national accelerator laboratory

FNAL-Pub-74/58-THY
June 1974

# Renormalization Group Sum Rules and the Construction of Massless Field Theories in 4-$\epsilon$ Dimensions

R. L. SUGAR* and A. R. WHITE
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

## ABSTRACT

We study massless $\phi^4$ field theory and the Reggeon calculus with Pomeron intercept one, in 4-$\epsilon$ dimensions. We present sum rules which give the full propagator and the bare mass (or intercept) as integrals over the remaining (finite) renormalization constants of these theories. When an infra-red stable Gell-Mann-Low eigenvalue exists these sum rules can be used to extract the infra-red behavior of the propagator. They can also be used to show that the perturbation series is an asymptotic expansion for small values of the coupling constant and large values of the momentum. The sum rules can be combined with the Schwinger-Dyson equations for each theory to give a perturbative construction of the Green's functions which is free of infra-red divergences.

*Permanent address: Physics Department, University of California, Santa Barbara, CA. 93106.

Operated by Universities Research Association Inc. Under Contract with the United States Atomic Energy Commission

# I. INTRODUCTION

Massless field theories in $4-\epsilon$ dimensions are of great interest in both solid state and high-energy physics. In the study of critical point phenomena the field theory of major interest is relativistic $\phi^4$ theory (analytically continued to the Euclidean region).[1-3] In the high energy Pomeranchuk problem the relevant field theory is the "non-relativistic" $\psi^3$ theory, better known as the Reggeon calculus.[4-6] The unifying feature of these problems is that in both cases the development of long-range order leads to scaling laws for the correlation (Green's) functions in the infra-red region. In both cases the critical exponents and scaling functions can be directly calculated using renormalization group techniques. However, the construction of these theories in perturbation theory is (for finite rational $\epsilon$) plagued with infra-red divergences.[2,3,7] If these field theories are renormalizable at all in $4-\epsilon$ dimensions, then they are super-renormalizable. That is infra-red divergences can be related only to the mass renormalization. From dimensional analysis the bare mass $m_0$ in the $\phi^4$ theory is related to the coupling constant $g_0$ by

$$m_0 = g_0^{\frac{1}{\epsilon}} f(\epsilon) \tag{1}$$

and so it gives rise to terms non-analytic in $g_0$, and hence to divergences of perturbation theory. $m_0$ also contains an essential singularity at

$\epsilon = 0$, and so in the usual $\epsilon$-expansion of the theory is taken to be zero. It is desirable, therefore, to have a method for constructing these theories which avoids the $\epsilon$-expansion, as well as the difficulties of perturbation theory.

In this paper we present sum rules, valid in both theories, which give both the bare mass and the full propagator as integrals over the finite renormalization constants of the theories. The integral representation for the propagator explicitly displays the anomalous dimension infra-red behavior when a stable Gell-Mann-Low eigenvalue is present (which in perturbation theory is the case in both theories, at least for small $\epsilon$). It can also be used to show that the bare perturbation expansion is an asymptotic expansion valid for small values of the coupling constant or large values of the momentum. We further show that our sum rules can be combined with Schwinger-Dyson equations for each of the theories to give an iterative construction procedure which is free of infra-red divergences. The question of the convergence of this iteration procedure goes beyond that of the convergence of the perturbation series for massive theories because of the non-analyticity of $m_0$. However, at each step of the calculation the approximation for $m_0$ is systematically improved, as is explained in the text. As a result, one may be optimistic about the convergence of our procedure. If it does converge, both theories are renormalizable and continuable in $\epsilon$.

We shall concentrate on the application of our results to $\phi^4$ in this paper and give only a brief treatment of the Reggeon calculus. A more extensive treatment of the Reggeon Calculus can be found in a companion paper.[8] Previous discussions of the existence and construction of massless $\phi^4$, in $4-\epsilon$ dimensions, have been based on taking the zero mass limit of the massive theory. Symanzik[7] has shown that a non-zero bare mass is obtained in this limit and that the bare massive propagator can be used to obtain a perturbation expansion of Green's functions which is free of infra-red singularities. Unfortunately, to use this expansion, it is necessary to know the bare mass. Symanzik[9] has given an integral representation for the bare mass in terms of the functions appearing in the Callan-Symanzik equation for the massive theory, but it is not clear whether this can be used perturbatively. Also Symanzik's expansion cannot be used to study the infra-red behavior of the theory. Parisi[3] has given an integral representation for the Green's functions of the massless theory, in terms of the $\phi^2$-inserted functions of the massive theory, which can be used to study infra-red behavior.

A major advantage of our approach is that we work entirely within the massless theory. This is a particular advantage in the Reggeon Calculus where the massive theory (the mass corresponds to the displacement of the Pomeranchuk intercept below one) also runs into difficulties in perturbation theory, due to the presence of a tachyon arising from the pure imaginary coupling constant.[8] By employing our sum rule iteratively, we obtain, at each stage of our perturbation construction, a propagator which vanishes at zero momentum with anomalous dimension. Our sum rules can also be used to express the bare mass in terms of the parametric functions of the renormalization group. The formula obtained is analagous to that obtained by Symanzik,[9] except that Symanzik's formula involves the parametric functions of the Callan-Symanzik equation for the massive theory.

The essential problem in constructing massless field theories (in $4-\epsilon$ dimensions) is that the divergences which appear in finite order perturbation theory only disappear or can be properly subtracted after an infinite set of diagrams is summed. Therefore, it is necessary to re-organize the perturbation expansion in some way. This could perhaps be done by selective re-organization and summation of Feynman diagrams and we discuss this in detail for the Reggeon Calculus in Ref. 8. The approach we use in this paper is based on the renormalization group.

We first calculate the parametric functions of the renormalization group using lowest (non-trivial) order perturbation theory. The renormalization constants of the theory are related to these functions by well-known formulae and so can be calculated next. Using the expressions obtained we employ our sum rules to calculate an improved propagator and four-point function. These are then substituted into the Schwinger-Dyson equations to obtain the next order calculation of the parametric functions. We therefore have an iteration process which is free of infra-red divergences and which, apart from mass counter-terms (which are non-analytic in $g_0$), gives Green's functions that coincide with perturbation theory to the order we are calculating. In effect we use the renormalization group to sum partial contributions from higher-order perturbation theory which produce the mass counter-term and avoid the infra-red divergences. This procedure works for $0 < \epsilon < 1 + \frac{3}{2}\eta$ for the $\phi^4$ theory, where $\eta$ (positive) is the anomalous dimension of the propagator. There is no such restriction in the Reggeon Calculus.

In Sec. II, we derive our sum rules and develop the necessary renormalization group analysis. We also give our formula for the bare mass and show that our formula for the propagator reduces to the bare propagator at large momentum.

In Sec. III, we discuss the infra-red divergences and present our iteration procedure using the Schwinger-Dyson equations together with the skeleton expansions of higher Green's functions. We use the skeleton expansions to build up all Green's functions, employing the complete propagator and four-point function as building blocks.

Section IV contains a rather formal general treatment of the Reggeon Calculus. This is simply meant to illustrate that our methods can be extended to this more complicated problem. For a more detailed and less formal analysis of the Reggeon Calculus we refer the reader to Ref. 8.

## II. SUM RULES AND RENORMALIZATION CONSTANTS

We shall construct the massless theory directly. Therefore, we work with "massless" unrenormalized and renormalized Lagrangians

$$\mathscr{L}_u = -\frac{1}{2}[\partial_u \phi_0(x)]^2 - \frac{g_0}{4!} : \phi_0^4(x): + \frac{1}{2}\delta m^2 \phi_0^2(x) \tag{2}$$

$$\mathscr{L}_R = -\frac{1}{2} Z_3 [\partial_u \phi(x)]^2 - Z_1 \frac{g}{4!} : \phi^4(x): + \frac{1}{2} Z_3 \delta m^2 \phi^2(x) \tag{3}$$

$Z_3$ is the wave-function renormalization constant, $Z_1$, is the coupling constant renormalization constant and $\delta m^2$ is the mass counter-term ($= -m_0^2$). So

$$\phi_0(x) = Z_3^{\frac{1}{2}} \phi(x) \tag{4}$$

$$g_0 = Z_1 Z_3^{-2} g \tag{5}$$

and if $\Gamma_u^{(N)}(P_i;g_0)$ and $\Gamma_R^{(N)}(P_i;g)$ are respectively unrenormalized and renormalized (amputated) Green's functions then

$$\Gamma_R^{(N)}(P_i;g) = Z_3^{\frac{N}{2}} \Gamma_u^{(N)}(P_i;g_0) \tag{6}$$

The renormalization conditions we choose to specify $Z_1$, $Z_3$ and $\delta m^2$ are

$$\Gamma_R^{(2)}(P^2;g)\Big|_{P^2=0} = 0 \tag{7}$$

$$\left.\frac{\partial i\Gamma_R^{(2)}}{\partial P^2}(P^2;g)\right|_{P^2=-\mu^2} = 1 \qquad (8)$$

$$\left. i\Gamma_R^{(4)}(P,\theta_k;g)\right|_{\substack{P^2=-\mu^2 \\ \theta_k=\mu_k}} = g \qquad (9)$$

P is some convenient momentum (which we shall take to be $(P_1+P_2)/2$) $\theta_k$ are a suitable set of dimensionless parameters formed from the ratios $P_i \cdot P_j / P^2$, $i,j = 1, \ldots, 4$. Note that we have not specified the $\mu_k$. The only condition on them is that they do not correspond to exceptional values of the four-momentum variables.[7] Varying the $\mu_k$ will simply lead to a redefinition of $Z_1$, and so (9) can be regarded as giving $Z_1$ a dependence on the $\mu_k$.

Conventionally we would impose the above renormalization conditions order by order in a perturbation expansion in the renormalized coupling constant g. This would determine $Z_1$, $Z_3$ and $\delta m^2$ to the relevant order. In particular, we would expect that $\delta m^2$ could be determined by imposing (7) directly on $\Gamma_u^{(2)}(P;g_0)$. Throughout the paper we shall define integrals by analytic continuation in $\epsilon$ and use dimensional regularization to remove ultra-violet divergences. Then, from dimensional considerations alone, a general diagram of order k in $g_0$, which contains no mass renormalization counter-terms, must have the form

$$\Gamma^{(2)}_{uk}(P, g_0) = a_k(\epsilon)\, g_0^k (-P^2)^{1 - \frac{k\epsilon}{2}} \tag{10}$$

From this we see that working to order k, with $k\epsilon < 2$ we would determine $\delta m^2 = 0$, whereas for $k\epsilon > 2$ it is not possible to evaluate $\delta m^2$ perturbatively.

If we use dimensional regularization to remove ultra-violet divergences in the theory then these divergences will give rise to poles in the $a_k(\epsilon)$ appearing in (10) at $k\epsilon = 2$. In fact, in the limit $P^2 \to 0$, each Feynman diagram of order k also becomes infra-red divergent for $k\epsilon > 2$. This shows the inter-relationship between the infra-red and ultra-violet divergences. We have to show that the poles can be absorbed into $\delta m^2$ with (7) preserved. Their presence actually proves that $\delta m^2$ cannot be zero.

From (1) it follows that in the presence of $\delta m^2$ the unrenormalized propagator can be written in the form

$$i\Gamma^{(2)}_u(P;g_0) = P^2 \sum_{k,m=0} \frac{b_{km}(\epsilon)}{1 - \frac{k\epsilon}{2} - m} \left[g_0(-P^2)^{-\frac{\epsilon}{2}}\right]^k (g_0^{\frac{2}{\epsilon}} P^{-2})^m + \delta m^2 . \tag{11}$$

($b_{00} = 1$ and $b_{10} = b_{01} = 0$) where we have now displayed the poles coming from ultra-violet divergences explicitly. From (6) and (8)

$$\left.\frac{\partial}{\partial P^2}\, i\Gamma^{(2)}_u(P;g_0)\right|_{P^2 = -\mu^2} = \frac{1}{Z_3} \tag{12}$$

$$= -\sum_{k,m=0}^{\infty} b_{km}(\epsilon)\,(\mu^2)^{-(\frac{\epsilon}{2}k+m)}\,(g_0)^{k+\frac{2}{\epsilon}m} \tag{13}$$

Notice that $\Gamma_u^{(2)}(P,g_0)$ can be written in the form $g_0^{\frac{2}{\epsilon}}$ times a function of $g_0^{2/\epsilon}/P^2$. Therefore, if we write

$$Z_3 = Z_3(x) \qquad\qquad x = g_0^{2/\epsilon}/\mu^2 \tag{14}$$

then

$$i\Gamma_u^{(2)}(P;g_0) = 4g_0^{2/\epsilon}\int_0^{g_0^{2/\epsilon}/-P^2} \frac{dx}{x^2}\left[\frac{1}{Z_3(x)} - 1\right] + P^2 + \delta m^2 \tag{15}$$

The $P^2$ term has to be dealt with explicitly to obtain convergence at $x=0$, since $Z_3^{-1}(x) \to 1 + b_{20}x^{\epsilon}$. Therefore, (15) converges at $x=0$ for $\epsilon > 1$. For $\epsilon < 1$ the integral can be defined by analytic continuation from $\epsilon > 1$. From (6) and (15) we obtain

$$\delta m^2 = -g_0^{2/\epsilon}\int_0^{\infty} \frac{dx}{x^2}\left[\frac{1}{Z_3(x)} - 1\right] \tag{16}$$

and

$$i\Gamma_u^{(2)}(P^2;g_0) = -g_0^{2/\epsilon}\int_{g_0^{2/\epsilon}/-P^2}^{\infty} \frac{dx}{x^2}\,\frac{1}{Z_3(x)} \tag{17}$$

Note that the poles present in (11) result from the $x=0$ end point of integration in (15), and are absent in (17). Equation (7) is also satisfied by (17) and so we have achieved our object--provided that the integral converges at $\infty$. This we will discuss shortly. Equations (16) and (17) are our promised sum rules for the bare mass ($m_0^2 = -\delta m^2$) and the inverse propagator.

To give a complete construction of the theory in the next section, we shall also need an analogous expression to (17) for $\Gamma_u^{(4)}$. However, $\Gamma_u^{(4)}$ is specified by the simple renormalization condition (9). It is not necessary to use a derivative condition, since the coupling constant renormalization is finite for all values of $\epsilon$ in the range that we are interested in. The analogue of (17) is therefore obtained by noting from (5) and (6) that

$$\Gamma_u^{(4)}(P^2, \theta_k, g_0)\Big|_{\substack{P^2 = -\mu^2 \\ \theta_k = \mu_k}} = Z_3^{-2} g \tag{18}$$

$$= \frac{g_0}{Z_1(g_0^{2/\epsilon}/\mu^2, \mu_k)} \tag{19}$$

and so

$$\Gamma_u^{(4)}(P^2, \theta_k, g) = g_0/Z_1\,(g_0^{2/\epsilon}/P^2, \theta_k) \tag{20}$$

This seems a trivial observation but becomes significant when we use renormalization group apparatus to obtain integral representations for $Z_1$ and $Z_3$. Next, therefore, we give a brief development of the necessary (standard) apparatus.[2]

First we define the dimensionless coupling constant

$$u = g/\mu^{\epsilon} \tag{21}$$

Next we define

$$\beta(u) = \mu \frac{\partial u}{\partial \mu}\bigg|_{g_0, \mu_k \text{ fixed}} \tag{22}$$

$$\gamma(u) = \mu \frac{\partial}{\partial \mu} \ell n\, Z_3 \bigg|_{g_0 \text{ fixed}} \tag{23}$$

In the following we shall generally supress the dependence of $\beta$ and $Z_1$ on the $\mu_k$. Using

$$u\mu^{-\epsilon} = Z_3^2\, Z_1^{-1}\, g_0 \tag{24}$$

we obtain

$$\beta(u) = -\epsilon \left\{ \frac{\partial}{\partial u} \ell n \left[ u Z_1(u) Z_3^{-2}(u) \right] \right\}^{-1} \tag{25}$$

$$\gamma(u) = \beta(u) \frac{\partial}{\partial u} \ell n\, Z_3(u) \tag{26}$$

This last relation gives

$$Z_3(u) = e^{\int_0^u du' \gamma(u')/\beta(u')} \tag{27}$$

We note the familiar fact that $Z_3 \to 0$ at the first zero, $\bar{u}$, of $\beta$ if

$$\gamma(\bar{u})/\beta'(\bar{u}) > 0 \tag{28}$$

From (25)

$$Z_1(u)Z_3^{-2}(u) = e^{\int_0^u du' \left[-\frac{\epsilon}{\beta} - \frac{1}{u'}\right]} \tag{29}$$

Therefore, if

$$\beta'(\bar{u}) > 0 \tag{30}$$

$$Z_1(u)Z_3^{-2}(u) \xrightarrow[u \to \bar{u}]{} \infty \tag{31}$$

So from (24) the interval $[0, \infty)$ in $g_0$ maps onto the internal $[0, \bar{u})$ in u. In particular if

$$\beta \underset{u \to \bar{u}}{\sim} b(u - \bar{u}) \tag{32}$$

then, defining $\eta = \gamma(\bar{u})$

$$Z_3 \sim (\bar{u} - u)^{\eta/b} \tag{33}$$

and

$$Z_1 Z_3^{-2} \underset{u \to \bar{u}}{\sim} (\bar{u} - u)^{-\epsilon/b} \tag{34}$$

But

$$x = \left(\frac{g_0}{\mu^2}\right)^{2/\epsilon} = u^{2/\epsilon} e^{2/\epsilon \int_0^u du' \left[-\frac{\epsilon}{\beta} - \frac{1}{u'}\right]} \tag{35}$$

$$\underset{u \to \bar{u}}{\sim} (\bar{u} - u)^{-2/b} \tag{36}$$

Therefore

$$Z_3 \underset{x \to \infty}{\simeq} x^{-\eta/2} \tag{37}$$

$$Z_1 \underset{x\to\infty}{\sim} x^{-\eta + \epsilon/2} \tag{38}$$

and so if a zero of β exists satisfying (28) and (30) then (17) will converge provided that

$$\eta < 2 \tag{39}$$

Also we obtain

$$i\Gamma_u^{(2)}(P^2;g_0) \underset{P^2\to 0}{\sim} -g_0^{2/\epsilon} \int_{g_0^{2/\epsilon}/-P^2}^{\infty} \frac{dx}{x^2}\, x^{\eta/2} \tag{40}$$

$$\sim g_0^{\eta/\epsilon}(-P^2)^{1-\eta/2} \tag{41}$$

and from (20) and (38)

$$\Gamma_u^{(4)}(P^2,\theta_k;g_0) \underset{P^2\to 0}{\sim} g_0^{2\eta/\epsilon}\,(-P^2)^{\epsilon/2-\eta} \tag{42}$$

(40) and (42) demonstrate that our sum rules correctly give the anomalous dimension infra-red behavior of $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ when β has an appropriate zero. That is when an infra-red stable Gell-Mann-Low eigenvalue is present.

It is important to note that, when the integral representation of (17) exists, the perturbation series is an asymptotic expansion for both small values of $g_0$, and large values of $-P^2$. To see this we choose an $\bar{x}$ small enough so that the series expansion for $Z_3^{-1}(x)$ given in Eq. (13)

can be used for $g_0^{2/\epsilon}/(-P^2) \leq x \leq \bar{x}$. Integrating term by term over this interval we find that the leading contributions come from those terms with m=0 and $k< \frac{2}{\epsilon}$. They reproduce the perturbation series to order $\frac{2}{\epsilon}$, with a correction term of order $g_0^{2/\epsilon}$. Similarly the integral from $\bar{x}$ to infinity yields a contribution of order $g_0^{2/\epsilon}$. Notice that to the order the asymptotic series is valid one need not calculate any diagrams with mass renormalization counterterms. Thus

$$i\Gamma_u^{(2)}(P, g_0) \xrightarrow[\substack{g_0 \to 0 \\ \text{or } P^2 \to -\infty}]{} - \sum_{k=0}^{(2/\epsilon)} \frac{b_{k0}}{1-\frac{k\epsilon}{2}} (-P^2)^{1-\frac{k\epsilon}{2}} \qquad (43)$$

where $(\frac{2}{\epsilon})$ denotes the largest integer less than $\frac{2}{\epsilon}$. For $\frac{2}{\epsilon}$ equal to an integer, the propagator develops logarithmic dependence on $P^2$ and $g_0$ as was pointed out by Symanzik.[7] For example, for $\epsilon \lesssim 1$

$$i\Gamma_u(P^2, g_0) \xrightarrow[\substack{g_0 \to 0 \\ \text{or } P^2 \to -\infty}]{} P^2 - b_{20}\, g_0^{2/\epsilon} \int_{g_0^{2/\epsilon}/-P^2}^{x} dx\, x^{1/\epsilon - 2} \qquad (44)$$

$$\underset{\epsilon \cong 1}{=} P^2 - b_{20}\, g_0 \,\ln(-P^2/g_0) \qquad (45)$$

The fact that the full propagator goes to the bare one in the ultra-violet limit is particularly significant for the Reggeon Calculus, as we discuss in Sec. IV.

Using the above results $\delta m^2$ can be expressed in terms of $\gamma$ and $\beta$. From (35)

$$dx = -\frac{2}{\beta} x \, du \tag{46}$$

so that

$$\frac{dx}{x^2} = -\frac{2}{\beta x} du \tag{47}$$

which from (16) gives

$$\delta m^2 = g_0^{2/\epsilon} \int_0^{\bar{u}} du \, \frac{2u^{-2/\epsilon}}{\beta(u)} \, e^{\int_0^u du' \left[\frac{2}{\beta(u')} + \frac{1}{u'}\right]} \left[ e^{-\int_0^u du' \gamma(u')/\beta(u')} - 1 \right] \tag{48}$$

This expression compares directly with that of Symanzik,[9] except that whereas $\gamma$ and $\beta$ in (48) are renormalization group functions, Symanzik's formula involves the corresponding functions of the Callan-Symanzik equation for the massive theory. This explains why our expression is a little simpler in form than Symanzik's.

To express $\Gamma_u^{(2)}$ in terms of $\gamma$ and $\beta$ it is first necessary to invert (24) to express u as a function of $g_0 \mu^{-\epsilon}$ (for fixed $\mu_k$). If the solution of this equation is

$$u = Z^{-1}(g_0 \mu^{-\epsilon}) \tag{49}$$

(this defines $Z^{-1}$ as a function) then

$$\Gamma_u^{(2)}(P^2; g_0) = -g_0^{2/\epsilon} \int_{Z^{-1}\left[g_0(-P^2)^{-\epsilon/2}\right]}^{\bar{u}} du \, \frac{2u^{-2/\epsilon}}{\beta(u)} \, e^{\int_0^u du' \left[\frac{2-\gamma}{\beta} + \frac{1}{u'}\right]} \tag{50}$$

or alternatively

$$\Gamma_R^{(2)}(P^2;g) = -Z_1^{2/\epsilon+1}(u)\, Z_3^{-4/\epsilon}(u)\, g^{2/\epsilon}$$

$$\times \int_{Z^{-1}\left[g(-P^2)^{-\epsilon/2} Z_3^{-2} Z_1\right]}^{\bar{u}} du\;\; 2u^{(-2/\epsilon)/\beta(u)}\; e^{\int_0^u du' \left[\frac{2-\gamma}{\beta} + \frac{1}{u'}\right]} \tag{51}$$

## III. INFRA-RED DIVERGENCES AND CONSTRUCTION OF THE THEORY

We begin by considering some simple examples of Feynman diagrams which contain the infra-red divergences we wish to avoid. Consider the simple bubble diagram $I_0(k)$ of Fig. 1. From dimensional analysis

$$I_0(k) = g_0^2\, k^{-\epsilon}\, c_0(\epsilon) \tag{52}$$

Next consider the diagram $I_n(P^2)$ shown in Fig. 2

$$I_n(P^2) \propto g_0 \int \frac{d^D k}{(P+k)^2} \left[\frac{I_0(k)}{g_0}\right]^n \tag{53}$$

$$= g_0 (c_0 g_0)^n \int \frac{d^D k}{(P+k)^2}\, k^{-n\epsilon} \tag{54}$$

This last integral has two divergence problems. The ultra-violet divergence (D1) occurs at $(n+1)\epsilon = 2$, and gives rise to the poles in $a_k(\epsilon)$ at $k\epsilon = 2$ referred to in the last section. They have to be cancelled by $\delta m^2$ as we found in going from (15) to (17). Equation (54) is also infra-red divergent (D2), for finite $P^2$, at $(n+1)\epsilon = 4$ and $\epsilon = 2$.[10]

Further sets of divergences of the form (D2) at $(n+1) = 2m\epsilon$, $n, m = 1, 2, 3, \ldots$ can be obtained by considering the iteration within a diagram, of more complicated substructures.

The divergences $D_2$ are clearly the result of performing an illegal perturbation expansion in that

$$\sum_n I_n (P) \propto \int \frac{d^D k}{(k+P)^2} \frac{I_0(k)}{1 - I_0(k)} \tag{55}$$

which is not infra-red divergent. It is clear, therefore, that in order to avoid both the divergences (D2) and to define $\delta m^2$ (that is avoid the divergences (D1)) we need a technique for performing at least partial sums to all orders in perturbation theory. We propose to use our results of the last section to do this as follows.

We first set up Schwinger-Dyson equations for $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ as shown in Figs. 3, 4 and 5. Figure 5 represents the skeleton expansion of $\Gamma_u^{(6)}$ in terms of $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$. This has to be inserted into the last diagram of Fig. 4 to obtain a closed equation for $\Gamma_u^{(4)}$. The equation shown in Fig. 4 can be obtained by considering all the possible initial interactions for particle 1. If $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ have been calculated to a given order in $g_0$, the equations of Figs. 3-5 can be used to calculate to the next order. The higher Green's functions also have skeleton expansions in terms of $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ so that once these functions have been calculated the higher Green's functions can be calculated to the required order.

Further, if $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ respectively satisfy (41) and (42), then one sees from power counting that no infra-red divergences result if they are substituted into the right hand sides of Figs. (3) and (4) provided $2 + \eta > \epsilon$. A more stringent restriction arises from the skeleton expansion of $\Gamma_u^{(6)}$. Here diagrams in which three internal particles repeatedly interact will develop infra-red singularities unless $1 + \frac{3}{2}\eta > \epsilon$.[10] The problem is analagous to the one arising from the interaction of two internal particles discussed in connection with (53) and (54). The latter problem is avoided because at each step of the calculation we use a propagator and a four-point function which involve an infinite sum over powers of $g_0$. If this series were truncated we would obtain integrals of the form (54) and the (D2) divergences would reappear. If one wishes to work in dimensions for which $\epsilon \geq 1 + \frac{3}{2}\eta$, it is necessary to construct $\Gamma_u^{(6)}$ nonperturbatively so that it has its appropriate infra-red behavior at each order of the calculation. Here we shall restrict ourselves to $1 + \frac{3}{2}\eta > \epsilon$. Since $\eta$ is positive definite this includes the physically interesting point D = 3.

We therefore propose the following procedure:

1) <u>Calculate $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ to order $g_0^2$ in perturbation theory.</u> This involves calculating Fig. 6 for $\Gamma_u^{(2)}$ and Fig. 1 for $\Gamma_u^{(4)}$. Both diagrams are infra-red convergent for $0 < \epsilon < 2$.

2) <u>Calculate $Z_1$, $Z_3$, $\gamma$ and $\beta$ to order $g_0^2$.</u>

3) <u>Use (27) and (29) to calculate new expressions for $Z_1$ and $Z_3$.</u>

(these constants will now become infinite series in u or $g_0$).

4) Use (17) and (20) to calculate an improved $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$. If expressed in the form (15) and (20) these will agree with the expressions obtained in 1) to order $g_0^2$. $\Gamma_u^{(2)}$ will also contain terms proportional to $g_0^{2/\epsilon}$. Since $\beta(u)$ has a zero to this order in $g_0^2$, $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ will satisfy (41) and (42).

5) Use the improved $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ and the equations of Figs. 3-5 to calculate $Z_1$, $Z_3$, $\gamma$ and $\beta$ to next order in the skeleton expansion.

6) Again calculate a new $Z_1$ and $Z_3$ using (27) and (29).

7) Finally calculate a further improved $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$.

This procedure can now be iterated to arbitrarily high order in the skeleton expansion. At each stage of the iteration $\Gamma_u^{(2)}$ and $\Gamma_u^{(4)}$ coincide with perturbation theory to the given order, apart from terms involving powers of $g_0^{2/\epsilon}$, which correspond to mass counterterm insertions.

We should note the above construction procedure is likely to be dependent, for its success, on the existence of a zero of $\beta$ at each order of the iteration. In the usual $\epsilon$-expansion of $\beta(u, \epsilon)$ this is only true for small $\epsilon$, because of the oscillating nature of higher order contributions to $\beta$. Some technique, such as the use of Pade' approximates,[3] has to be used to restore the zero. If a zero is really present in $\beta$ for finite $\epsilon$ we might expect our iteration procedure to preserve this zero at all orders of the iteration (at least for a larger range in $\epsilon$ than the $\epsilon$-expansion). Firstly $\beta$ becomes an infinite series in u after the first iteration, and so its asymptotic behavior for large u should not oscillate wildly as it does in perturbation theory. Secondly $\Gamma^{(2)}$ and $\Gamma^{(4)}$ acquire anomalous dimensions (close to the correct dimension for small $\epsilon$) after the first iteration and so this should give good convergence properties to the iteration.

As we have noted (17) and (20) ensure that $\Gamma^{(2)}$ and $\Gamma^{(4)}$ will acquire terms non-analytic in $g_0$ from our construction. This non-analyticity will be reflected in $\gamma(u, \epsilon)$ and $\beta(u, \epsilon)$ which will generate terms nonanalytic in both u and $\epsilon$ (of the form $u^{2/\epsilon}$, for example). It is interesting that the $\gamma$ and $\beta$ functions of the Callan-Symanzik equation for the massive theory are expected to be analytic in $\epsilon$. Symanzik has argued[7] that the critical exponents of the massless theory (e.g., $\eta$ of the last section) can be obtained from the massive theory and so have to be analytic in $\epsilon$.

We have not been able to prove that the nonanalyticity in $\epsilon$ of $\gamma$ and $\beta$ cancels in $\eta$, but the following argument suggests that it will.

By definiti on

$$\eta(\epsilon) = \gamma\left[\bar{u}(\epsilon), \epsilon\right] \tag{56}$$

and $u \to \bar{u}$ corresponds to $g_0 \to \infty$. From (17) and (20) $g_0 \to \infty$ is equivalent to $P^2 \to 0$ and

$$\Gamma_R^{(2)}(\bar{u}) = Z_1 \Gamma_u^{(2)} \tag{57}$$

$$\underset{g_0 \to \infty}{\sim} (g_0/\mu^{\epsilon})^{-\eta/\epsilon} g_0^{\eta/\epsilon} (-P^2)^{1-\eta/2} \tag{58}$$

$$= \mu^{\eta} (-P^2)^{1-\eta/2} \tag{59}$$

Similarly

$$\Gamma_R^{(4)}(\bar{u}) \sim \mu^{2\eta} (-P^2)^{\epsilon/2-\eta} \tag{60}$$

So in $\Gamma_R^{(2)}(\bar{u})$ and $\Gamma_R^{(4)}(\bar{u})$ the nonanalyticity in $g_0$ disappears and if $\eta(\epsilon)$ is analytic in $\epsilon$ so will $\Gamma_R^{(2)}(\bar{u})$ and $\Gamma_R^{(4)}(\bar{u})$ be. Now one method of computing $\eta(\epsilon)$ is to determine it self-consistently using the skeleton expansion.[11] We cannot use Eqs. (59) and (60) in the skeleton expansion for all values of the loop momenta since they would lead to ultra-violet divergences. However, for small values of the loop momenta, which one expects to control the infra-red behavior,[11] we can use Eqs. (59) and (60) (i.e., we can take the limit $g_0 \to \infty$ through the phase space integrations of Figs. 3-5). As a result, we would not expect to encounter any explicit nonanalyticity in a self-consistent determination of $\eta(\epsilon)$.

The above argument, coupled with the fact that the perturbation series is an asymptotic expansion for small $g_0$, helps to justify the $\epsilon$-expansion calculation of the critical exponents. If one wishes to compute them to nth order in $\epsilon$ then it is merely necessary to imagine working with $\epsilon < 2/n$. The renormalization group functions can then be calculated in perturbation theory without ever encountering graphs with mass counterterms. Of course, at the end of the calculation one can attempt to continue the results to larger values of $\epsilon$.

Finally we consider what can be said explicitly about the convergence of our proposed procedure. The problem goes beyond the familiar one of whether or not the infinite series arising from the skeleton expansion of Figs.4 and 5 converges. The additional difficulty is that our procedure only reproduces exactly the perturbation theory diagrams with no mass counterterms. (the terms $b_{k0}$ in (11).) To be able to calculate the diagrams with one or more mass counterterms ($b_{km}$, $m > 1$) exactly one must know $\delta m$ exactly and this is only possible when the whole perturbation series is summed and $\delta m^2$ determined self-consistently. However, it should be noted that our approximation for $\delta m^2$ is systematically improved at each step of the calculation. With each interation an additional $b_{k0}$ is calculated exactly thereby improving the small x behavior of $Z_3(x)^{-1}$. In fact after n iterations the residues of the poles in $\delta m^2$ at $\epsilon = 1, \frac{2}{3}, \ldots \frac{2}{n+1}$ are given exactly. In addition since the critical indices are given correctly to order $\epsilon^n$ after n iterations, the large x behavior of $Z_3(x)^{-1}$ is also being systematically improved. As a result, one can have real hope that our iteration procedure does converge. Clearly, if this is the case, we will have a massless solution to the Schwinger-Dyson equations.

## IV. THE REGGEON CALCULUS

The "non-relativistic" unrenormalized Lagrangian we consider is

$$\mathscr{L}_u = \frac{i}{2}\left[\psi_0^+(\vec{x},t)\frac{\partial}{\partial t}\psi_0(\vec{x},t)\right] - \alpha_0' \nabla\psi_0^+(\vec{x},t)\nabla\psi_0(\vec{x},t)$$

$$-\frac{\lambda_0}{2}\left[\psi_0^+(\vec{x},t)\psi_0(\vec{x},t)^2 + \text{h.c.}\right] + \delta\Delta\,\psi_0^+(\vec{x},t)\psi_0(\vec{x},t) \tag{61}$$

$\psi(\vec{x},t)$ is a field in $D(=4-\epsilon)$ space dimensions and one time dimension. The Feynman rules for this theory are given explicitly in Refs. 5 and 8. The conjugate variables to $\vec{x}$ and $\vec{t}$ are the transverse momentum $\underline{k}$ and energy E. The bare inverse propagator is

$$\Gamma_{u0}^{(1,1)}(E,k^2) = E - \alpha_0 k^2 \tag{62}$$

Note that the theory is not crossing symmetric and our notation for (amputated) Green's functions distinguishes incoming and outgoing particles--that is, Pomerons. In conventional angular momentum plane notation $1-E=j$-- the angular momentum, and $k^2 = -t$--the square of the momentum transfer. Therefore, $\alpha_0'$ is the bare slope of the Pomeron at $t=0$. $\lambda_0 = ir_0$ is the pure imaginary triple-Pomeron coupling and $\delta\Delta$ is the "mass" counterterm (minus the displacement of the bare Pomeron intercept below one). The renormalized Lagrangian is

$$\mathcal{L}_R = \frac{i}{2} Z_3 \left[ \psi^+ \overleftrightarrow{\frac{\partial}{\partial t}} \psi \right] - Z_2 \alpha' \nabla \psi^+ \nabla \psi$$

$$- \frac{i}{2} Z_1 r \left[ \psi^+ \psi^2 + \psi^{+2} \psi \right] + Z_3 \, \delta \Delta \psi^+ \psi \tag{63}$$

So $Z_3$ is the wave-function renormalization, $Z_2$, the renormalization of the slope and $Z_1$ the renormalization of the coupling constant.

$$\psi_0(\vec{x}, t) = Z_3^{\frac{1}{2}} \psi(\vec{x}, t) \tag{64}$$

$$\alpha_0' = Z_2 Z_3^{-1} \alpha \tag{65}$$

$$r_0 = Z_1 Z_3^{-3/2} r \tag{66}$$

If $\Gamma_u^{(M,N)}$ and $\Gamma_R^{(M,N)}$ are respectively unrenormalized and renormalized M Pomeron to N Pomeron Green's functions (amputated) then

$$\Gamma_R^{(M,N)} = Z_3^{(M+N)/2} \, \Gamma_u^{(M,N)} \tag{67}$$

The renormalization conditions we impose are

$$\Gamma_R^{(1,1)} (E, k^2) \Big|_{E, k^2 = 0} = 0 \tag{68}$$

$$\frac{\partial i \Gamma_R^{(1,1)}}{\partial E} (E, k^2) \Bigg|_{\substack{E = -E_N \\ k = k_N}} = 1 \tag{69}$$

$$\left. \frac{\partial i\Gamma_R^{(1,1)}}{\partial k^2}(E,k^2) \right|_{\substack{E=-E_N \\ k=k_N}} = -\alpha' \tag{70}$$

$$\left. \Gamma_R^{(1,2)}(E,k,\phi_m) \right|_{\substack{E=-E_N \\ k=k_N \\ \phi_m=\nu_m}} = -ir/(2\pi)^{\frac{D+1}{2}} \tag{71}$$

In (71) E and k are convenient energy and momentum variables (which we take to be those of the incoming Pomeron) and the $\phi_m$ are dimensionless parameters formed from the variables $E_i/E$ and $k_i \cdot k_j/k^2$. Equations (68) - (71) differ from the renormalization conditions imposed by Abarbanel and Bronzan[5] only in that we do not put $k_N = 0$ and the $\nu_m$ are not specified. Our more general conditions are chosen because we wish to build up the complete E and $k^2$ behavior of $\Gamma_u^{(1,1)}$ and $\Gamma_u^{(1,2)}$ using the techniques of Sec. II.

There are now two dimensionless parameters we can define (the relevant dimensional analysis can be found in Ref. 5).

$$g = \frac{r}{(\alpha')^{D/4} E_N^{\epsilon/4}}, \quad y = \frac{\alpha'}{r}(k_N^2)^{\epsilon/4} \tag{72}$$

If we also define

$$g_0 = \frac{r_0}{(\alpha_0')^{D/4}} \qquad y_0 = \frac{\alpha_0'}{r_0} \tag{73}$$

then dimensional arguments enforce

$$\delta\Delta = (g_0^2)^{2/\epsilon} \; \Delta(\epsilon) \tag{74}$$

where $\Delta(\epsilon)$ is a dimensionless function of $\epsilon$ only.

The generalization of (11) is

$$i\Gamma_u^{(1,1)}(E, k^2, g_0, y_0) = E \sum_{\substack{k,n=0\\ m=n}}^{\infty} a_{kmn}(\epsilon)\left[g_0^2(-E)^{-\epsilon/2}\right]^k \left(g_0^{4/\epsilon} E^{-1}\right)^m \left(y_0^{4/\epsilon} k^2\right)^n + \delta\Delta \tag{75}$$

As in the $\phi^4$ theory, ultra-violet divergences give rise to poles in the $a_{kmn}$.[8] However, now they occur only in the $a_{kmo}$, so we write

$$a_{kmo} = \frac{b_{km}}{1 - \frac{\epsilon}{2}k - m} \tag{76}$$

Because of the non-relativistic kinematics there are no infra-red singularities of the type (D2) discussed in Sec. II.

From (67) and (69)

$$\left.\frac{\partial i\Gamma_u^{(1,1)}}{\partial E}\right|_{\substack{E=-E_N\\ k^2=k_N^2}} = \frac{1}{Z_3\left(\frac{g_0^{4/\epsilon}}{E_N}, y_0^{4/\epsilon} k_N^2\right)} \tag{77}$$

and from (65), (67) and (70)

$$\left.\frac{\partial i\Gamma_u^{(1,1)}}{\partial k^2}\right|_{\substack{E=-E_N \\ k^2=k_N^2}} = \frac{-\alpha_0}{Z_2\left(\frac{g_0^{4/\epsilon}}{E_N}, \; y_0^{4/\epsilon} k_N^2\right)} \tag{78}$$

Note that the poles appearing in (76) are absent from both $Z_3$ and $Z_2$.

If we now write

$$x_1 = \frac{g_0^{4/\epsilon}}{E_N} \quad , \quad x_2 = y_0^{4/\epsilon} k_N^2 \tag{79}$$

then from (76) and (78) we obtain the analogue of (15)

$$i\Gamma_u^{(1,1)}(E, k^2; g_0, y_0) = -g_0^{4/\epsilon} \int_{\vec{0}}^{\left(-\frac{g_0^{4/\epsilon}}{E}, \; + y_0^{4/\epsilon} k^2\right)} d\vec{x} \cdot \left[\frac{1}{\vec{Z}} - \vec{e}\right] + E - \alpha_0 k^2 + \delta\Delta$$

where

$$\frac{1}{\vec{Z}} = \left(\frac{1}{Z_3}, \; \frac{-\alpha_0 (y_0 g_0)^{-4/\epsilon}}{Z_2}\right) \tag{81}$$

and

$$d\vec{x} = \left(\frac{dx_1}{x_1^2}, \; dx_2\right) \tag{82}$$

$$\vec{e} = \left[1, \; -\alpha_0 (g_0 y_0)^{-4/\epsilon}\right] \tag{83}$$

Note that the poles in (76) arise from the $\vec{0}$ end-point in (80). Imposing (68) we obtain

$$\delta\Delta = -g_0^{4/\epsilon} \int_{\vec{0}}^{\infty} d\vec{x} \cdot \left[\frac{1}{\vec{Z}} - \vec{e}\right] \tag{84}$$

and

$$i\Gamma_u^{(1,1)}(E, h^2, g_0, y_0) = -g_0^{4/\epsilon} \int_{\left(-\frac{g_0^{4/\epsilon}}{E},\, y_0^{4/\epsilon} k^2\right)}^{\infty} d\vec{x} \cdot \frac{1}{\vec{Z}} \tag{85}$$

Equations (84) and (85) are the generalization to the Reggeon Calculus of our sum rules (16) and (17). From (66), (67) and (71) we also obtain the analogue of (20), that is

$$\Gamma_u^{(1,2)}(E, k^2, \phi_m, g_0, y_0) = \frac{-ir/(2\pi)^{\frac{D+1}{2}}}{Z_1\left(\phi_m, \frac{g_0^{4/\epsilon}}{E}, y_0^{4/\epsilon} k^2\right)} \tag{86}$$

The necessary renormalization group analysis we develop as follows. Define

$$\gamma_E = E_N \frac{\partial \ln Z_3}{\partial E_N}\Bigg|_{\substack{k_N, g_0, y_0 \\ \phi_m \text{ fixed}}} \qquad \gamma_K = k_N^2 \frac{\partial \ln Z_3}{\partial k_N^2}\Bigg|_{\substack{E_N, g_0, y_0 \\ \phi_m \text{ fixed}}} \tag{87}$$

$$\beta_E = E_N \frac{\partial g}{\partial E_N} \qquad \beta_K = k_N^2 \frac{\partial g}{\partial k_N^2} \tag{88}$$

$$\zeta_E = E_N \frac{\partial y}{\partial E_N} \qquad \zeta_K = k_N^2 \frac{\partial y}{\partial k_N^2} \tag{89}$$

From which we obtain

$$\gamma_E = \beta_E \frac{\partial \ln Z_3}{\partial g} + \zeta_E \frac{\partial \ln Z_3}{\partial y} \tag{90}$$

$$\gamma_K = \beta_K \frac{\partial \ln Z_3}{\partial g} + \zeta_K \frac{\partial \ln Z_3}{\partial y} \tag{91}$$

If we define

$$\vec{g} = (g, y), \quad \vec{\gamma} = (\gamma_E, \gamma_K) \tag{92}$$

$$\beta = \beta_E \zeta_K - \beta_K \zeta_E \tag{93}$$

and

$$M = \begin{pmatrix} \zeta_K & -\zeta_E \\ -\beta_K & \beta_E \end{pmatrix} \tag{94}$$

then the inverse of (90) and (91) is

$$\frac{\partial \ln Z_3}{\partial \vec{g}} = \frac{M \cdot \vec{\gamma}}{\beta} \tag{95}$$

So that

$$\ln Z_3 = \int_0^{\vec{g}} d\vec{g} \, \frac{M \cdot \vec{\gamma}}{\beta} \tag{96}$$

From (65) and (66) we obtain

$$g_0 E_N^{-\epsilon/4} = g Z_1 Z_3^{-\frac{1}{2} - \epsilon/4} Z_2^{-D/4} = g Z_1 Z_3^{-\frac{1}{2}} Z_2^{-1} \left(\frac{Z_2}{Z_3}\right)^{\epsilon/4} \tag{97}$$

$$y_0 (k_N^2)^{\epsilon/4} = y Z_1^{-1} Z_3^{\frac{1}{2}} Z_2 \tag{98}$$

Therefore

$$g_0 y_0 \left(\frac{k_N^2}{E_N}\right)^{\epsilon/4} = g\,y\left(\frac{Z_2}{Z_3}\right)^{\epsilon/4} \tag{99}$$

and

$$\cdot E_N \frac{\partial}{\partial E_N} \ell n\left[gy\, Z_2^{\epsilon/4}\, Z_3^{-\epsilon/4}\right] = -\frac{\epsilon}{4} \tag{100}$$

$$k_N^2 \frac{\partial}{\partial k_N^2} \ell n\left[gy\, Z_2^{\epsilon/4}\, Z_3^{-\epsilon/4}\right] = \frac{\epsilon}{4} \tag{101}$$

and so if $\vec{i} = (1, -1)$, then

$$g\,y\,Z_2^{\epsilon/4}\,Z_3^{-\epsilon/4} = e^{-\frac{\epsilon}{4}\int^{\vec{g}} d\vec{g}\cdot\frac{M\cdot\vec{i}}{\beta}} \tag{102}$$

$$Z_2 Z_3^{-1} = e^{-\int_0^{\vec{g}} d\vec{g}\cdot\left(\frac{M\cdot\vec{i}}{\beta} + \frac{4}{\epsilon}\,\vec{g}^{-1}\right)} \tag{103}$$

where $\vec{g}^{-1} = (g^{1}, y^{-1})$

From (98), if $\vec{j} = (0, 1)$

$$Z_1^{-1} Z_3^{\frac{1}{2}} Z_2 = e^{\frac{\epsilon}{4}\int_0^{\vec{g}} d\vec{g}\cdot\left(\frac{M\cdot\vec{j}}{\beta} - \frac{4}{\epsilon}\frac{\vec{j}}{y}\right)} \tag{104}$$

To calculate the quantities defined in (87)-(89) it is necessary to calculate the diagrams of Figs. 7 and 8. The details of these calculations can be found elsewhere . Here we use only the general form of the result

$$\beta_E = -\frac{\epsilon}{4} g + b_1 g^3 + b_2 (gy)^{4/\epsilon} g^3 + \ldots \qquad \beta_K = b_3 (gy)^{4/\epsilon} g^3 + \ldots \tag{105}$$

$$\zeta_E = -d_1\, y g^2 + \ldots \qquad \zeta_K = \frac{\epsilon}{4} y + \ldots \tag{106}$$

$$\gamma_E = -c_1\, g^2 + \ldots \qquad \gamma_K = -c_2 (yg)^{4/\epsilon} g^2 + \ldots \tag{107}$$

$b_1, b_2, b_3$ depends on $\phi_m$ but they and $c_1, c_2, d_1$ are all positive and independent of g and y.

From (105) we note that $\beta_E$ has a zero at y=0, $g = g_1 = \sqrt{\frac{\epsilon}{4b_1}} + 0(\epsilon)$. From (105) - (107) we obtain

$$\vec{M}\cdot\vec{\gamma} \underset{y\to 0}{\sim} (\zeta_K \gamma_E, \ -\gamma_E\beta_K + \gamma_K\beta_E) \tag{108}$$

$$\beta \underset{y\to 0}{\sim} \beta_E \zeta_K \tag{109}$$

and

$$\frac{-\gamma_E\beta_K + \gamma_K\beta_E}{\beta_E\zeta_K} \xrightarrow[y\to 0]{} 0 \tag{110}$$

Therefore

$$\int_0^{\vec{g}} d\vec{y}\cdot \frac{M\cdot\vec{\gamma}}{\beta} \xrightarrow[y\to 0]{} \int_0^{g} dg' \frac{\gamma_E}{\beta_E} \tag{111}$$

Similarly

$$\frac{M\cdot\vec{i}}{\beta} \xrightarrow[y\to 0]{} \left(\frac{1}{\beta_E} + \frac{\zeta_E}{\zeta_K}\frac{1}{\beta_E} g, -\frac{1}{\zeta_K}\right) \tag{112}$$

and so

$$\int_0^{\vec{g}} d\vec{g}\cdot\left(\frac{\vec{M}\cdot\vec{i}}{\beta} + \frac{4}{\epsilon}\vec{g}^{-1}\right) \xrightarrow[y\to 0]{} \int_0^{g} dg'\left[\frac{1}{\beta_E}\left(1 + \frac{\zeta_E}{\zeta_K}\right) + \frac{4}{\epsilon g}\right] \tag{113}$$

Finally

$$\frac{\epsilon}{4}\int_0^{\vec{g}} d\vec{g}\cdot\left(\frac{M\cdot\vec{j}}{\beta}-\frac{4}{\epsilon}\frac{\vec{j}}{y}\right)\xrightarrow[y\to 0]{}\frac{\epsilon}{4}\int_0^{g} dg'\ \frac{-\zeta_E}{\zeta_K\beta_E} \tag{114}$$

Now from (72), (89) and (106)

$$\frac{\zeta_E}{\zeta_K}\xrightarrow[y\to 0]{}\frac{4}{\epsilon}\left\{E_N\frac{\partial}{\partial E_N}\left[\frac{(\alpha')^{\epsilon/4}}{g}\right]\Big/\frac{(\alpha')^{\epsilon/4}}{g}\right\}_{k_N=0}-1 \tag{115}$$

$$=\frac{1}{\alpha'}E_N\frac{\partial\alpha'}{\partial E_N}\bigg|_{k_N=0}-\frac{4}{\epsilon}g^{-1}E_N\frac{\partial g}{\partial E_N}\bigg|_{k_N=0}-1 \tag{116}$$

$$\xrightarrow[g\to g_1]{}\frac{1}{\alpha'}E_N\frac{\partial\alpha'}{\partial E_N}-1=-Z \tag{117}$$

where Z has the same definition as in Ref. 5. If we now write $\beta_E \underset{\substack{g\to g_1\\ y\to 0}}{\simeq} b_0(g-g_1)$ then from (96), (103), (104), (111), (113) and (114)

$$Z_3 \underset{\substack{y\to 0\\ g\to g_1}}{\sim} (g_1-g)^{\eta/b_0} \qquad (\eta=\gamma(g_1)) \tag{118}$$

$$Z_3^{-1}Z_2\sim(g_1-g)^{(Z-1)/b_0} \tag{119}$$

$$Z_1^{-1}Z_3^{\frac{1}{2}}Z_2\sim(g_1-g)^{\frac{\epsilon}{4}Z/b_0} \tag{120}$$

Therefore from (97) and (98)

$$g_0^{4/\epsilon}E_N^{-1}\sim g_1^{4/\epsilon}(g_1-g)^{-1/b_0} \tag{121}$$

$$y_0^{4/\epsilon}\ k_N^2 \sim y^{4/\epsilon}\,(g_1 - g)^{Z/b_0} \tag{122}$$

$$\sim y^{4/\epsilon}\ E_N^Z\,(g_1/g_0)^{4Z/\epsilon} \tag{123}$$

Note that (118) and (121) show that (85) will certainly converge if the large x part of the integral is taken along the $x_2 = 0$ line, and if,

$$\eta < 1 \tag{124}$$

(121) and (122) with $E_N \to -E$ and $k_N^2 \to k^2$ can now be substituted into (118)-(120) to give the leading forms of $Z_1, Z_2, Z_3$ which can be substituted into (85) and (86) to give the behavior of $\Gamma_u^{(1,1)}$ and $\Gamma_u^{(1,2)}$ as $E, k^2 \to 0$. Note that for $E, k^2 \to 0$ to correspond to $g \to g_1$, $y \to 0$ we must have

$$k^2(-E)^{-Z} \to 0 \tag{125}$$

From (85) we obtain

$$i\Gamma_u^{(1,1)}(E,k^2) = g_0^{4/\epsilon} \int_{(-g_0^{4/\epsilon} E^{-1})}^{\infty} \frac{dx_1}{x_1^2 Z_1(x_1, x_2=0)} + \int_0^{(y_0^{4/\epsilon}\ k)} \frac{dx_2 \alpha_0(y_0)^{-4/\epsilon}}{Z_2\left(-g_0^{4/\epsilon} E^{-1}, x_2\right)} \tag{126}$$

$$\underset{\substack{E \to 0 \\ k^2(-E)^Z \to 0}}{\sim} \quad g_0^{4\eta/\epsilon}\,(-E)^{1-\eta}\left[1 + c\ \alpha_0 g_0^{4Z/\epsilon}\,\frac{k^2}{(-E)^Z} + \ldots\right] \tag{127}$$

where c is constant. From (86) we obtain

$$\Gamma_u^{(1,2)}(E,k^2) \underset{\substack{E\to 0\\ k^2/(-E)^Z \to 0}}{\sim} \frac{-ir_0}{(2\pi)^{\frac{D+1}{2}}}(-g_0^{-4/\epsilon}E)^{1-\frac{3}{2}\eta - Z(1-\frac{\epsilon}{4})} \qquad (128)$$

Equations (127) and (128) confirm that our sum rules again give the correct infra-red behavior. Equation (127) is also consistent with the general scaling law obtained by Abarbanel and Bronzan[5] that

$$-i\Gamma_u^{(1,1)} \underset{\substack{E\to 0\\ k^2\to 0}}{\sim} (-E)^{1-\eta}\, \phi[k^2(-E)^{-Z}] \qquad (129)$$

We have obtained only the leading behavior for $\Gamma_u^{(1,2)}$, although the condition (125) does perhaps suggest a similar scaling law for $\Gamma_u^{(1,2)}$. Further discussion of the scaling laws can be found in Ref. 8. Here we simply note that to study the limit $k^2 \to 0$, $\frac{E^Z}{k^2} \to 0$ we could have followed an exactly analagous procedure from (72) onwards, simply taking as variables $g^{-1}$ and $y^{-1}$. The expansion of $\Gamma_u^{(1,1)}$ corresponding to (75) is obtained by expanding in $\left(\frac{1}{g_0 y_0}\right)^{4/\epsilon} = \frac{1}{\alpha_0'}$, rather than in $(g_0 y_0)^{4/\epsilon} = \alpha_0'$ as in (75). The validity of the scaling law then requires a relation between the new critical exponents obtained and those in (127).

We shall not give a detailed discussion of the infra-red divergences of the Reggeon Calculus here. This can be found in Ref. 8. Here we note only that the poles displayed in (76), which are related to divergences of the type (D1) discussed in Sec. III, are the only ones in the problem. There are no infra-red divergences of the type (D2) in the calculus because of the non-relativistic kinematics. Since the bare propagator is linear in E, rather than quadratic as in the relativistic case, the E contours of integration can always be distorted to avoid the type (D2) divergences. Although it is not essential, it may help to speed the convergence to construct the theory iteratively using the Schwinger-Dyson equations of the theory. The necessary equations are shown in Figs. 9 and 10. It is straightforward to apply to these equations the iterative construction procedure described in Sec. III for $\phi^4$. The relevant sum rules are (85) and (86) and the necessary renormalization group apparatus is Eqs. (96), (103) and (104).

All the other results of Sec. II extend to the Reggeon Calculus in a simple way. Equations (84) and (85) can be used to express both $\delta\Delta$ and $\Gamma^{(1,1)}$ as integrals over the parametric functions defined in (87)-(89). Also from (85) it can be shown that the perturbation series is an asymptotic expansion for small $r_0$ as well as for large E and/or $k^2$.[8] The argument is essentially the same as the one given in Sec. II for the propagator in $\phi^4$ theory. As in the $\phi^4$ case the series is accurate to order $n < \frac{2}{\epsilon}$. In particular for $E, k^2 \to -\infty$ the full propagator goes to the bare one provided one stays away from the cuts, i.e., provided $\alpha' k^2/E < 2$. This result is important since it shows that for positive t (negative $k^2$) the propagator does contain a pole. It is only in this case that one is guaranteed that the solution to the field theory satisfies full multiparticle t-channel unitarity in the $\ell$-plane.[12]

## ACKNOWLEDGMENTS

We would like to thank H. D. I. Abarbanel, J. B. Bronzan, B. W. Lee and G. Parisi for helpful conversations.

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[10]We would like to thank G. Parisi for very useful discussions on the infra-red divergences of $\phi^4$ theory, in particular for pointing out to us both the (D2) divergences and the analagous divergences associated with the six-point function.

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## FIGURE CAPTIONS

Fig. 1 Feynman diagram for the integral $I_0(k)$.

Fig. 2 Feynman diagram for the integral $I_n(k)$.

Fig. 3 Schwinger-Dyson equation for the propagator.

Fig. 4 Schwinger-Dyson equation for the four-point function.

Fig. 5 Skeleton expansion for the six-point function.

Fig. 6 Lowest order contribution to the self-energy.

Fig. 7 Lowest order contribution to the Pomeron self-energy.

Fig. 8 Lowest order corrections to the triple-Pomeron vertex.

Fig. 9 Schwinger-Dyson equation for the Pomeron propagator.

Fig. 10 Skeleton expansion for the triple-Pomeron vertex.


FIG.1

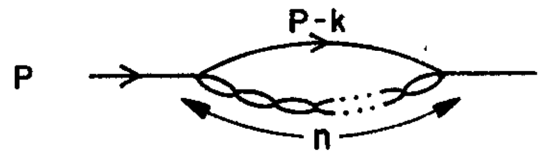


FIG.2

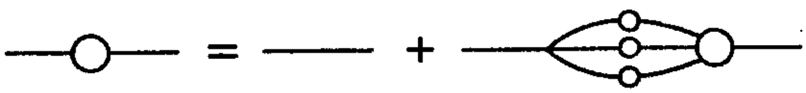

FIG.3

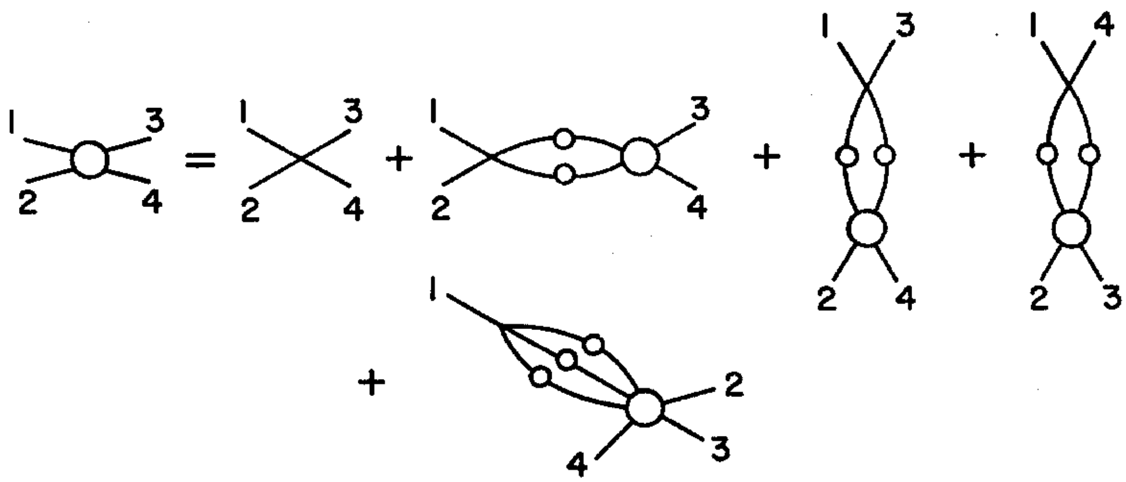


FIG.4

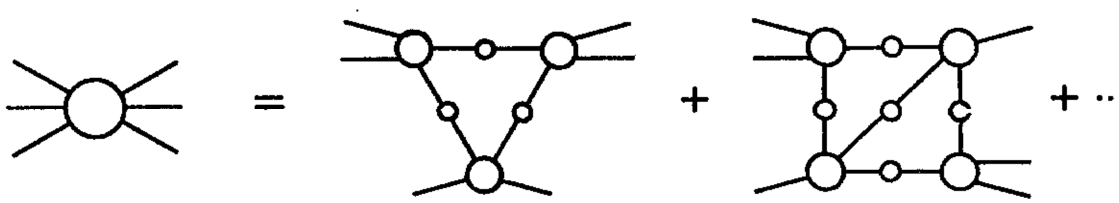

FIG.5

FIG. 6

FIG. 7

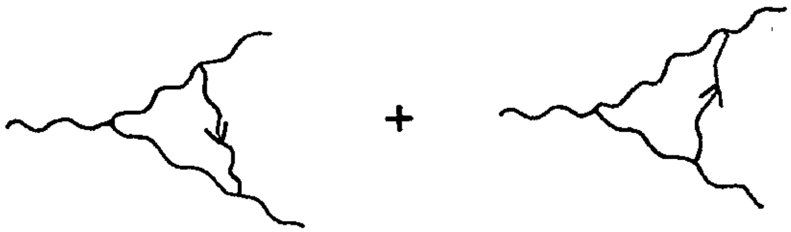

FIG. 8

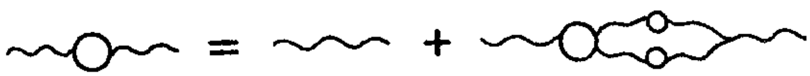

FIG. 9

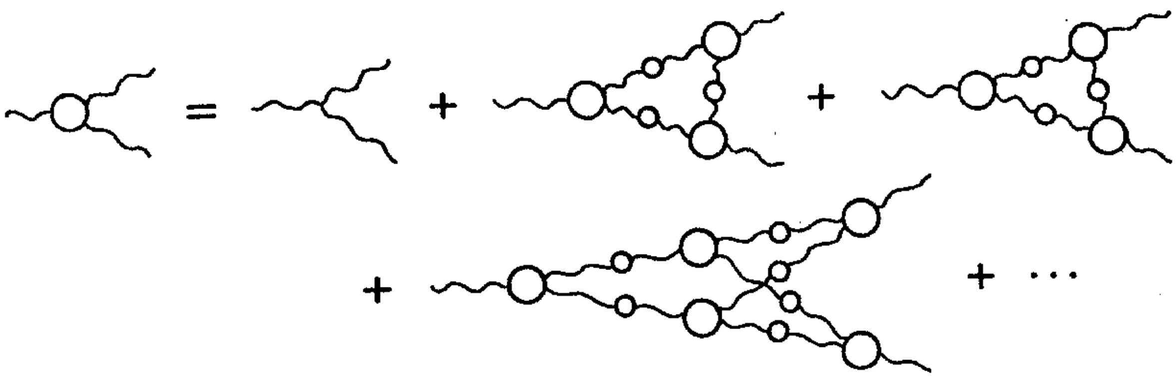

FIG. 10