, $\Pi_{\mu\nu}(q, r)$.
- Fig. 9 Left hand side of Eq. (32).
- Fig. 10 One loop diagrams for the inverse photon propagator.
- Fig. 11 One loop diagrams for the inverse ghost propagator.
- Fig. 12 One loop diagrams for the ghost ghost photon vertex.
- Fig. 13 One loop diagrams for the $(A_\mu)^4$ vertex.
- Fig. 14 One loop diagrams for the $(A_\mu)^3$ vertex.
- Fig. 15 Diagrammatic representation for Eq. (56).
- Fig. 16 Diagrammatic representation for

$$\left. \frac{\delta^2 \mathcal{F}_\alpha}{\delta \bar{\chi}_i \delta \chi_j} \right|_{\Phi = \chi = \bar{\chi} = 0} .$$
- Fig. 17 Diagrammatic representation for $K_{\eta i}[\Phi, \chi, \bar{\chi}]$.

- Fig. 18 Diagrammatic representation for $\left[\delta K_{\bar{n}} / \delta \bar{\chi}_m \right]_{\Phi=\chi=\bar{\chi}=0}$.
- Fig. 19 A suspect renormalization part of diagram in Fig. 18.
- Fig. 20 Diagrammatic representation of Σ' .
- Fig. 21 Additional subtractions needed for $\Delta_{\mu}(p)$.

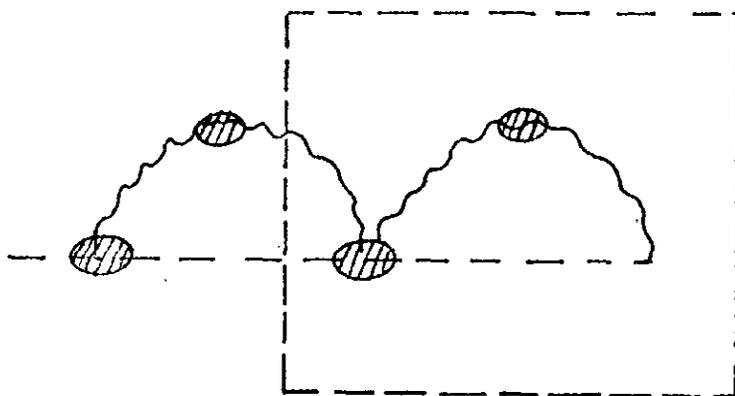


FIG. 4

$$G^{(3)}(p, q; r\mu) = \xi Y p_\mu + Y \left\{ \begin{array}{l} (r\mu) \\ \downarrow \end{array} \right. \text{diagram} \text{ with } p \text{ and } -q \text{ arrows}$$

The diagram part of the equation shows a horizontal dashed line with arrows pointing right, labeled with momentum p at the start and $-q$ at the end. There are two shaded vertices on this line. A wavy line connects the first vertex to a third shaded vertex above the line. A fourth shaded vertex is also above the line, connected to the second vertex on the line. A curly bracket labeled $(r\mu)$ with a downward arrow points to the wavy line.

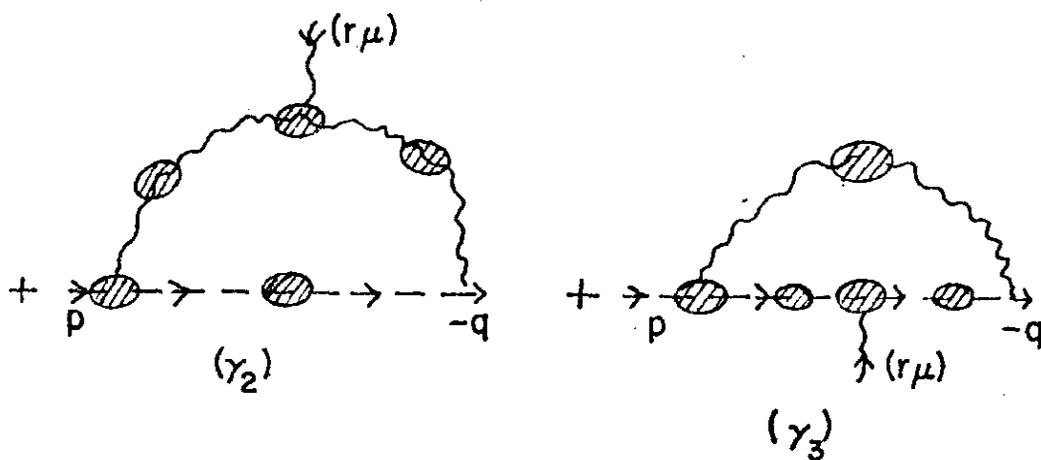


FIG. 5

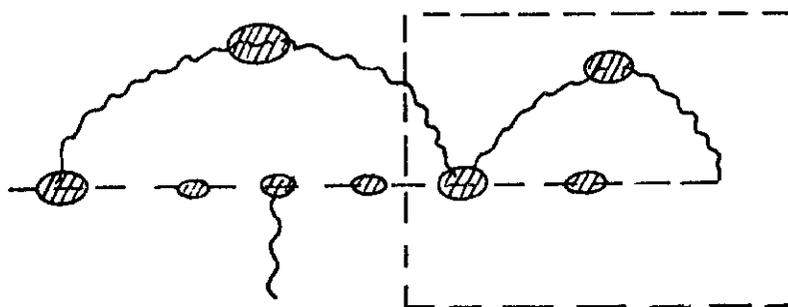
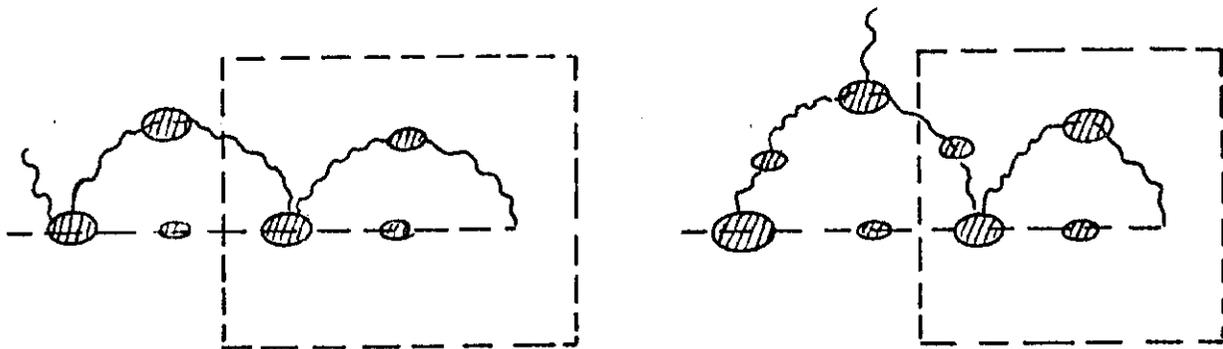


FIG. 6

Expression	Diagram (Fourier transform)	Symbol	Overall Subtraction
$-\frac{1}{\xi} \xi_{ij}^a \frac{\delta \Delta_{ij}}{\delta \Phi_k} \Big _{\Phi=0}$		$\Delta_{\mu}(p) \equiv p_{\mu} \Delta(p)$	$A p_{\mu}$
$\frac{1}{\xi} \xi_{ij}^a \frac{\delta^2}{\delta \Phi_k \delta \Phi_l} (i \Delta_{ij}) \Big _{\Phi=0}$		$2 \square_{\mu\nu}(q, r)$	$B g_{\mu\nu}$
$+$		$\Pi_{\mu\nu}(q, r)$	$C g_{\mu\nu}$

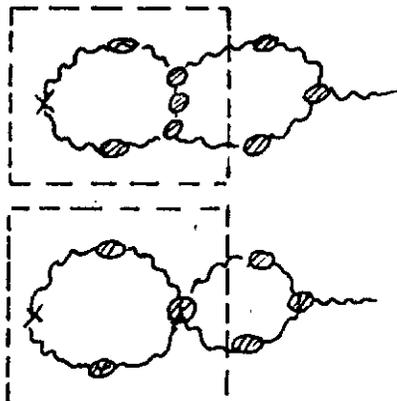
FIG. 7

Diagram

Subdiagrams which are renormalization parts

Subtraction
(the dotted square contains r -loops)

$\Delta\mu(p)$



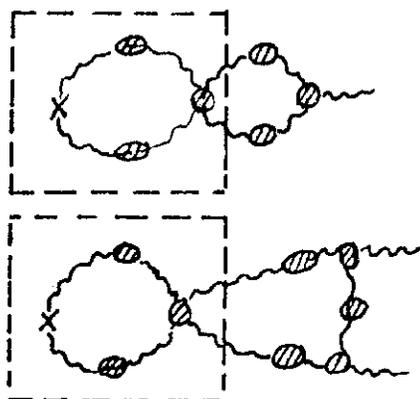
$$-Br [\Delta\mu(p)]_{n-r}$$

$$-\frac{Cr}{2} [\Delta\mu(p)]_{n-r}$$

$\square_{\mu\nu}(q,r)$

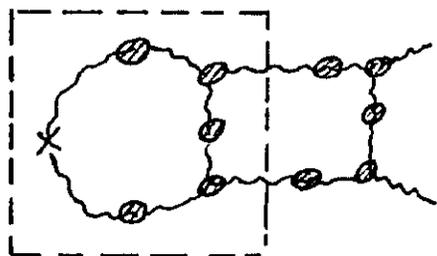
$\Pi_{\mu\nu}(q,r)$

- no -



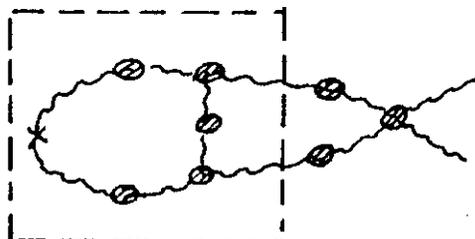
$$-\frac{1}{2} Cr [\Pi_{\mu\nu}]_{n-r}$$

$$-Cr [\square_{\mu\nu}]_{n-r}$$



$$-2Br [\square_{\mu\nu}]_{n-r}$$

and crossed diagram



$$-Br [\Pi_{\mu\nu}]_{n-r}$$

FIG. 8

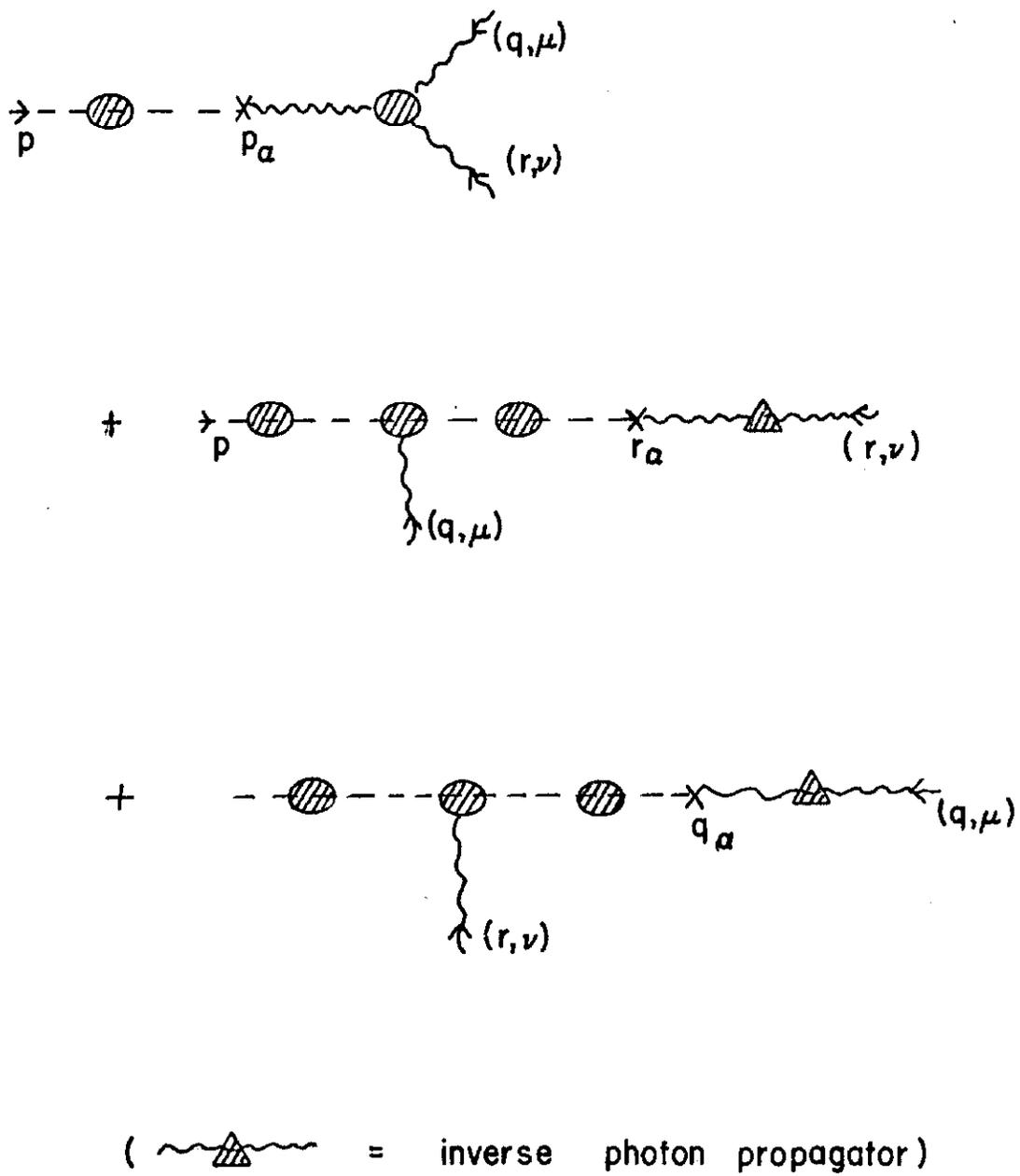


FIG. 9

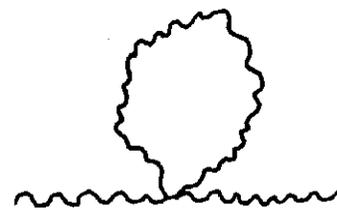
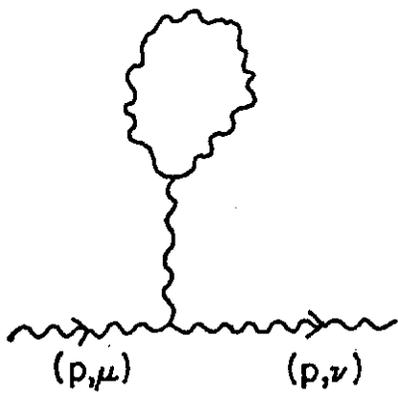
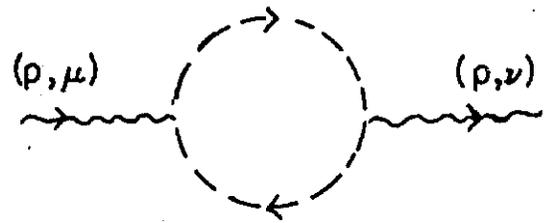
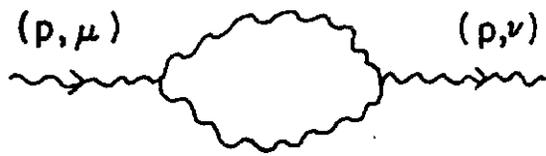


FIG. 10

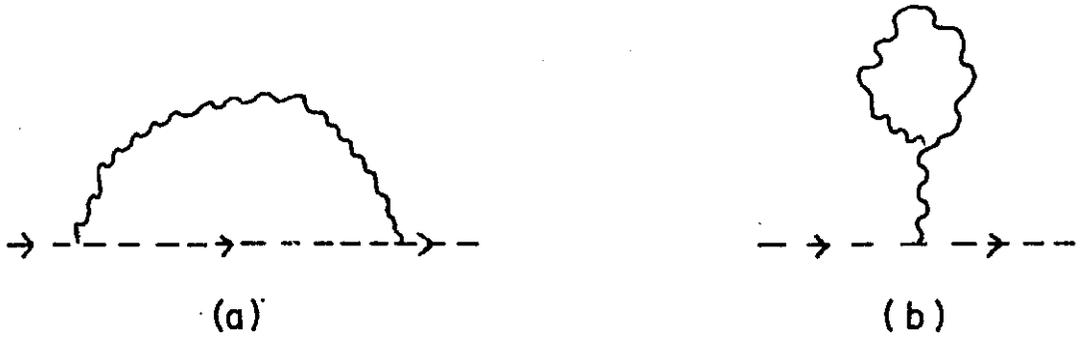


FIG. 11

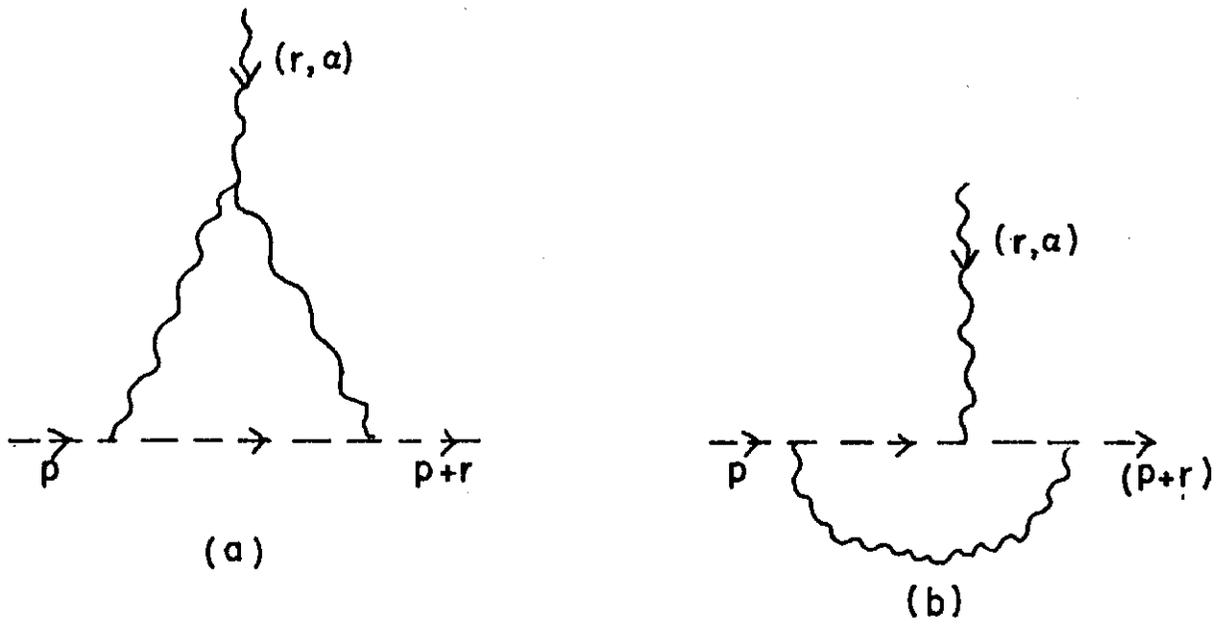
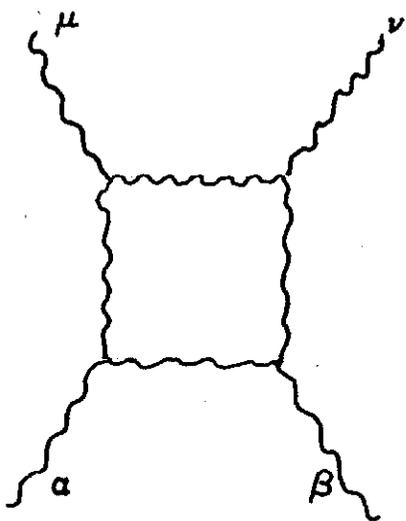
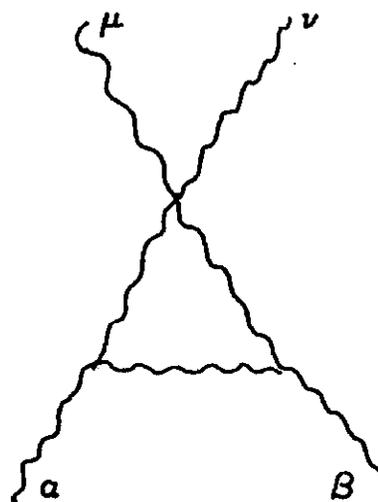


FIG. 12



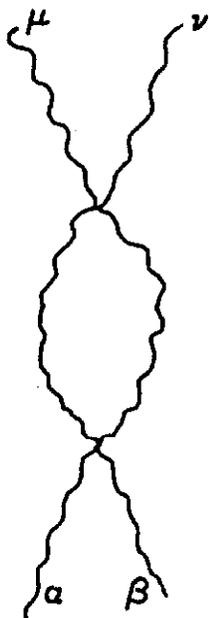
and two more permutations

(a)



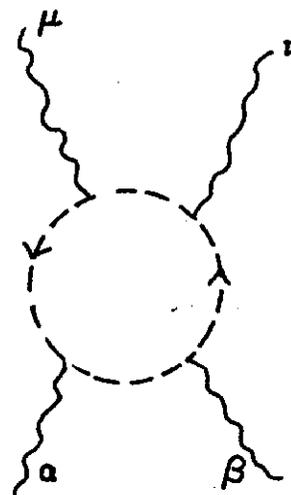
and five more permutations

(b)



and two more permutations

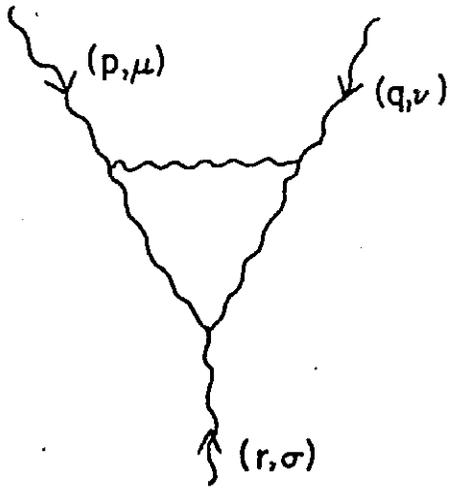
(a)



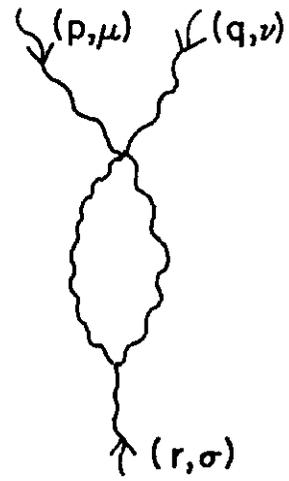
and five more permutations

(b)

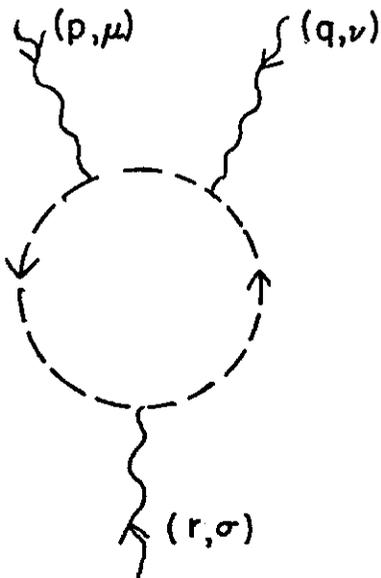
FIG. 13



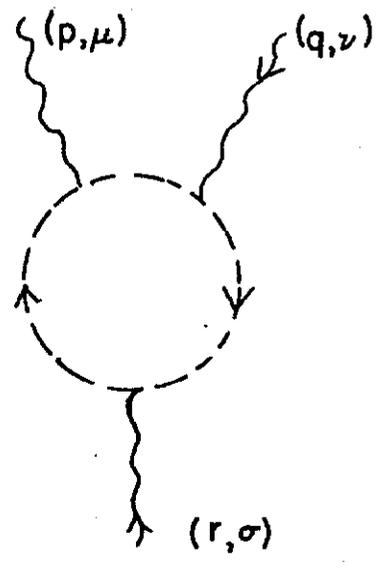
(a)



and
two more permutations
(b)

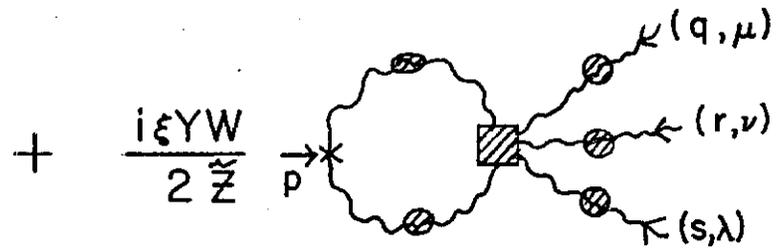
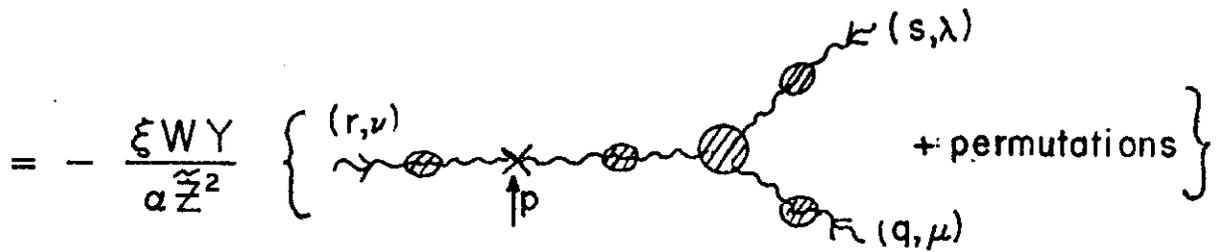
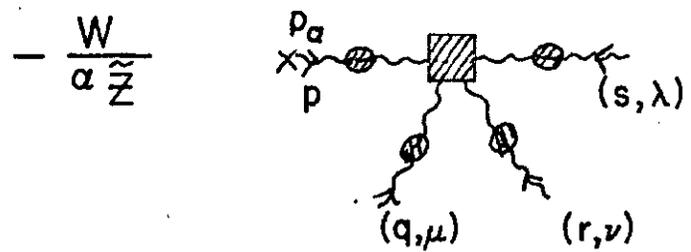
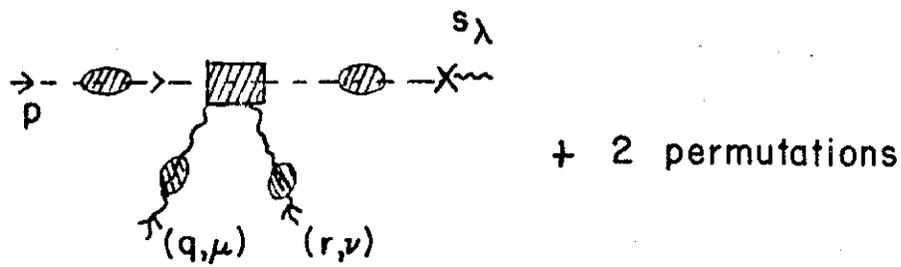


(c)



(d)

FIG. 14



(A shaded box stands for connected truncated Green's functions.)

FIG, 15

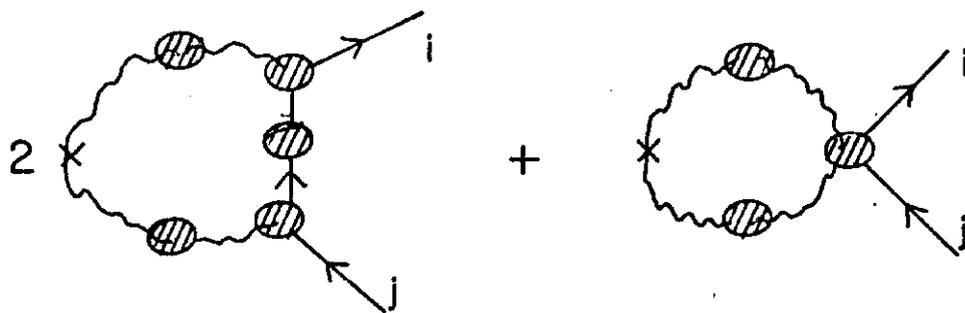


FIG. 16

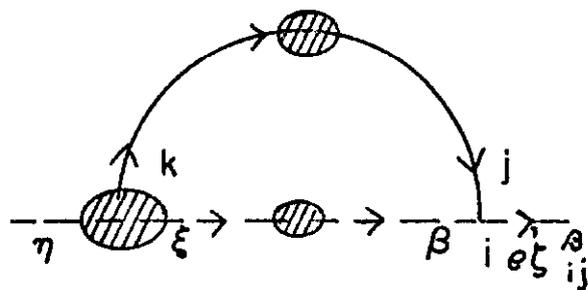


FIG. 17

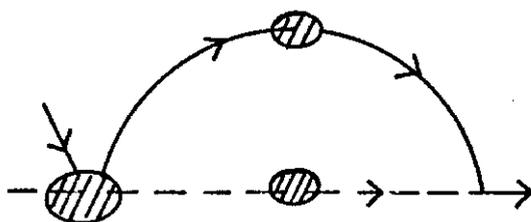


FIG. 18

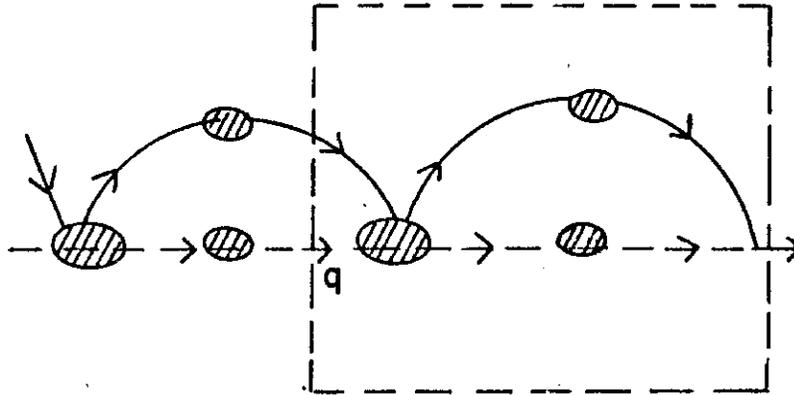


FIG. 19

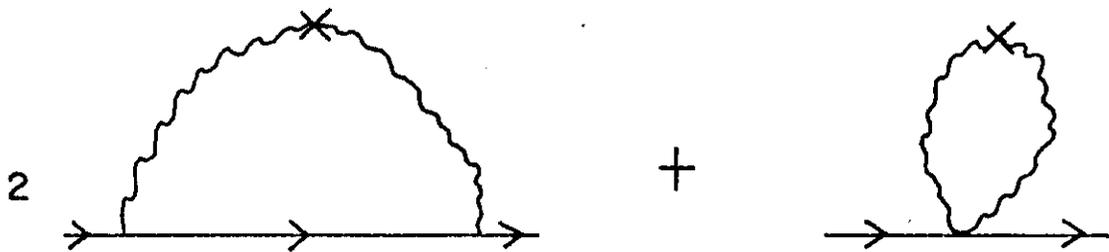
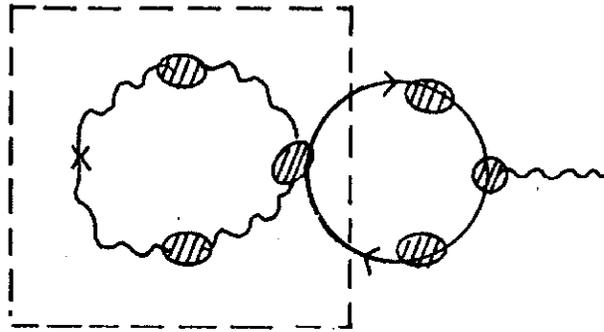
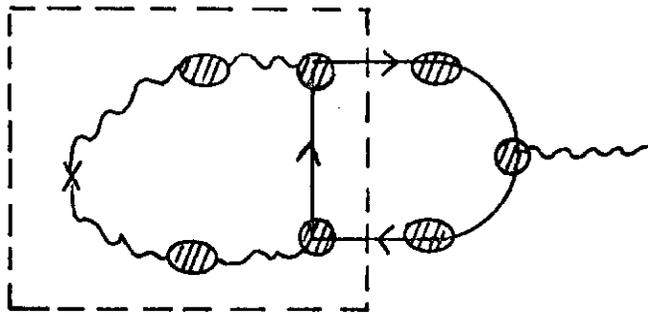


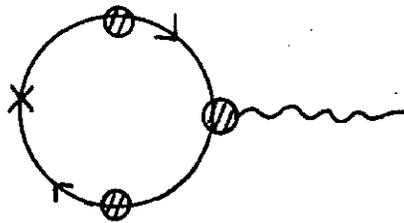
FIG. 20



(a)



(b)



(c)

FIG. 21