

Construction of the Reggeon Calculus in  $4-\epsilon$  Dimensions

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ABSTRACT

We study the Reggeon field theory in  $4-\epsilon$  dimensions. When the Pommeranchuk singularity has intercept one the theory can not be renormalized order by order in the perturbation series. Nevertheless we are able to develop systematic techniques for constructing the Pommeranchuk Green's functions. An integral representation is obtained for the Pommeranchuk propagator which allows us to explicitly display its infra-red ( $l \approx 1$ ,  $t \approx 0$ ) behavior and to show that the perturbation series is an asymptotic expansion for small values of the coupling constant and for large values of the angular momentum or momentum transfer. We also obtain an integral representation for the intercept renormalization counter-term. We find that for the renormalized Pommeranchuk singularity to have intercept one, the bare Pommeranchuk pole must have intercept greater than one.

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## I. INTRODUCTION

Gribov's Reggeon field theory<sup>1</sup> provides an elegant framework for the study of high energy reactions. In this theory Reggeons are treated as quasi-particles and associated with fields in two space and one "time" dimension. The space variables are conjugate to the transverse momentum,  $\vec{k}$ , and the "time" to the "energy" variable,  $E = 1 - \ell$ , where  $\ell$  is the angular momentum.

The field theory approach is most useful in discussing problems involving the Pommeranchuk singularity where interactions among  $\ell$  - plane poles and cuts are important. In these problems it seems necessary to take into account the full Reggeon unitarity relations.<sup>2</sup> The field theory has the advantage of guaranteeing that these relations are satisfied identically.

The purpose of this paper is to develop systematic techniques for constructing solutions to the Reggeon field theory when the Pommeranchuk singularity has intercept one. In this case the theory has a non-trivial infra-red ( $E, k^2 \approx 0$ ) behavior, and the standard renormalization program cannot be carried through order by order in perturbation theory. The problem is analogous to the one encountered in relativistic field theories with massless scalar particles.

Recently considerable progress has been made in studying the infra-red behavior of the Reggeon field theory by making use of the

renormalization group and the  $\epsilon$ -expansion.<sup>3,4</sup> Here one studies the field theory in  $D = 4 - \epsilon$  space dimensions. We shall do likewise. Our construction of the Pomeron propagator leads directly to the renormalization group scaling law when an infra-red stable Gell-Mann-Low eigenvalue exists. More importantly, we are able to justify the neglect of intercept renormalization in the  $\epsilon$ -expansion calculations.

In this paper we consider only a bare triple Pomeranchuk coupling and a linear trajectory function. Higher order couplings and higher powers in  $k^2$  in the trajectory function are generated by the interaction. Wilson has shown that in the analogous problem of Euclidean  $\phi^4$  theory the inclusion of such terms directly into the bare Lagrangian will in general not disturb the infra-red behavior of the theory.<sup>5</sup> The techniques we develop in this paper are also applicable to massless  $\phi^4$  theory in  $4 - \epsilon$  dimensions. A discussion of  $\phi^4$  can be found in a companion paper.<sup>6</sup>

In Sec. II we discuss the structure of a general perturbation theory diagram which contributes to the Pomeron self-energy part. In general the interactions among the Pomerons will lead to a shift in their intercept. This can be compensated by adding an intercept renormalization counter term to the Lagrangian just as one adds a mass counter term in conventional field theories. When the Pomeron intercept is below one the counter term can be adjusted at each order in perturbation theory so that the intercept takes on its physical value.

However, when the intercept is at one the counter term develops a branch point at zero value of the coupling constant, and it is no longer possible to carry out the intercept renormalization order by order in perturbation theory. Another difficulty with the perturbation theory is that as the Pomeron intercept approaches one, the second order approximation to the propagator develops a tachyon (an  $l$ -plane pole to the right of one). This spurious pole disappears only after an infinite class of diagrams has been summed. Despite these difficulties it is possible to systematically construct the renormalized Pomeron propagator by making use of the skeleton expansion for the self-energy part. For  $\epsilon > 0$  the field theory is super-renormalizable, and once the renormalized propagator has been constructed, all Green's functions can in principle be calculated.

In Sec. III we develop an alternative method for constructing the renormalized propagator which appears to have considerable calculational advantages. We derive integral representations for the intercept renormalization counter term and for the full propagator in terms of the other (finite) renormalization constants in the theory. The latter can be calculated systematically using renormalization group techniques. This approach has several advantages. It enables us to explicitly display the infra-red behavior of the propagator; it allows us to see how the tachyon, which is present in lowest order perturbation theory, is removed from the full solution to the field theory; and it

enables us to show that the perturbation series is an asymptotic expansion when the coupling constant becomes small for fixed values of  $E$  and  $k^2$ , and when  $E$  or  $k^2$  become large for fixed values of the coupling constant.

From the integral representation for the intercept counter term we are able to show that the intercept of the bare Pommeranchuk pole lies above one. We should emphasize that this bare Pommeranchuk pole is not necessarily experimentally observable at low energies, since there are many effects responsible for the complete intercept renormalization which are not accurately represented by the Reggeon calculus.

In Sec. IV we review our results and briefly discuss their application to the  $\epsilon$ -expansion and to direct calculations in two space dimensions.

## II. PERTURBATION THEORY AND THE SKELETON EXPANSION

In the Reggeon field theory a non-interacting Reggeon is treated as a quasi-particle with energy-momentum dispersion relation

$$E = 1 - \alpha_0(-\vec{k}^2) . \quad (1)$$

where  $E = 1-l$  and  $\alpha_0(-\vec{k}^2)$  is the bare Regge trajectory. We take the bare Pomeron trajectory to be linear and write

$$\alpha_0(-\vec{k}^2) = \alpha(0) - \alpha_0' \vec{k}^2 . \quad (2)$$

The free Lagrangian density is then given by

$$\begin{aligned} \mathcal{L}_0(\vec{x}, t) = & \frac{i}{2} \psi_0^+(\vec{x}, t) \frac{\overleftrightarrow{\partial}}{\partial t} \psi_0(\vec{x}, t) - \alpha_0' \vec{\nabla} \psi_0^+(\vec{x}, t) \cdot \vec{\nabla} \psi_0(\vec{x}, t) \\ & - \Delta \psi_0^+(\vec{x}, t) \psi_0(\vec{x}, t) \quad . \end{aligned} \quad (3)$$

Here  $\psi_0(\vec{x}, t)$  is the bare Pomeron field and  $\Delta = 1 - \alpha(0)$  is the intercept gap. We shall work in  $D = 4 - \epsilon$  space dimensions keeping in mind that the point of physical interest is  $D = \epsilon = 2$  .

The interactions Lagrangian density will be taken to be

$$\begin{aligned} \mathcal{L}_I(\vec{x}, t) = & -\frac{1}{2} i r_0 \left[ \psi_0^+(\vec{x}, t) \psi_0(\vec{x}, t)^2 + \psi_0^+(\vec{x}, t)^2 \psi_0(\vec{x}, t) \right] \\ & + \delta \Delta \psi_0^+(\vec{x}, t) \psi_0(\vec{x}, t) \quad . \end{aligned} \quad (4)$$

It follows from Gribov's signature analysis<sup>1</sup>, or equivalently the negative sign of the two Pomeron cut<sup>2, 7</sup>, that the bare triple Pomeron coupling constant,  $r_0$ , is real.  $\delta \Delta$  is the intercept renormalization counter term which is to be adjusted so that the intercept gap,  $\Delta$ , retains its physical value.

Our task is to construct the Green's functions,  $G^{n,m}(E_i, \vec{k}_i)$ , for  $n$  incoming and  $m$  outgoing Pomerons. The rules for evaluating the contribution of an individual Feynman diagram to  $G^{n,m}$  are as follows:

1. For each Pomeron of momentum  $\vec{k}$  and energy  $E$  use the bare propagator

$$G_0(E, \vec{k}^2) = i [E - \alpha_0' \vec{k}^2 - \Delta + i\epsilon]^{-1}$$

2. For each triple Pomeron vertex put a factor of  $r_0 / (2\pi)^{(D+1)/2}$  .
3. For each intercept renormalization counter term put a factor of  $i\delta\Delta$  .
4. For each two Pomeron loop with both momenta in the same direction multiply by  $\frac{1}{2}$  .
5. Conserve energy and momentum at all vertices.
6.  $\int d^D k dE$  around each closed loop.

Let us start by studying the structure of the Pomeron propagator in perturbation theory. It is convenient to write

$$i \Gamma^{1,1}(E, k^2) = i G^{1,1}(E, k^2)^{-1} = E - \alpha_0' k^2 - \Delta - \Sigma(E, k^2) + \delta\Delta . \quad (5)$$

$\Sigma(E, k^2)$  is the proper self-energy part. The lowest order diagram that contributes to  $\Sigma(E, k^2)$  is the bubble graph shown in Fig. 1. It gives<sup>8</sup>

$$\begin{aligned} \Sigma_2(E, k^2) &= \frac{-i}{2} \frac{r_0^2}{(2\pi)^{D+1}} \int d^D k' dE' \left[ E' - \alpha_0' k'^2 - \Delta + i\epsilon \right]^{-1} \\ &\quad \left[ E - E' - \alpha_0' (\vec{k} - \vec{k}')^2 - \Delta + i\epsilon \right]^{-1} \\ &= c r_0^2 (\frac{1}{2}\alpha_0' k^2 + 2\Delta - E)^{1-\epsilon/2} , \end{aligned} \quad (6)$$

with

$$c = \frac{\Gamma(-1+\epsilon/2)}{2(8\pi\alpha_0')^{D/2}} . \quad (7)$$

Clearly if the Pomeron intercept is to remain at  $\alpha(0)$  one must have

$$\Gamma^{1,1}(\Delta, 0) = 0 \quad , \quad (8)$$

which requires that

$$\delta\Delta_2 = cr_0^2 \Delta^{1-\epsilon/2} \quad . \quad (9)$$

Many of the difficulties with the perturbation theory already appear in this second order calculation. First notice that the limits  $\Delta \rightarrow 0$  and  $\epsilon \rightarrow 2$  do not commute. In fact for  $\Delta = 0$  and  $\epsilon \geq 2$  there is no choice for  $\delta\Delta_2$  such that Eq. (8) holds. Furthermore there is a second solution to the equation

$$\Gamma^{1,1}(E, 0) = 0 \quad (10)$$

This spurious solution occurs for  $E < \Delta$  when  $\Delta^{\epsilon/2} < (-1+\epsilon/2)cr_0^2$  and for  $E < 0$  when  $\Delta^{\epsilon/2} < -cr_0^2 (2^{1-\epsilon/2}-1)$ . In the second case the propagator has a tachyon, an  $\ell$ -plane pole to the right of  $\ell = 1$ . As we shall see, this tachyon is not present in the full solution to the field theory. However, its presence in second order perturbation theory indicates that the  $\Delta$  goes to zero limit is a subtle one. In higher orders of perturbation theory one will find tachyon-tachyon and Pomeron-tachyon cuts. The tachyon pole and cuts disappear only after an infinite class of diagrams are summed.

To analyze higher order terms in the expansion of  $\Sigma(E, k^2)$  it is convenient to use Rayleigh-Schroedinger perturbation theory (which is equivalent to performing all the  $E$ -integrations by picking up propagator poles). Then momentum, but not energy, is conserved

at each vertex, and one integrates only over the loop momenta. For each triple Pomeron vertex one now puts a factor of  $r_0/(2\pi)^{D/2}$  and for each  $n$  Pomeron intermediate state the free  $n$ -Pomeron Green's function

$$G_n(E, \vec{k}_1, \dots, \vec{k}_n) = i \left[ E - n\Delta - \alpha'_0 \sum_{i=1}^n k_i^2 + i\epsilon \right]^{-1} . \quad (11)$$

Let us consider a diagram of order  $2n$  in the triple Pomeron coupling but with no intercept counter terms. Making use of the Feynman identity to combine the Green's function denominators one can write the contribution of such a diagram in the form<sup>9</sup>

$$\begin{aligned} \Sigma_n(E, k^2) &= (-)^{n+1} \left( r_0^2 / (2\pi)^D \right)^n \frac{1}{2} \int \prod_{i=1}^n d^D k_i \int_0^1 \prod_{i=1}^{2n-1} dz_i \\ &\cdot \delta \left( 1 - \sum_1^{2n-1} z_i \right) \left[ \sum_{i,j=1}^n A_{ij}(z) \alpha'_0 \vec{k}_i \cdot \vec{k}_j + a(z) k^2 + b(z) \Delta - E - i\epsilon \right]^{-(2n-1)} . \quad (12) \end{aligned}$$

Here  $A_{ij}(z)$  is a positive semi-definite, symmetric matrix,  $a(z) > 0$  and  $b(z) \geq 2$  for all allowed values of the  $z_i$ . Notice that  $\Sigma_n(E, k^2)$  has no singularities as a function of  $\epsilon$  arising from the infra-red ( $k_i^2 \approx 0$ ) region of integration even when  $\Delta = 0$ . Such infra-red singularities are present in relativistic field theories with massless scalar particles which makes the analysis of these theories more difficult.<sup>6</sup>

The only singularities in  $\Sigma_n(E, k^2)$  as a function of  $\epsilon$  are poles arising from ultra-violet divergences. Performing the momentum integrations in Eq. (12) gives

$$\Sigma_n(E, k^2) = (-)^{n+1} \left( r_0^2 / (4\pi\alpha_0') \right)^{D/2} n \frac{\Gamma(-1+n\epsilon/2)}{\Gamma(2n-1)} \quad (13)$$

$$\cdot \int_0^1 \prod_{i=1}^{2n-1} dz_i \delta\left(1 - \sum_1^{2n-1} z_i\right) \left[ \det A(z) \right]^{-D/2} \left[ a(z)k^2 + b(z)\Delta - E - i\epsilon \right]^{1-n\epsilon/2}.$$

The pole at  $\epsilon = 2/n$ , which arises from the overall ultra-violet divergence of the diagram, appears explicitly in the gamma-function in Eq. (13). If the topology of the diagram is such that one of the internal lines contains a self-energy insertion of order  $2m$ , then there will be an additional pole at  $\epsilon = 2/m$ . It will arise from a vanishing of  $\det A(z)$  when one or more of the Feynman parameters is zero. Some specific examples illustrated in Figs. 2 and 3 are worked out in the appendix.

From simple power counting one sees that for  $\epsilon > 0$  the only ultra-violet divergences in the theory are those that arise in the self-energy part. Once they have been removed by performing the intercept renormalization all Green's functions are finite. The renormalization can be carried out order by order in perturbation theory for  $\Delta > 0$ , but not for  $\Delta = 0$  which is the theory of interest. To see this let us set  $\Delta = 0$  in Eq. (13). We can then write

$$\Sigma_n(E, 0) = (-)^n \left( r_0^2 / \alpha_0' D/2 \right)^n (-E)^{1-n\epsilon/2} C_n (1-n\epsilon/2)^{-1}. \quad (14)$$

Clearly for  $n \geq 2/\epsilon$  there is no choice for  $\delta \Delta_n$  such that  $\Sigma_n(0, 0) - \delta \Delta_n = 0$ .

This is the generalization of the difficulty encountered in

the second order diagram for  $\epsilon \geq 2$ . The problem is that when

$\Delta = 0$ ,  $\delta \Delta$  has a branch point at  $r_0 = 0$ . This follows from the

fact that when  $\Delta = 0$  the only quantity with the dimensions of energy is

$\left( r_0^2 / \alpha_0' D/2 \right)^{2/\epsilon}$ . As a result,  $\delta \Delta$  must have the form

$$\delta \Delta = \left( r_0^2 / \alpha_0' D/2 \right)^{2/\epsilon} f(\epsilon), \quad (15)$$

where  $f(\epsilon)$  is a dimensionless quantity independent of  $r_0$  and  $\alpha_0'$ .

It is clear from Eq. (15) that one must develop non-perturbative techniques for calculating  $\delta \Delta$ . One approach is to make use of the skeleton expansion for the self-energy part. One starts by summing all graphs in which the internal lines have no self-energy insertions and no intercept counter term insertions. Examples are the ladder graphs shown in Fig. 2. For this class of graphs there is no difficulty in carrying out the intercept renormalization after the summation has been performed, since the only singularity of each diagram is the overall ultra-violet divergence. One then obtains a first order approximation to the propagator. The next step is to recalculate the skeleton graphs using the improved propagator in place of the bare one, and to again

renormalize after summing the graphs. By carrying out this process an infinite number of times one generates all the diagrams in the theory. At each step of the calculation the inverse propagator is finite and vanishes at  $E = k^2 = 0$ .

Let us consider the first round calculation in more detail. We can write

$$\Sigma^{(1)}(E, 0) = \sum_{n=1}^{\infty} (-)^n \left( r_0^2 / \alpha_0' D/2 \right)^n (-E)^{1-n\epsilon/2} D^{(1)}(n) (1-n\epsilon/2)^{-1}, \quad (16)$$

where the  $D^{(1)}(n)$  are positive and have no singularities in  $n$  for  $n \geq 2/\epsilon$ . If the  $D^{(1)}(n)$  are sufficiently well behaved for large values of  $n$  as is indicated by the examples of Appendix I and by renormalization group calculations, then the series can be summed using the Sommerfeld-Watson transformation:

$$\Sigma^{(1)}(E, 0) = \frac{i}{2} \int_c \frac{dn}{\sin \pi n} \left( r_0^2 / \alpha_0' D/2 \right)^n (-E)^{1-n\epsilon/2} D^{(1)}(n) (1-n\epsilon/2)^{-1}. \quad (17)$$

The contour  $c$  circles all of the positive integers in the clockwise direction. Let us now pull back the integration contour. The right most singularity of the integral that one encounters is the pole at  $n = 2/\epsilon$ , so one can write

$$\Sigma^{(1)}(E, 0) = \left( r_0^2 / \alpha_0' D/2 \right)^{2/\epsilon} \frac{2}{\epsilon} \frac{\pi}{\sin \pi \frac{2}{\epsilon}} D^{(1)} \left( \frac{2}{\epsilon} \right) + \bar{\Sigma}^{(1)}(E, 0), \quad (18)$$

where  $\bar{\Sigma}^{(1)}(E, 0)$  vanishes as  $E$  goes to zero. As a result,

$$\delta \Delta^{(1)} = \left( r_0^2 / \alpha_0' D/2 \right)^{2/\epsilon} \frac{2}{\epsilon} \frac{\pi}{\sin \pi \frac{2}{\epsilon}} D^{(1)} \left( \frac{2}{\epsilon} \right) . \quad (19)$$

The task of calculating the  $D^{(1)}(n)$  is a formidable one, to say nothing of the problem of carrying out the higher order iterations. In the next section we shall present a more practical calculational scheme. The point we wish to emphasize here is the interplay between the ultra-violet and infra-red singularities. It is the fact that the over-all ultra violet divergence of the graphs occurs at the same value of  $\epsilon$  that the infra-red difficulties set in that makes our construction work. This will continue to happen in the higher order iterations.

Although the skeleton expansion does not seem to be of practical use, the Sommerfeld -Watson representation of the self-energy part does. For example, in the next section we shall show that for small values of  $\epsilon$  the full self-energy part can be written in the form of Eq. (17) with  $D^{(1)}(n)$  replaced by

$$C(n) = \left( \frac{6}{(8\pi)^2 \epsilon} \right)^n \frac{\Gamma(n + \frac{1}{6})}{n! \Gamma(\frac{1}{6})} . \quad (20)$$

In this approximation

$$\delta \Delta = \left[ 6r_0^2 / (8\pi)^2 \alpha_0' D/2 \right]^{2/\epsilon} \frac{2}{\epsilon} \frac{\pi}{\sin \pi \frac{2}{\epsilon}} \frac{\Gamma(\frac{2}{\epsilon} + \frac{1}{6})}{\Gamma(1 + \frac{2}{\epsilon}) \Gamma(\frac{1}{6})} . \quad (21)$$

The pole at  $n = 0$  arising from the vanishing of  $\sin \pi n$  cancels the

linear term in  $E$  in the bare propagator, so the small  $E$  behavior of  $\Gamma^{(1,1)}(E, 0)$  is controlled by the right most pole in  $\Gamma(n+\frac{1}{6})$ . One finds

$$i\Gamma^{(1,1)}(E, 0) \xrightarrow{E \rightarrow 0} - \left( 6r_0^2 / (8\pi)^2 \alpha_0^{D/2} \epsilon \right)^{-\frac{1}{6}} (-E)^{1+\epsilon/12} \quad (22)$$

in agreement with the results of references 3 and 4.

### III. INTEGRAL REPRESENTATIONS FOR THE POMERON PROPAGATOR

Up to now we have concentrated on the intercept renormalization. However, in order to derive integral representations for the Pomeron propagator it will be useful to carry out the wave function, coupling constant and slope parameter renormalizations. We start by introducing the renormalized Pomeron field operator  $\psi(\vec{x}, t)$ ,

$$\psi(\vec{x}, t) = Z_3^{-\frac{1}{2}} \psi_0(\vec{x}, t) \quad . \quad (23)$$

Then the Lagrangian density of Eqs. (3) and (4) can be rewritten in the form

$$\begin{aligned} \mathcal{L} = & \frac{i}{2} Z_3 \psi^+(\vec{x}, t) \frac{\overleftrightarrow{\partial}}{\partial t} \psi(\vec{x}, t) - Z_2 \alpha_0 \vec{\nabla} \psi^+(\vec{x}, t) \vec{\nabla} \psi(\vec{x}, t) \\ & - \frac{i}{2} r Z_1 [ \psi^+(\vec{x}, t) \psi(\vec{x}, t)^2 + \psi^+(\vec{x}, t)^2 \psi(\vec{x}, t) ] \\ & + Z_3 \delta \Delta \psi^+(\vec{x}, t) \psi(\vec{x}, t) \quad . \quad (24) \end{aligned}$$

For the remainder of this paper we shall take  $\Delta = 0$ .  $Z_1, Z_2$  and  $Z_3$  will be referred to respectively as the coupling constant, slope

and wave function renormalization constants.

$$\alpha' = Z_3 Z_2^{-1} \alpha'_0 \quad (25)$$

$$r = Z_3^{3/2} Z_1^{-1} r_0 \quad (26)$$

The  $Z_i$  are determined by the following normalization conditions on the renormalized propagator and three point function

$$\left. \frac{\partial}{\partial E} i \Gamma_R^{1,1}(E, k^2) \right|_{\substack{E = -E_N \\ k^2 = k_N^2}} = 1 \quad , \quad (27)$$

$$\left. \frac{\partial}{\partial k^2} i \Gamma_R^{1,1}(E, k^2) \right|_{\substack{E = -E_N \\ k^2 = k_N^2}} = -\alpha' \quad , \quad (28)$$

$$\left. \Gamma_R^{1,2}(E_1, \vec{k}_1, E_2, \vec{k}_2, E_3, \vec{k}_3) \right|_{\substack{E_1 = 2E_2 = 2E_3 = -E_N \\ \vec{k}_1 = 2\vec{k}_2 = 2\vec{k}_3 = \vec{k}_N}} = ir / (2\pi)^{\frac{D+1}{2}} \quad . \quad (29)$$

Here  $(-E_N, k_N^2)$  is a general point away from the singularities of the Green's functions. Our normalization conditions differ from those of Abarbanel and Bronzan in that we have not set  $k_N^2 = 0$ . This generalization is necessary in order for us to study the  $k^2$  dependence of the propagator. With the above definitions, the renormalized proper vertex function for  $n$  incoming and  $m$  outgoing Pomerons,  $\Gamma_R^{n,m}$ , is related to the unrenormalized one by

$$\Gamma_R^{n,m}(E_i, \vec{k}_i, r, \alpha', E_N, k_N^2) = Z_3^{\frac{n+m}{2}} \Gamma^{n,m}(E_i, \vec{k}_i, r_0, \alpha'_0) . \quad (30)$$

Let us now turn to the problem of finding an integral representation for the Pomeron propagator. We start with the unrenormalized self-energy part with  $k^2=0$  . From dimensional analysis it can be written in the form

$$\Sigma(E, 0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[ r_0^2 / \alpha'_0 \right]^{n+2m} (-E)^{1-n\frac{\epsilon}{2}-m} \frac{C_{n,m}}{n\frac{\epsilon}{2}+m-1} . \quad (31)$$

The  $(n, m)$  term in the sum arises from diagrams with  $n$  triple Pomeron vertices and  $m$  intercept renormalization counter terms. We have used Eq.(15) for  $\delta\Delta$  and absorbed  $f(\epsilon)$  into  $C_{n,m}$  . The pole in  $\epsilon$  arising from the over all ultra-violet divergence of the graphs has been displayed explicitly in Eq. (31) . The  $C_{n,m}$  , which are functions of  $\epsilon$  only, also contain poles in  $\epsilon$  , but they cancel among themselves when the sum is taken. It is convenient to introduce the auxiliary function

$$F(x) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} x^{n\frac{\epsilon}{2}+m} C_{n,m} . \quad (32)$$

Then Eq. (31) can be rewritten in the form

$$\Sigma(E, 0) = \left[ r_0^2 / \alpha'_0 \right]^{2/\epsilon} \int_0^{x_1} \frac{dx}{x} F(x) , \quad (33)$$

where

$$x_1 = \left( r_0^2 / \alpha_0' \right)^{D/2} (-E)^{-1} . \quad (34)$$

The integral in Eq. (33) is well defined for  $\epsilon > 2$  . It can be continued to smaller values of  $\epsilon$  by making use of Eq. (32) to explicitly remove the poles at  $\epsilon = \frac{2}{n} (1-m)$  . The residues of these poles can be calculated in perturbation theory.

The function  $F(x)$  can be related to the wave function renormalization constant,  $Z_3$  , provided we carry out the renormalization at  $k_N^2 = 0$  . In that case

$$\begin{aligned} Z_3^{-1}(x_{1N}) &= \left. \frac{\partial}{\partial E} i \Gamma^{1,1}(E, k^2) \right|_{\substack{E = -E_N \\ k^2 = 0}} \\ &= 1 - F(x_{1N}) , \end{aligned} \quad (35)$$

where

$$x_{1N} = \left( r_0^2 / \alpha_0' \right)^{D/2} E_N^{-1} . \quad (36)$$

That  $Z_3$  is a function of  $x_{1N}$  only follows from the fact that this is the only dimensionless variable one can make from  $r_0, \alpha_0'$  and  $E_N$  . We now have

$$\delta\Delta = \Sigma(0, 0) = \left( r_0^2 / \alpha_0' \right)^{D/2} \int_0^\infty \frac{dx}{x^2} \left[ 1 - Z_3^{-1}(x) \right] , \quad (37)$$

and from Eq. (5)

$$i\Gamma^{1,1}(E, 0) = - \left( r_0^2 / \alpha_0' \right)^{2/\epsilon} \int_{x_1}^{\infty} \frac{dx}{x} Z_3^{-1}(x) \quad . \quad (38)$$

Notice that the bare propagator has been cancelled by the term in  $\Sigma(E, 0)$  which is linear in  $E$  .

The next step is to evaluate  $Z_3$  . This can be done using renormalization group techniques. Following Abarbanel and Bronzan<sup>3</sup> we introduce a dimensionless renormalized coupling constant

$$g = r \alpha' E_N^{-D/4 - \epsilon/4} \quad , \quad (39)$$

and the renormalization group functions

$$\beta = E_N \frac{\partial}{\partial E_N} g \Bigg|_{\substack{r_0 \\ \alpha_0'} \text{ fixed}} \quad , \quad (40)$$

$$\gamma = E_N \frac{\partial}{\partial E_N} \ln Z_3 \Bigg|_{\substack{r_0 \\ \alpha_0'} \text{ fixed}} \quad (41)$$

$$\zeta = E_N \frac{\partial}{\partial E_N} \alpha' \Bigg|_{\substack{r_0 \\ \alpha_0'} \text{ fixed}} \quad . \quad (42)$$

Since  $g$  is the only dimensionless renormalized quantity in the problem when  $k_N^2 = 0$  ,  $\beta$  ,  $\gamma$  ,  $\zeta/\alpha'$  and the  $Z_i$  must be functions of  $g$

only. This tells us that Eqs. (41) and (42) can be written in the form

$$\gamma = \beta \frac{d}{dg} \ln Z_3 \quad , \quad (43)$$

and

$$\zeta = \alpha' \beta \frac{d}{dg} \ln(Z_2^{-1} Z_3) \quad . \quad (44)$$

Furthermore, from Eqs. (26) and (39) we find that

$$g^{-1} + \frac{\epsilon}{4} \beta^{-1} = \frac{d}{dg} \ln \left( Z_3^{3/2-D/4} Z_1^{-1} Z_2^{D/4} \right) \quad . \quad (45)$$

Since the  $Z_i$  are all one at  $g = 0$  our final results are

$$Z_3(g) = \exp \left[ \int_0^g dg' \gamma(g') \beta^{-1}(g') \right] \quad (46)$$

$$Z_2^{-1}(g) Z_3(g) = \exp \left[ \int_0^g dg' (\zeta / \alpha') \beta^{-1}(g') \right] \quad , \quad (47)$$

and

$$r_0 \alpha_0^{-D/4} E_N^{-\epsilon/4} = g \exp \left[ - \int_0^g dg' (g')^{-1} + \frac{\epsilon}{4} \beta^{-1}(g') \right] \quad . \quad (48)$$

Several important results can be read off without detailed calculations. First since  $\gamma$  and  $\beta$  are real analytic functions,  $Z_3$  is positive definite for real values of  $g$ . As a result, we see from Eq. (38) that  $\Gamma^{1,1}(E, 0)$  cannot vanish in the range  $-\infty < E < 0$ . Thus the tachyon which we found in second order perturbation theory is not present in the full propagator. Its cancellation is clearly a non-

perturbative effect.

Next let us suppose that  $\beta(g)$  has an infra-red stable zero, i. e., a zero of the form

$$\beta(g) \xrightarrow{g \rightarrow g_1} b(g-g_1) \tag{49}$$

with  $b > 0$ . This is the case at least for small values of  $\epsilon$ .<sup>3,4</sup> Then for  $g \lesssim g_1$  Eqs. (46) and (48) yield

$$x_{1N} \simeq c_1 (g_1 - g)^{-1/b} \tag{50}$$

$$Z_3 \simeq c'_3 (g_1 - g)^{\gamma/b} = c_3 x_{1N}^{-\gamma}, \tag{51}$$

where  $\gamma \equiv \gamma(g_1)$  and the  $c$ 's are constants. We now see that the integrals in Eqs. (37) and (38) will converge provided

$$\gamma < 1. \tag{52}$$

Notice that the limit  $g \rightarrow g_1$  requires either  $r_0^2 \rightarrow \infty$  or  $E_N \rightarrow 0$ . The small  $E$  behavior of  $\Gamma^{1,1}(E, 0)$  is just

$$i\Gamma^{1,1}(E, 0) \xrightarrow{E \rightarrow 0} -c_3 \left( r_0^2 / \alpha_0^{D/2} \right)^{\frac{2}{\epsilon} \gamma} (-E)^{1-\gamma} \tag{53}$$

in agreement with the renormalization group result.<sup>3,4</sup>

The small  $r_0$  and large  $E$  behaviors of  $\Gamma^{1,1}(E, 0)$  can also be read off from Eq. (38). Let us denote by  $\bar{x}_1$  a value of  $x$  small enough so that the series expansion for  $Z_3^{-1}(x)$  given by Eqs. (32)

and (35) can be used for  $x_1 \leq x \leq \bar{x}_1$ . Then in this range one can integrate the series term by term. The dominant contribution will come from those terms in Eq. (32) with  $m = 0$  and  $n < \frac{2}{\epsilon}$ . They will just give rise to the first  $n$  terms in the perturbation expansion of the propagator plus a correction of order  $(r_0^2/\alpha_0^{D/2})^{2/\epsilon}$ . The integral from  $\bar{x}_1$  to infinity will also be of order  $(r_0^2/\alpha_0^{D/2})^{2/\epsilon}$ . Thus the perturbation series provides an asymptotic expansion of the propagator for large  $E$  or small  $r_0$ . However, it is only legitimate to work to order  $n < \frac{2}{\epsilon}$ . No diagrams with intercept counter terms need be included. This result is crucial for the  $\epsilon$ -expansion calculations of the critical exponents.<sup>3,4</sup> They are generally assumed to have a power series expansion about  $\epsilon = 0$ . If one wishes to calculate the first  $n$  terms in this expansion one need merely imagine working with  $\epsilon < 2/n$ . One can then calculate the renormalization group functions in perturbation theory, and no diagrams with intercept renormalization counter terms will enter. At the end of the calculation one can of course attempt to continue the answer to  $\epsilon > 2/n$ .

In order to proceed further it is necessary to compute the renormalization group functions. In the one loop approximation one need only calculate the bubble diagram of Fig. 1 and the vertex graphs shown in Fig. 4. We find that

$$\beta(g) = -\frac{\epsilon}{4}g(1-g^2/g_1^2) \quad (54)$$

$$\gamma(g) = \gamma g^2 / g_1^2 \quad (55)$$

$$\zeta / \alpha' = \delta g^2 / g_1^2 \quad . \quad (56)$$

Here  $g_1$ ,  $\gamma$  and  $\delta$  are functions of  $\epsilon$  only and  $\gamma$  and  $\delta$  are negative. The one loop approximation gives the correct behavior of the renormalization group functions to leading order in  $\epsilon$ . Working to this order one finds<sup>3,4</sup>

$$\frac{g_1^2}{(8\pi)^2} = \epsilon/6 \quad (57)$$

$$\gamma = -\epsilon/12 \quad (58)$$

$$\delta = -\epsilon/24 \quad . \quad (59)$$

From Eqs. (46) and (48) we see that in the one loop approximation

$$x_{1N}^{\epsilon/2} = g^2 (1 - g^2 / g_1^2)^{-1} \quad (60)$$

and<sup>10</sup>

$$Z_3 = \left(1 - g^2 / g_1^2\right)^{\frac{2}{\epsilon} \gamma} = \left(1 + x_{1N}^{\epsilon/2} / g_1^2\right)^{-\frac{2}{\epsilon} \gamma} \quad . \quad (61)$$

Notice that the region  $-g_1 \leq g \leq g_1$  corresponds to  $-\infty < r_0 < \infty$ .

This will always be the case if an infra-red stable zero exists since

$\beta(g)$  has the form  $-\frac{\epsilon}{4} g f(g^2)$  with  $f(0) = 1$ . Substituting Eq. (61) into

Eq. (37) gives

$$\delta\Delta = \gamma \left( r_0^2 / \alpha_0' D / 2 g_1^2 \right)^{2/\epsilon} \frac{2}{\epsilon} \frac{\Gamma\left(1 - \frac{2}{\epsilon}\right) \Gamma\left[\frac{2}{\epsilon}(1 - \gamma)\right]}{\Gamma\left(1 - \frac{2}{\epsilon} \gamma\right)} \quad , \quad (62)$$

which reduces to Eq. (21) in the small  $\epsilon$  limit. For small values of  $\epsilon$  we also find that

$$i\Gamma^{1,1}(E,0) = -\left(r_0^2/\alpha_0^{D/2}\right)^{2/\epsilon} \int_{x_1}^{\infty} dx x^{-2} \left(1+x^{\epsilon/2}/g_1^2\right)^{-\frac{2}{\epsilon}\gamma}$$

$$\approx E \left[1+6r_0^2(-E)^{-\epsilon/2}/(8\pi)^2 \alpha_0^{D/2} \epsilon\right]^{-1/6} \quad (63)$$

This result was first obtained by Bronzan using the renormalization group directly.<sup>11</sup> He has shown that it corresponds to summing all terms in the theory of order  $(r_0^2/\epsilon)^n$ , but neglecting those of order  $\epsilon^m (r_0^2/\epsilon)^n$ . One sees directly from Eq. (63) that there is a tachyon in second order perturbation theory but not in the full theory.

From Eq. (62) we see that  $\delta\Delta$  develops poles at  $\epsilon = \frac{2}{n}$ ,  $n = 1, 2, \dots$  and so the bare intercept is undefined at these values of  $\epsilon$ , and in particular at the physical value  $\epsilon = 2$ . This is because we are calculating within a pure renormalizable field theory. In practice, of course, there will be a natural cut-off in the angular momenta and momentum transfers we consider provided by, say, secondary trajectories and the two-pion threshold. This cut-off does not affect the infra-red behavior of the theory, but it does give a meaning to the bare intercept. The introduction of the cut-off can be carried out using the renormalization group method of Wilson.<sup>5</sup>

This is essentially the method used by Migdal, Polyakov and Ter-Martirosyan.<sup>4</sup> In this approach higher-order couplings can also be simply introduced from the start since the renormalizability of the theory is not a requirement when the cut-off is present. As we discussed in the introduction the critical exponents should be the same, whether or not higher-order couplings are introduced.

If we suppose there is a cut-off  $\Lambda$  in the angular momentum then this cut-off will apply also to the representation of  $\delta\Delta$  given in Eq. (37).

That is (at least to leading order in  $\Lambda$ )

$$\delta\Delta(\Lambda) = \left(r_0^2/\alpha_0'^{D/2}\right)^{2/\epsilon} \int_{\left(r_0^2/\alpha_0'^{D/2}\right)^{2/\epsilon} \Lambda^{-1}}^{\infty} dx x^{-2} \left[1-Z_3^{-1}(x)\right]. \quad (64)$$

At least in the small  $\epsilon$  limit  $Z_3 \geq 1$  and so we see that  $\delta\Delta > 0$  and hence  $\alpha_0(0) - 1 > 0$ , where  $\alpha_0(0)$  is the bare Pomeron intercept. At  $\epsilon = 2$

$$\delta\Delta = \frac{r_0^2}{\alpha_0'} \int_{\frac{r_0^2}{\alpha_0' \Lambda}}^{\bar{x}_1} \frac{dx}{x^2} \left[ \frac{x}{16\pi} + o(x^2) \right] + \frac{r_0^2}{\alpha_0'} \int_{\bar{x}_1}^{\infty} \frac{dx}{x^2} \left[ 1-Z_3^{-1}(x) \right] \quad (65)$$

$$= \frac{-1}{16\pi} \frac{r_0^2}{\alpha_0'} \ln \left[ \frac{r_0^2}{\alpha_0' \Lambda} \right] + o\left(\frac{r_0^2}{\alpha_0'}\right). \quad (66)$$

Similarly one can show that  $\delta\Delta > 0$ , for general  $\epsilon$ , when  $\left(\frac{r_0^2}{\alpha_0' D/2}\right)^{2/\epsilon} \frac{1}{\Lambda}$  is sufficiently small. Equation (66) agrees with that obtained by Migdal, Polyakov and Ter-Martirosyan.<sup>4</sup> Note, however, that corrections to Eq. (66) are

$0\left(\frac{r_0^2}{\alpha_0'}\right)$  and not  $0\left(\frac{r_0^4}{\alpha_0'^2}\right)$  as suggested in Ref. 4. It should be clear from our discussion that we do not agree with the suggestion in Ref. 4, that  $\delta\Delta$  be determined perturbatively by a self-consistency equation. The second-order self-consistency equation given by Migdal et al. is

$$-\delta\Delta + \frac{1}{16\pi} \frac{r_0^2}{\alpha_0'} \ln \left[ \frac{r_0^2}{\alpha_0'[-\delta\Delta]} \right] = 0 \quad (67)$$

which does not have a real solution for  $\delta\Delta$ .

We can now envision a systematic calculation of the propagator. We were able to calculate the renormalization group functions in the one loop approximation because that did not involve the intercept renormalization counter term. We were then led to the first order approximation to  $\delta\Delta$  given in Eq. (62). This in turn can be used to construct an improved propagator which is exact to order  $g^2$  and has the Pommeranchuk singularity intercept at one. The improved propagator can in turn be used to calculate a new approximation to the renormalization group functions in which all one and two loop diagrams are treated correctly.<sup>12</sup> This procedure can in principle be repeated indefinitely.<sup>13</sup> In the n-loop approximation the critical exponents will be given correctly to nth order in  $\epsilon$ .

In order to carry out this program it is necessary to calculate  $\Gamma^{1,1}(E, k^2)$  for  $k^2 \neq 0$ . It can be expressed in terms of the variable

$x_1$  defined in Eq. (34) and

$$x_2 = \alpha'_0 k^2 / (-E) \quad . \quad (68)$$

Introducing the vector notation  $\vec{x} = (x_1, x_2)$  and  $\vec{\nabla} = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$  and noting that the origin of the  $(x_1, x_2)$  - plane corresponds to  $r_0 = 0$  , we can write the self-energy part as a line integral

$$\Sigma(E, k^2) = \int_{\vec{0}}^{\vec{x}} d\vec{x} \cdot \vec{\nabla} \cdot \Sigma \quad . \quad (69)$$

In order to evaluate  $\vec{\nabla} \Sigma$  we must carry out the renormalization at a general point  $E_N, k_N^2 > 0$  . Then from Eqs. (27) and (28) we have

$$Z_3^{-1} = 1 - \frac{\partial}{\partial E} \Sigma(E, k^2) \Big|_{\substack{E = -E_N \\ k^2 = k_N^2}} \quad (70)$$

$$Z_2^{-1} = 1 + \frac{1}{\alpha'_0} \frac{\partial}{\partial k^2} \Sigma(E, k^2) \Big|_{\substack{E = -E_N \\ k^2 = k_N^2}} \quad , \quad (71)$$

so

$$\begin{aligned} \frac{\partial}{\partial x_1} \Sigma(x_1, x_2) \Big|_{\substack{x_1 = x_{1N} \\ x_2 = x_{2N}}} &= \left( r_0^2 / \alpha'_0 \right)^{2/\epsilon} x_{1N}^{-2} \\ &\cdot \left[ \left( 1 - Z_3^{-1} \right) + x_{2N} \left( 1 - Z_2^{-1} \right) \right] \end{aligned} \quad (72)$$

and

$$\left. \frac{\partial}{\partial x_2} \Sigma(x_1, x_2) \right|_{\substack{x_1 = x_{1N} \\ x_2 = x_{2N}}} = - \left( r_0^2 / \alpha_0' \right)^{D/2} x_{1N}^{-1} (1 - Z_2^{-1})^{2/\epsilon} , \quad (73)$$

where  $x_{1N}$  is defined in Eq. (36) and

$$x_{2N} = \alpha_0' k_N^2 / E_N . \quad (74)$$

That the  $Z_i$  can be expressed as functions of  $x_{1N}$  and  $x_{2N}$  only follows from dimensional analysis. They can also be written in terms of two dimensionless renormalized parameters which we shall choose to be  $g$  and  $14$

$$y = \alpha_0' k_N^2 / E_N . \quad (75)$$

The intercept renormalization counter term is now given by

$$\delta \Delta = \int_{\vec{0}}^{\vec{\infty}} d\vec{x}' \cdot \vec{\nabla} \Sigma . \quad (76)$$

The integral in Eq. (76) can run over any path from the origin to infinity lying in the first quadrant of the  $(x_1, x_2)$  plane. Eq. (37) corresponds to the special choice of having the contour integral run along the  $x_1$ -axis. The full propagator takes the form

$$i \Gamma^{1,1}(E, k^2) = \left( r_0^2 / \alpha_0' \right)^{D/2} \int_{\vec{x}}^{\vec{\infty}} d\vec{x}' \cdot \vec{A}(x_1', x_2') / x_1'^2 , \quad (77)$$

with

$$A_1(x_1, x_2) = - [ Z_3^{-1}(x_1, x_2) + x_2 Z_2^{-1}(x_1, x_2) ] \quad (78)$$

and

$$A_2(x_1, x_2) = x_1 Z_2^{-1}(x_1, x_2) \quad . \quad (79)$$

Just as in Eq. (38) the bare propagator has been cancelled by the terms in  $\Sigma(E, k^2)$  which do not involve the  $Z_i$ . Equation (77) reduces to Eq. (38) when  $k^2 = 0$  provided one again takes the path of integration along the  $x_1$ -axis . It will often be convenient to express Eq. (77) as a one dimensional integral by simultaneously scaling  $E$  and  $k^2$ .<sup>15</sup> Writing

$$x_1 = x_{1N} \xi^{-1} \quad (80)$$

$$x_2 = x_{2N} \xi^a \quad (81)$$

gives

$$\begin{aligned} i\Gamma^{1,1}(E, k^2) &= -E_N \int_0^{\xi} d\xi' \left[ Z_3^{-1}(\xi') + (a+1)x_{2N} \xi'^a Z_2^{-1}(\xi') \right] \\ &= - \int_0^{-E} dE'_N Z_3^{-1}(E'_N, k_N'^2) - \int_0^{\alpha_0 k^2} d(\alpha_0 k_N'^2) Z_2^{-1}(E'_N, k_N'^2) \quad . \quad (82) \end{aligned}$$

In the last line of Eq. (82) the integrals are to be taken along the path

$$E'_N/E_N = (k_N'^2/k_N^2)^{1/(1+a)} \quad .$$

The  $Z_i$  can again be calculated using renormalization group techniques. There are now two  $\beta$ -functions.

$$\beta_E(g, y) = E_N \left. \frac{\partial}{\partial E_N} g \right|_{\substack{r_0, \alpha'_0, W_N \\ \text{fixed}}} \quad (83)$$

$$\beta_W(g, y) = W_N \left. \frac{\partial}{\partial W_N} g \right|_{\substack{r_0, \alpha'_0, E_N \\ \text{fixed}}} \quad (84)$$

where  $W_N = k_N^2/E_N$ . Similarly, we define  $\gamma_E, \gamma_W, \zeta_E$  and  $\zeta_W$  in analogy with Eqs. (43) and (44). In keeping with Eqs. (80) and (81) we let  $E_N \rightarrow E_N \xi$  and  $k_N^2 \rightarrow k_N^2 \xi^{a+1}$ . The  $\xi$  dependence of  $g$  and  $y$  can be determined from the differential equations ( $t \equiv \ln \xi$ )

$$\frac{dg(t)}{dt} = \beta_E[g(t), y(t)] + a\beta_W[g(t), y(t)] \quad (85)$$

and

$$\frac{dy(t)}{dt} = y(t) \left\{ a + \alpha'(t)^{-1} \left[ \zeta_E(g(t), y(t)) + a\zeta_W(g(t), y(t)) \right] \right\} \quad (86)$$

with the boundary conditions

$$g(0) = g \quad (87)$$

$$y(0) = y \quad (88)$$

The renormalization constants of interest can be obtained from the

differential equations

$$\frac{\partial}{\partial t} \ln Z_3(t) = \gamma_E(t) + a\gamma_W(t) \quad (89)$$

and

$$\frac{\partial}{\partial t} \ln [ Z_2(t)^{-1} Z_3(t) ] = \alpha'(t)^{-1} [ \zeta_E(t) + a\zeta_W(t) ] \quad (90)$$

with

$$Z_i(0) = Z_i \quad (91)$$

Since  $\xi \rightarrow \infty$  corresponds to  $r_0 \rightarrow 0$  we also know that

$$Z_i(\infty) = 1 \quad (92)$$

In order to see how the calculation goes let us consider the one loop approximation. In order to simplify the algebra we shall work in the small  $\epsilon$  limit. Then the renormalization group functions can be written in the form

$$\gamma_E = \gamma G^2/g_1^2 \quad (93)$$

$$\gamma_W = \gamma \frac{1}{2} y (1 + \frac{1}{2} y)^{-1} G^2/g_1^2 \quad (94)$$

$$\zeta_E/\alpha' = \delta G^2/g_1^2 \quad (95)$$

$$\zeta_W/\alpha' = \delta \frac{1}{2} y (1 + \frac{1}{2} y)^{-1} G^2/g_1^2 \quad (96)$$

$$\beta_E = -\frac{\epsilon}{4} g [ 1 - G^2/g_1^2 ] \quad (97)$$

$$\beta_W = \frac{\epsilon}{4} g \frac{1}{2} y (1 + \frac{1}{2} y)^{-1} G^2 / g_1^2 , \quad (98)$$

with

$$G^2 = g^2 (1 + \frac{1}{2} y)^{-\epsilon/2} . \quad (99)$$

$g_1$ ,  $\gamma$  and  $\delta$  are again given by Eqs. (57)-(59). Eqs. (85) and (86) can be rewritten in the form

$$\frac{\partial}{\partial t} G(t)^2 = -\frac{\epsilon}{2} G(t)^2 \left[ 1 + a \frac{1}{2} y(t) \left( 1 + \frac{1}{2} y(t) \right)^{-1} \right] \left[ 1 - G(t)^2 / g_1^2 \right] \quad (100)$$

and

$$\frac{\partial}{\partial t} y(t) = y(t) \left\{ a + \delta \left[ G(t)^2 / g_1^2 \right] \left[ 1 + a \frac{1}{2} y(t) \left( 1 + \frac{1}{2} y(t) \right)^{-1} \right] \right\} . \quad (101)$$

As long as the quantity  $\left\{ 1 + a \frac{1}{2} y(t) \left[ 1 + \frac{1}{2} y(t) \right]^{-1} \right\}$  remains positive,  $G(t)^2$  will approach the fixed point limit  $g_1^2$  as  $t \rightarrow -\infty$  ( $\xi \rightarrow 0$ ) . Combining Eq. (100) with Eqs. (89), (90) and (91) gives

$$\frac{\partial}{\partial t} \ln Z_3(t) = -\frac{2}{\epsilon} \gamma \frac{\partial}{\partial t} G(t)^2 / g_1^2 , \quad (102)$$

so

$$Z_3(t) = \left[ 1 - G(t)^2 / g_1^2 \right]^{\frac{2}{\epsilon} \gamma} . \quad (103)$$

Similarly we find that

$$Z_2(t) = \left[ 1 - G(t)^2 / g_1^2 \right]^{\frac{2}{\epsilon} (\delta - \gamma)} . \quad (104)$$

There are three cases of interest. First, if  $a < -\delta = \frac{\epsilon}{24}$  ,

then  $y(t) \xrightarrow[t \rightarrow -\infty]{} 0$  . From Eq. (100)

$$1-G(t)^2/g_1^2 \xrightarrow[t \rightarrow -\infty]{} C_0 e^{\frac{\epsilon}{2}t}, \quad (105)$$

where  $C_0$  depends on  $g$  and  $y$ , but not on  $t$ . Equation (82) then yields

$$i\Gamma^{1,1}(E, k^2) \xrightarrow[E, k^2 \rightarrow 0]{} -E_N C_0^{-\frac{2}{\epsilon} \gamma} \xi^{1-\gamma} \left[ 1 + x_{2N} C_0^{\frac{2}{\epsilon} \delta} \xi^{a+\delta} \right]. \quad (106)$$

Only terms of leading order in  $\epsilon$  have been retained in Eq. (106). The renormalized propagator takes the form

$$i\Gamma_R^{1,1}(E, k^2) \xrightarrow[E, k^2 \rightarrow 0]{} -E_N C_0^{-\frac{2}{\epsilon}} (-E/E_N)^{1-\gamma} \left[ 1 + C_0^{\frac{2}{\epsilon} \delta} \rho \right] \quad (107)$$

where

$$\rho = \frac{\alpha' k^2}{E_N (-E/E_N)^{1-\delta}}, \quad (108)$$

and

$$C_0' = C_0 \left[ 1 - G(0)^2/g_1^2 \right]^{-1}. \quad (109)$$

Equation (107) is in agreement with the result of Abarbanel and Bronzan.<sup>3</sup> Again, it holds for  $a < -\delta$ , i.e.,  $\rho \ll 1$ .

For  $a > -\delta$ ,  $y(t) \xrightarrow[t \rightarrow -\infty]{} \infty$ . Then

$$1-G(t)^2/g_1^2 \xrightarrow[t \rightarrow -\infty]{} C_\infty e^{\frac{\epsilon}{2}(1+a)t}, \quad (110)$$

and

$$i \Gamma_R^{1,1}(E, k^2) \xrightarrow[t \rightarrow -\infty]{} -\alpha' k_N^2 C_\infty'^{-\frac{2}{\epsilon}(\gamma-\delta)} (k^2/k_N^2)^{1-\gamma+\delta} \cdot \left[ 1 + C_\infty'^{-\frac{2}{\epsilon}\delta} \rho^{-1-\delta} \right], \quad (111)$$

for  $\rho \gg 1$ . Again  $C_\infty$  and  $C_\infty' = C_\infty (1-G(0)^2/g_1^2)^{-1}$  depend only on  $g$  and  $y$ . Finally for  $a = -\delta + 0(\epsilon^2)$ ,  $y(t)$  approaches a constant as  $t \rightarrow -\infty$ , as does the scaling variable,  $\rho$ . Working to leading order in  $\epsilon$ , the  $y$  dependence in Eq.(100) can be dropped and we find

$$1-G(t)^2/g_1^2 \xrightarrow[t \rightarrow -\infty]{} \left[ G(0)^2/g_1^2 \right]^{-1} \left[ 1-G(0)^2/g_1^2 \right] e^{\frac{\epsilon}{2}t}, \quad (112)$$

and

$$i \Gamma_R^{1,1}(E, k^2) \xrightarrow[E, k^2 \rightarrow 0]{} -E_N \left[ G(0)^2/g_1^2 \right]^{\frac{2}{\epsilon}\gamma} (-E/E_N)^{1-\gamma} \cdot \left\{ 1 + \left[ G(0)^2/g_1^2 \right]^{\frac{2}{\epsilon}\delta} \rho \right\} \\ = -\alpha' k_N^2 \left[ G(0)^2/g_1^2 \right]^{\frac{2}{\epsilon}(\gamma-\delta)} (k^2/k_N^2)^{1-\gamma+\delta} \cdot \left\{ 1 + \left[ G(0)^2/g_1^2 \right]^{\frac{2}{\epsilon}\delta} \rho^{-1-\delta} \right\}, \quad (113)$$

for all values of  $\rho$ .

Equations (107), (111) and (113) provide an example of a general relation among the scaling indices. If we write

$$i \Gamma_R^{1,1}(E, k^2) = (-E/E_N)^{1-\gamma} F(\rho) = (k^2/k_N^2)^\nu G(\rho) \quad , \quad (114)$$

then

$$\nu = \frac{1-\gamma}{1-\delta} \quad , \quad (115)$$

in agreement with our results to leading order in  $\epsilon$  .

If we go beyond the one loop approximation we will still obtain scaling results analogous to Eqs. (107), (111) and (113) provided the effective  $\beta$ -function  $\beta_E + a\beta_W$  , has an infra-red stable zero. The scaling indices  $\gamma$  and  $\delta$  will again be given by the values of  $\gamma_E + a\gamma_W$  and  $(\zeta_E + a\zeta_W)/\alpha'$  at the Gell Mann-Low zero. The integral representation of Eq. (82) will converge provided  $\gamma < 1$  and  $[\gamma - (a + \delta)] < 1$  . Under these circumstances we can again show that the perturbation series is an asymptotic expansion both for small values of  $r_0$  and for large values of  $-E$  and (or)  $k^2$  . Again the perturbation series is valid only to order  $n < \frac{2}{\epsilon}$  . The argument is essentially the same as for the  $k^2 = 0$  case.

#### IV. DISCUSSION

In this paper we have studied the problem of constructing solutions to the Reggeon field theory when the Pommeranchuk singularity has intercept one. We have seen that this theory cannot be renormalized order by order in the perturbation series. In fact our integral representation

for the intercept renormalization counter term

$$\delta \Delta = \left( r_0^2 / \alpha_0' \right)^{D/2} \int_0^\infty \frac{dx}{x^2} \left[ 1 - Z_3(x)^{-1} \right] \quad (116)$$

shows explicitly that  $\delta \Delta$  has a branch point at  $r_0 = 0$ . This is the case even for  $\epsilon = 2$ . The integral in Eq. (116) has a pole in  $\epsilon$  at this point so  $\delta \Delta$  is well defined only if one introduces a cut-off into the theory. One then finds that

$$\delta \Delta = - \frac{r_0^2}{16\pi\alpha_0'} \ln(r_0^2 / \alpha_0' \Lambda) + o(r_0^2 / \alpha_0') \quad (117)$$

At least for small values of  $\epsilon$ , or  $(r_0^2 / \alpha_0' \Lambda)^{D/2} \frac{1}{\Lambda}$ ,  $\delta \Delta$  is positive which means that the intercept of the bare Pommeranchuk pole must be greater than one.

We have given a non-perturbative prescription for constructing the Pommeranchuk propagator by use of the integral representation

$$i \Gamma^{1,1}(E, k^2) = - \int_0^\xi d\xi' E_N Z_3^{-1}(\xi') + \alpha_0' k_N^2 (a+1) \xi'^a Z_2^{-1}(\xi') \quad (118)$$

where  $\xi = (-E/E_N) = \left( k^2 / k_N^2 \right)^{\frac{1}{1+a}}$ . The renormalization constants  $Z_2$  and  $Z_3$  which enter into Eqs. (116) and (118) can be computed systematically using renormalization group techniques as is described in Sec. III.

Equation (118) is useful for a variety of purposes. In Sec. III we saw how it could be used to derive the infra-red behavior of the propagator. It also allowed us to see how the tachyon which is present in second order in the perturbation series is removed from the exact solution.

The tachyon arose from a cancellation between the bare inverse propagator and the second order approximation to the self-energy part. However, our integral representation for  $\Sigma(E, k^2)$  contained linear terms in  $E$  and  $k^2$  which exactly cancelled the bare inverse propagator and left us with Eq. (118) which is explicitly free of tachyons. The cancellation of the bare inverse propagator is also crucial for the infra-red behavior, since it goes to zero for small  $E$  and  $k^2$  more slowly than the full inverse propagator. One might be concerned that this cancellation would be destroyed if one altered the large  $E$  and  $k^2$  behavior of the theory by, for example, introducing a cut-off or introducing  $E$  and  $k^2$  behavior into  $r_0$ .<sup>3</sup> Such effects must be taken into account in a more general formulation of the Reggeon field theory. In fact under these circumstances we can obtain integral representations analogous to the ones given in Sec. III, and again demonstrate the cancellation of the bare inverse propagator. The infra-red behavior of the theory appears to be unchanged provided  $r_0$  approaches a constant for small values of  $E$  and  $k^2$ .

We have also used our integral representation for the propagator to show that the perturbation series is an asymptotic expansion for small values of  $r_0$  as well as for large values of  $E$  and/or  $k^2$ . The perturbation series is accurate only to order  $n < 2/\epsilon$ , and to this order one can neglect all diagrams involving intercept renormalization

counter terms. The small  $r_0$  result is important for the  $\epsilon$ -expansion calculations of the critical exponents.<sup>3,4</sup> If one wishes to calculate them to  $n$ th order in  $\epsilon$ , then one can imagine working with  $\epsilon < 2/n$  and use perturbation theory to calculate the renormalization group functions. Of course at the end one may attempt to continue the answers to larger values of  $\epsilon$ .<sup>16</sup>

The large  $(E, k^2)$  result is also of considerable importance. The full propagator will approach the bare one for large negative values of  $E$  and  $k^2$  provided one stays away from the cuts, i. e., provided  $\alpha' k^2/E < 2$ . This means that for large positive values of  $t$  (negative values of  $k^2$ ) the leading  $\ell$ -plane singularity will be a pole. It is only in this case that one is guaranteed that the solution to the field theory satisfies full multi-particle  $t$ -channel unitarity in the  $\ell$ -plane.<sup>2</sup>

Since the order  $\epsilon^2$  calculations of the critical exponents indicate that the  $\epsilon$ -expansion is not converging rapidly, it seems worthwhile to ask whether the approach presented here can be used to perform practical calculations directly in two dimensions. It is crucial for our procedure that an infra-red stable Gell Mann-Low eigenvalue exist at each step of the calculation. As we have seen, this is the case in the one loop approximation. If one wishes to go beyond the one loop approximation in two dimensions, it is not possible to use perturbation theory to calculate the renormalization group functions. Instead one

must use the latest approximation to the propagator at each step of the calculation. This may actually be an advantage. Since the improved propagator has the type of infra-red behavior that one expects in the exact solution, our iteration scheme may converge more rapidly than the perturbation series for small values of  $\epsilon$ . Even if this is the case, the higher order calculations will certainly be difficult.

#### ACKNOWLEDGMENTS

We would like to thank J. B. Bronzan for innumerable discussions and for making his unpublished work available to us. We would also like to thank H. D. I. Abarbanel and B. W. Lee for helpful conversations.

## APPENDIX

We have seen in the text that for many purposes it is sufficient to study the self-energy part at  $k^2 = 0$ . For simplicity we shall do so here.

Let us start with the bubble diagrams of Fig. 3. We denote by

$\Sigma_{n_1 k_1; n_2 k_2}$  the contribution to  $\Sigma$  of the diagrams in which the upper and lower lines have  $n_1$  and  $n_2$  inserted bubbles and  $k_1$  and  $k_2$  intercept counter terms respectively. From Eq. (6) we see that

$$\begin{aligned} \Sigma_{n_1 k_1; n_2 k_2}(E, 0) &= \frac{i}{2} (Cr_0^2)^{n_1+n_2} \frac{r_0^2}{(2\pi)^{D+1}} \frac{(n_1+k_1)!}{n_1! k_1!} \frac{(n_2+k_2)!}{n_2! k_2!} (-\delta \Delta)^{k_1+k_2} \\ &\cdot \int dE' d^D q \left[ \alpha_0' q^2 + \Delta - E' - i\epsilon \right]^{-(n_1+k_1+1)} \\ &\left[ \alpha_0' q^2 + \Delta - E + E' - i\epsilon \right]^{-(n_2+k_2+1)} \left[ \frac{1}{2} \alpha_0' q^2 + 2\Delta - E' - i\epsilon \right]^{n_1(1-\epsilon/2)} \\ &\cdot \left[ \frac{1}{2} \alpha_0' q^2 + 2\Delta - E + E' - i\epsilon \right]^{n_2(1-\epsilon/2)}. \end{aligned} \quad (A1)$$

The factorials count the number of ways of ordering the bubbles and counter terms on each line. Making use of the identity

$$x^\alpha = -\frac{\sin \pi \alpha}{\pi} \int_0^\infty dx' \frac{x'^\alpha}{x' + x} \quad (A2)$$

and introducing Feynman parameters to combine denominators with the same  $E$  dependence we find

$$\begin{aligned}
 \sum_{n_1 k_1; n_2 k_2} (E, 0) &= \frac{i}{2} (Cr_0^2)^{n_1+n_2} \frac{r_0^2}{(2\pi)^{D+1}} \frac{(n_1+k_1)! (n_2+k_2)!}{n_1! k_1! n_2! k_2!} (-\delta \Delta)^{k_1+k_2} \\
 &\cdot \frac{\sin \pi n_1 (1-\epsilon/2)}{\pi} \cdot \frac{\sin \pi n_2 (1-\epsilon/2)}{\pi} \\
 &\cdot \int_0^1 dz_1 z_1^{n_1+k_1} \int_0^1 dz_2 z_2^{n_2+k_2} \int_0^\infty dx_1 x_1^{n_1(1-\epsilon/2)} \\
 &\int_0^\infty dx_2 x_2^{n_2(1-\epsilon/2)} \int dE' d^D q \\
 &\cdot \left[ z_1 \left( \alpha_0' q^2 + \Delta - E' \right) + (1-z_1) \left( x_1 + \frac{1}{2} \alpha_0' q^2 + 2\Delta - E' \right) - i\epsilon \right]^{-(n_1+k_1+2)} \\
 &\cdot \left[ z_2 \left( \alpha_0' q^2 + \Delta - E + E' \right) + (1-z_2) \left( x_2 + \frac{1}{2} \alpha_0' q^2 + 2\Delta - E + E' \right) - i\epsilon \right]^{-(n_2+k_2+2)}
 \end{aligned} \tag{A3}$$

The  $E'$  integration can now be done by closing the contour in either the upper or lower half plane and picking up the pole arising from the appropriate denominator. We next use the fact that

$$\int d^D q \left[ A q^2 + B \right]^{-N} = \pi^{D/2} A^{-D/2} B^{D/2-N} \frac{\Gamma(N-D/2)}{\Gamma(N)} \tag{A4}$$

to do the  $q$  integration. The  $x_1$  and  $x_2$  integrations then yield  $\beta$ -functions, and we finally find

$$\begin{aligned}
\sum_{n_1 k_1; n_2 k_2} (\mathbb{E}, 0) &= \left( Cr_0^2 \right)^{n_1+n_2+1} \frac{4^{D/2}}{\Gamma(-1+\epsilon/2)} \frac{\Gamma \left[ k_1+k_2-1+\frac{\epsilon}{2} (n_1+n_2+1) \right]}{\Gamma \left[ \left( -1+\frac{\epsilon}{2} \right) n_1 \right] \Gamma \left[ \left( -1+\frac{\epsilon}{2} \right) n_2 \right]} \\
&\cdot \frac{\left( -\delta \Delta \right)^{k_1+k_2}}{n_1! k_1! n_2! k_2!} \int_0^1 dz_1 \int_0^1 dz_2 \frac{z_1^{n_1+k_1}}{z_2^{n_2+k_2}} \\
&\cdot \left( 1-z_1 \right)^{-(1-\epsilon/2)n_1-1} \left( 1-z_2 \right)^{-(1-\epsilon/2)n_2-1} \\
&\cdot \left( 2+z_1+z_2 \right)^{-D/2} \left[ \left( 4-z_1-z_2 \right) \Delta - \mathbb{E} \right]^{- \left[ k_1+k_2-1+\frac{\epsilon}{2} (n_1+n_2+1) \right]} \quad (A5)
\end{aligned}$$

We see explicitly from Eq. (A5) that there is no difficulty in taking the limit  $\Delta \rightarrow 0$  provided  $\mathbb{E}$  is off the positive real axis. Then using the mean value theorem to remove the harmless factor  $\left( 2+z_1+z_2 \right)^{-D/2}$  from the integrand, we find

$$\begin{aligned}
\sum_{n_1 k_1; n_2 k_2} (\mathbb{E}, 0) &= A^{-D/2} \left( Cr_0^2 \right)^{n_1+n_2+1} \frac{4^{D/2}}{\Gamma(-1+\epsilon/2)} \\
&\frac{\Gamma \left[ k_1+k_2-1+\frac{\epsilon}{2} (n_1+n_2+1) \right]}{\Gamma \left( k_1+1+n_1 \frac{\epsilon}{2} \right) \Gamma \left( k_2+1+n_2 \frac{\epsilon}{2} \right)} \frac{(n_1+k_1)! (n_2+k_2)!}{n_1! k_1! n_2! k_2!} \\
&\left( -\delta \Delta \right)^{k_1+k_2} \left( -\mathbb{E} \right)^{1-k_1-k_2-\frac{\epsilon}{2} (n_1+n_2+1)} \quad , \quad (A6)
\end{aligned}$$

where  $2 \leq A \leq 4$ . Recalling that the graph is of order  $n = n_1+n_2+1$  in  $r_0^2$  and has  $m = k_1+k_2$  factors of  $\delta \Delta$ , we see that the  $\mathbb{E}$  dependence and the position of the pole arising from the over-all ultra-violet

divergence of the graph are in agreement with the general result given in the text. Clearly the position of the poles in  $\epsilon$  would be unaffected by taking  $\Delta$  and  $k^2$  different from zero.

In addition to the simple pole at  $\epsilon = 2(1-k_1-k_2)/(1+n_1+n_2)$  arising from the overall ultra-violet divergence of the diagram, there is a pole of order  $n_1+n_2+k_1+k_2$  at  $\epsilon = 2$ .  $n_1+n_2$  powers of  $(\epsilon-2)^{-1}$  come from the inserted bubbles and  $k_1+k_2$  powers from the intercept counter terms. Notice that in the neighborhood of  $\epsilon = 2$  the simple bubble graph is proportional to  $\frac{r_0^2}{\epsilon-2}$ . For  $\epsilon = 2$  this pole leads to a  $\ln(-E)$  contribution to  $\Sigma(E, 0)$  as  $E \rightarrow 0$ . On the other hand the graphs with  $N = n_1+n_2+k_1+k_2 > 0$  are proportional to  $r_0^2 \left[ r_0^2 / (\epsilon-2) \right]^N$  for  $\Delta > 0$ . At each order in the coupling constant, they are the most divergent graphs in the limit  $\epsilon \rightarrow 2$ , so one might hope that their sum would soften the small  $E$  behavior of the simple bubble graph and lead to a reasonable approximation for  $\Sigma$ . In  $\phi^4$  theory the corresponding set of graphs does soften the infra-red behavior and leads to the screening approximation.<sup>17</sup> Unfortunately, this is not the case in the Reggeon calculus because of the presence of the tachyon in the second order approximation to the propagator. The leading  $l$ -plane singularity of the sum of the  $\Sigma_{n_1 k_1; n_2 k_2}$  is a tachyon-tachyon cut, which has all of the bad features of the Pomeron-Pomeron cut of the simple bubble graph plus the added difficulty of having its intercept to the right of  $l = 1$ .

We now turn to the ladder graphs of Fig. 2. They are a subset of the skeleton graphs discussed in the text. Denoting by  $\Sigma_n$  the contribution to  $\Sigma$  of those graphs with  $n-1$  rungs on the ladder we have

$$\begin{aligned} \Sigma_n(E, 0) &= \frac{1}{2} \frac{(-ir_0^2)^n}{(2\pi)^{(D+1)n}} \int \prod_{i=1}^n d^D q_i dE_i \\ &\prod_{i=1}^n \left[ \frac{1}{2} E + E_i - \alpha'_0 q_i^2 + i\epsilon \right]^{-1} \left[ \frac{1}{2} E - E_i - \alpha'_0 q_i^2 + i\epsilon \right]^{-1} \\ &\prod_{i=1}^{n-1} \left\{ \left[ E_{i+1} - E_i - \alpha'_0 (\vec{q}_i - \vec{q}_{i+1})^2 + i\epsilon \right]^{-1} \right. \\ &\left. + \left[ E_i - E_{i+1} - \alpha'_0 (\vec{q}_i - \vec{q}_{i+1})^2 + i\epsilon \right]^{-1} \right\}. \end{aligned} \quad (A7)$$

The first product of Green's functions in Eq. (A7) arises from the sides of the ladder and the second product from the rungs. A sum of two Green's functions is associated with each rung because of the two possible directions of propagation of these Pomerons.

For simplicity we shall consider the case  $\epsilon \ll 1$ . Then the integrals in Eq. (A7) can be done in the following way. We first do the  $E_1$  integration. For the term containing the rung  $\left[ E_1 - E_2 - \alpha'_0 (\vec{q}_1 - \vec{q}_2)^2 + i\epsilon \right]^{-1}$  we close the  $E_1$  contour in the upper half plane, and for the rung  $\left[ E_2 - E_1 - \alpha'_0 (\vec{q}_1 - \vec{q}_2)^2 + i\epsilon \right]^{-1}$  we close in the lower half plane. In

either case we pick up only one pole from a Green's function associated with a side of the ladder, and we are left with a  $q_1$  integral of the form

$$\begin{aligned} & \frac{1}{(2\pi)^D} \int d^D q_1 \left[ 2\alpha'_0 q_1^2 - E - i\epsilon \right]^{-1} \left[ \alpha'_0 q_1^2 + \alpha'_0 (\vec{q}_1 - \vec{q}_2)^2 - E_2 - \frac{1}{2}E - i\epsilon \right]^{-1} \\ & = \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{(8\pi\alpha'_0)^{D/2}} \int_0^1 dz \left[ \alpha'_0 q_2^2 z(1-\frac{1}{2}z) - zE_2 - E(1-\frac{1}{2}z) - i\epsilon \right]^{-\epsilon/2}. \end{aligned} \quad (A8)$$

Now when we do the  $E_2$  integration we can ignore the contribution of the cut in  $E_2$  arising from the term in Eq. (A8), because the discontinuity across this cut is of order  $\epsilon$ . As a result, we can close the  $E_2$  contour so as to pick up a single pole arising from a side of the ladder. The  $q_2$  integration can then be done using Eqs. (A2) and (A4). Proceeding in this way we find that to leading order in  $\epsilon$

$$\sum_n \langle E, 0 \rangle = \frac{1}{4} \left[ -4r_0^2 / (8\pi\alpha'_0)^{D/2} \epsilon \right]^n \frac{(-E)^{1-n\epsilon/2}}{n! (1-n\epsilon/2)}. \quad (A9)$$

As a result, from Eqs. (35) and (38)

$$Z_3^{-1}(x) = \frac{3}{4} + \frac{1}{4} \exp\left[-4x^{\epsilon/2} / (8\pi)^{D/2} \epsilon\right], \quad (A10)$$

and

$$\begin{aligned} i \Gamma^{1,1}(E, 0) & = - \left( r_0^2 / \alpha'_0 \right)^{D/2} \int_{x_1}^{\infty} \frac{dx}{x^2} Z_3^{-1}(x) \\ & = E \left[ \frac{3}{4} + \frac{1}{4} \exp\left\{ - \frac{r_0^2}{(8\pi\alpha'_0)^{D/2} \epsilon} (-E)^{-\epsilon/2} \right\} \right]. \end{aligned} \quad (A11)$$

Again the tachyon, which is present in second order, disappears when the infinite set of graphs is summed. The essential singularity in  $\Gamma^{1,1}(E, 0)$  at  $E = 0$  was also found by Bronzan.<sup>18</sup> As we have seen in the text, it is not present in the full solution of the theory. The ladder graphs simply do not give a good indication of the small  $E$  behavior of the propagator.

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- <sup>7</sup>A. R. White, NAL-Pub-74/15-THY, to be published in Phys. Rev. D.
- <sup>8</sup>Throughout this paper we shall make use of dimensional regularization. All integrals we shall encounter are well defined for  $\epsilon > 2$ . After

the integration has been carried out we can continue the answers to the value of  $\epsilon$  of interest.

<sup>9</sup> Depending on the topology of the diagram, there may be additional factors of  $\frac{1}{2}$  arising from closed loops.

<sup>10</sup> Equation (20) follows directly from Eq. (61) coupled with Eq. (57)-(59).

<sup>11</sup> J. B. Bronzan (unpublished).

<sup>12</sup> Of course, we can treat the  $n$ -loop graphs by ordinary perturbation theory for  $n < \frac{2}{\epsilon}$ .

<sup>13</sup> The use of this iteration procedure to construct both massless  $\phi^4$  theory and the Reggeon calculus is described in more detail in Ref. 6.

<sup>14</sup> The second dimensionless variable that we use here differs from that used in Ref. 6. The variables used in Ref. 6 are more suitable for a general construction of the theory, emphasizing the symmetry in  $E$  and  $k^2$ . The variables used here seem to be more suitable for explicit calculation and in particular for a discussion of the scaling laws.

<sup>15</sup> This was pointed out to us by J. B. Bronzan (private communication).

<sup>16</sup> At first glance one might be concerned that the branch point in the propagator at  $g = 0$  would lead to nonanalyticity in the critical indices at  $\epsilon = 0$ ; however, this does not appear to be the case. The point is discussed in Ref. 6 for  $\lambda\phi^4$ , but the argument given there cannot be carried over directly due to the added complication that, in the present problem, the propagator depends on two variables. Probably the most

convincing way to argue that the critical indices have a power series expansion in  $\epsilon$  is to reformulate the problem using the Wilson approach to the renormalization group. (J. B. Bronzan, unpublished).

<sup>17</sup>R. A. Ferrell and D. J. Scalapino, PRL 29, 413 (1972).

<sup>18</sup>J. B. Bronzan, Phys. Rev. D7, 480 (1972).

FIGURE CAPTIONS

- Fig. 1                    Lowest order contribution to the self-energy part.
- Fig. 2                    A typical "ladder" graph.
- Fig. 3                    A typical "bubble" graph - x denotes an intercept  
counter-term.
- Fig. 4                    Lowest order contributions to the vertex function  
needed to calculate  $\beta(g)$  in the one loop approximation.

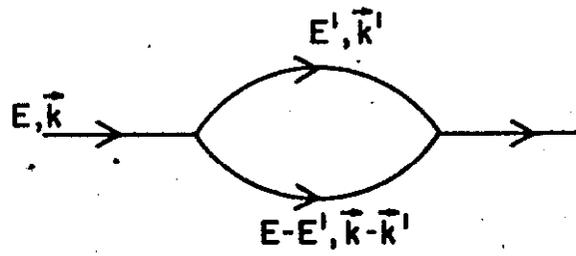


FIG. 1

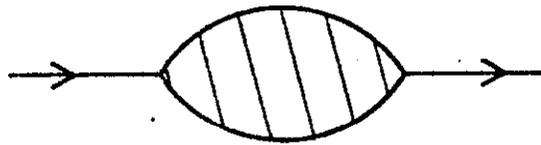


FIG. 2

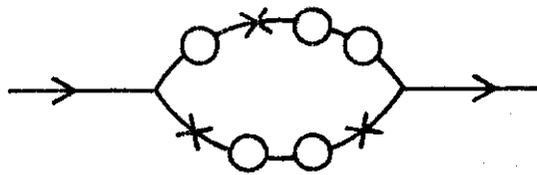


FIG. 3

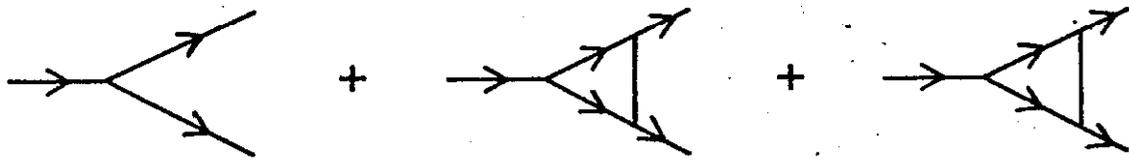


FIG. 4

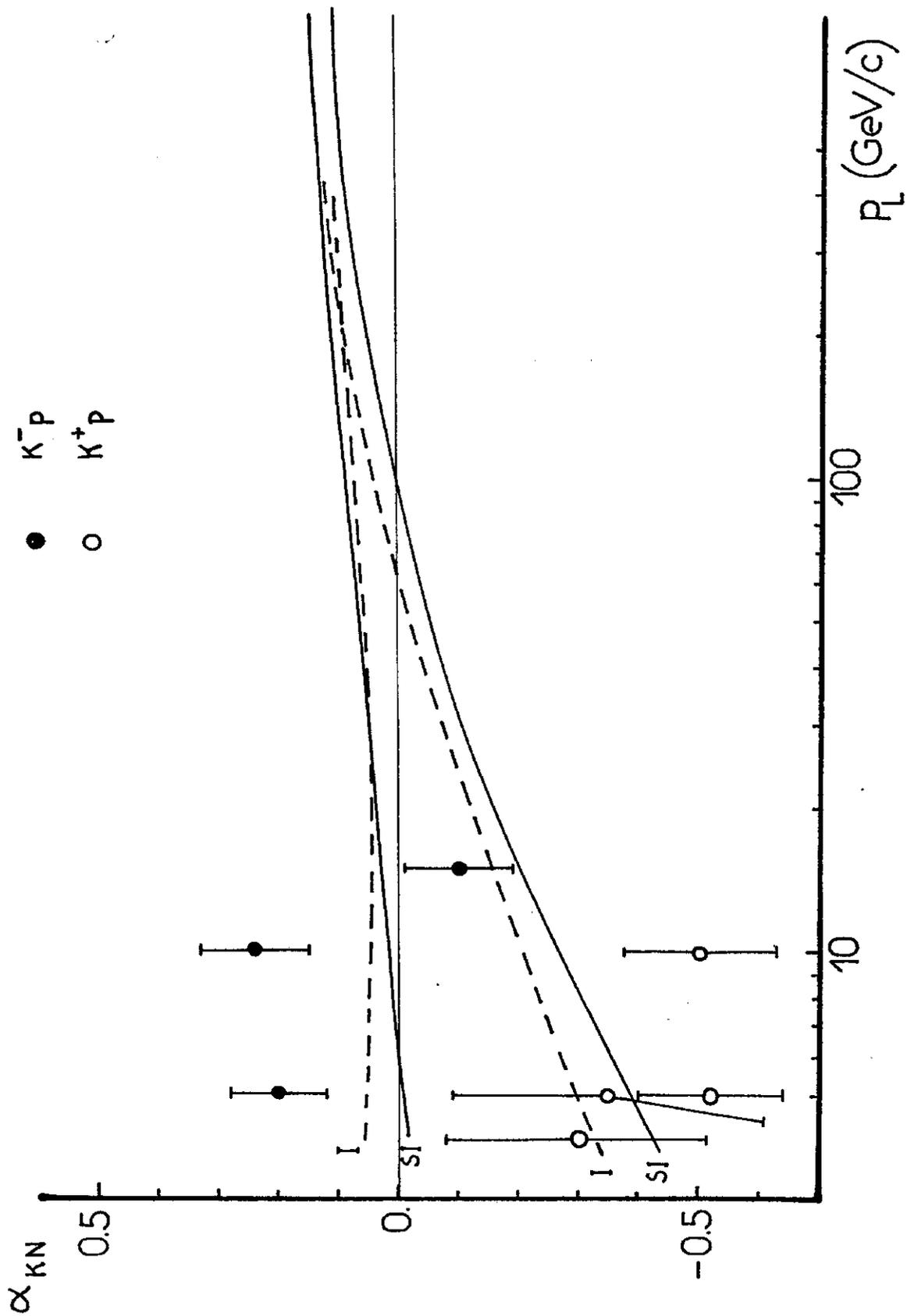


Fig. 6