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A Multiperipheral Model With Continued Cross Channel Unitarity*

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"A Multiperipheral Model ...."

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ABSTRACT

We derive, by using a spectral representation in momentum transfer, t, an integral equation, similar in structure to a multipheral equation, with continued cross channel unitarity, for the absorptive part for a composite particle scattering amplitude from a Bethe-Salpeter equation describing composite particle scattering in the s channel. At high energy in the t channel, the equation becomes homogeneous and has a Reggeized solution. We indicate how this equation may be solved using determinental techniques. We also show how the composite particle amplitude resulting from the original equation may be used to construct production and three body amplitudes. We also infer the possibility of studying, using the amplitude from the cross channel problem, the effect of extra unitarity on Reggeon-Reggeon-particle vertices.

I. INTRODUCTION

In recent years there has been much interest in ways of looking at multiparticle production phenomena and in models which attempt to set forth the basic dynamical mechanisms which give rise to such production. With regard to these considerations, many physicists have come to study what are known as inclusive reactions. These are reactions of the form

$$a + b \rightarrow c_1 + c_2 + \cdots + c_k + X \quad ,$$

where X denotes an unknown system of particles. Experimentally this process is realized by having a detection apparatus measure the momenta and types of particles c_1 to c_k , i.e., the distribution of one (or more) final particles is analyzed, all other particles being summed over.

The main pieces of information are single particle spectra, e.g., the momentum spectrum of the π^+ produced in p-p collisions. What is measured is

$$\frac{d^3\sigma}{d^3\underset{\sim}{p}}$$

in the laboratory frame, p being the momentum of the observed particle. From this,an invariant distribution function is defined,

$$E \frac{d^3\sigma}{d^3 \underset{\sim}{p}} = f(\underset{\sim}{p};s) \quad .$$

This invariant function was shown by Mueller [1] to be the missing mass (mass of the undetected particles) discontinuity of a three-to-three (six point) amplitude for forward scattering in analogy with the connection between the total cross section for a given process and the discontinuity in total energy of the elastic two body amplitude for the process. This is illustrated in Figs. 1 and 2.

If we wish to construct a model for a single particle inclusive distribution for a particular final state particle and compare the predictions of this model with experiment, we have two choices. First, we can try to calculate directly the single particle invariant function within the context of some model, or we can construct a model for a three-to-three scattering process, and using Mueller's Optical Theorem, construct the relevant single particle distribution. In addition, it is of interest to study high energy production amplitudes directly, i.e., two particles scattering into more than two particles (exclusively).

This paper will be concerned with the latter approach to single particle inclusive distributions as well as with the construction of a high energy production amplitude. We construct what we call a "multiperipheral" equation with continued cross channel unitarity which describes the scattering of two elementary particles into two composite particles, from an equation which in the cross channel (s channel) describes composite particle scattering. This equation will yield for us a high energy amplitude for the production of two composites.

If, in the composite particle problem, there is a mechanism for the formation of the composite in the initial state and the decay of the composite in the final state, then we can construct from the composite particle amplitude an amplitude describing three-to-three scattering. We have such a mechanism in the form of composite particle propagators and vertex functions for the decay and formation of a composite particle from two elementary particles (Fig. 3).

From the "multiperipheral" equation for the amplitude describing composite particle production, if we attach the appropriate vertex functions and propagators, we can construct an amplitude for the production of four particles. If we Reggeize the composite, we can construct an amplitude for two particles scattering into two Reggeons.

The organization of the paper is as follows: In Sec. II we discuss the derivation of equations describing composite-particle scattering, as was done by Freedman, Lovelace and Namyslowski [2] and by Aaron, Amado and Young [3] . Sec. III treats the formulation of our "multiperipheral" equation and the extraction of Regge behavior via a continuation of the Freedman, Lovelace, Namyslowski and Aaron, Amado, Young results to the cross channel by using a spectral representation in momentum transfer for the composite particle amplitude. Section IV presents an alternative derivation of the "multiperipheral" equation in terms of invariant variables. Section V is a discussion of our results.

II. COMPOSITE-PARTICLE SCATTERING WITH TWO BODY AND THREE BODY UNITARITY

Beginning with the Bethe-Salpeter equation for the three particle Green's function, Freedman, Lovelace and Namyslowski [2] derived a set of equations describing composite-particle scattering by using the Fadeev equations and by assuming that in the interaction of three particles, any two particle subsystem is dominated by bound states and resonances. Aaron, Amado and Young [3] derived a similar set of linear relativistic three body equations for the scattering of a particle from a bound state or correlated pair of others. Both Freedman, Lovelace, Namyslowsky and Aaron, Amado, Young combined the isobar idea with two and three-body unitarity as suggested by Blankenbecler and Sugar[4]. The resulting equations are such that the composite particle amplitude obeys two and three-body unitarity exactly in the interaction channel. The basic mechanisms for this lie in the potential term (which is chosen to have a particular three-particle cut) and in the composite particle propagator (which is chosen to have the appropriate two-particle and three-particle cuts). Let us now discuss this in more detail.

Two and Three-Body Unitarity ---
Structure of the Potential and the Propagator

For simplicity we consider the case of three spinless, identical particles and treat the elastic scattering of one particle from a spinless composite state of the other two.[5] The composite-particle scattering equations have the form (Fig. 4)

$$T(s) = B(s) + T(s)\,\tau(s)\,B(s) \tag{1}$$

$$T(s) = B(s) + B(s)\,\tau(s)\,T(s) \tag{2}$$

where $B(s)$ is the particle exchange term, $\tau(s)$ is the composite particle propagator and $T(s)$ is the amplitude describing the composite-particle scattering. The variable s is the square of the center of mass (c.m.) energy in the reaction channel.

To impose the unitarity conditions, we obtain the discontinuity in s of T by writing

$$B(s^-) = B(s^-) - B(s^+) + T(s^+) - B(s^+)\tau(s^+)T(s^+) \tag{3}$$

and

$$B(s^+) = B(s^+) - B(s^-) + T(s^-) - B(s^-)\tau(s^-)T(s^-) \quad , \tag{4}$$

Using Eqs. 1 and 4 we may write

$$\begin{aligned} T(s^-) &= B(s^-) + [\, B(s^-) - B(s^+)]\,\tau(s^-)T(s^-) \\ &+ T(s^+)\tau(s^-)T(s^-) - T(s^+)\tau(s^+)B(s^+)\tau(s^-)T(s^-) \quad . \end{aligned} \tag{5}$$

From Eqs. 2 and 3 we may write

$$\begin{aligned} T(s^-) &= B(s^-) + [\, B(s^-) - B(s^+)]\,\tau(s^-)T(s^-) \\ &+ T(s^+)\tau(s^-)T(s^-) - T(s^+)\tau(s^+)B(s^+)\tau(s^-)T(s^-) \quad . \end{aligned} \tag{6}$$

We then find

$$\begin{aligned} T(s^+) - T(s^-) &= T(s^+)\{\, \tau(s^+) - \tau(s^-)\}\, T(s^-) \\ &+\{\, I + T(s^+)\tau(s^+)\}\ \{B(s^+)-B(s^-)\}\,\{\, I+\tau(s^-)T(s^-)\} \quad . \end{aligned} \tag{7}$$

Now terms like

$$[\, B(s^+) - B(s^-)] + [\, B(s^+)-B(s^-)]\,\tau(s^-)T(s^-) + T(s^+)\tau(s^+)[\, B(s^+-B(s^-)]$$

correspond to cutting external lines, so they vanish. Therefore, we obtain

$$\begin{aligned} T(s^+)-T(s^-) &= T(s^+)[\,\tau(s^+) - \tau(s^-)]\, T(s^-) \\ &+ T(s^+)\tau(s^+)[\, B(s^+) - B(s^-)]\,\tau(s^-)T(s^-) \quad . \end{aligned} \tag{8}$$

The composite-particle scattering equations (with momentum labels as in Fig. 5 are

$$<p\,|\,T(s)\,|\,q> = <p\,|\,B(s)\,|\,q> + \frac{1}{(2\pi)^4}\int d^4k <p\,|\,B(s)\,|\,k>$$

$$\times\ \tau(\sigma_k) <k\,|\,T(s)\,|\,q> \tag{9}$$

with $\sigma_k = (p-k)^2$. The discontinuity of T satisfies the relation (Fig. 6):

$$<p\,|\,T(s^+)\,|\,q> - <p\,|\,T(s^-)\,|\,q>$$

$$= \frac{1}{(2\pi)^4}\int d^4k <p\,|\,T(s^+)\,|\,k> [\,\tau(\sigma_k^-) - \tau(\sigma_k^-)\,] <k\,|\,T(s^-)\,|\,q>$$

$$+ \frac{1}{(2\pi)^4}\int d^4k\, d^4k' <p\,|\,T(s^+)\,|\,k> \tau(\sigma_k^{\ +})$$

$$\times [\,<k\,|\,B(s^+)\,|\,k'> - <k\,|\,B(s^-)\,|\,k'>\,]\ \tau(\sigma_{k'}^{\ -}) <k'\,|\,T(s^-)\,|\,q> \ . \tag{10}$$

We want expressions for the discontinuities of $\tau(s)$ and $B(s)$ such that two and three-body unitarity is satisfied. Unitarity says

$$T_{fi} - T_{fi}^{\ +} = i\sum_n d\Omega_n\, T_{fn}\, T_{ni}^{\ +} = i\sum_n d\Omega_n\, T_{fn}^{\ +}\, T_{ni} \tag{11}$$

where

$$d\Omega_n = (2\pi)^4\, \delta^{(4)}(P_f - \sum_{i=1}^{n} q_i) \prod_{i=1}^{n} \frac{d^4q_i}{(2\pi)^4}\, 2\pi\, \delta^+(q_i^{\,2} - m_i^{\,2}) \tag{12}$$

is n-body phase space. Therefore, from two-body and three-body unitarity we have (Fig. 7)

$$<p\,|\,T(s^+)\,|\,q> - <p\,|\,T(s^-)\,|\,q>$$

$$= \frac{i}{(2\pi)^4}\int d^4k\, \delta^+(\sigma_k - m_R^{\,2})\, \delta^+(k^2 - m^2) <p\,|\,T(s^+) k> <k\,|\,T(s^-)\,|\,q> (2\pi)^2$$

$$+\frac{i}{(2\pi)^8}\int d^4k_1\, d^4k_2\, d^4k_3\, \delta(P-k_1-k_2-k_3)(2\pi)^3\, \delta^+(k_1^2-m^2)$$

$$\times\, \delta^+(k_2^2-m^2)\, \delta^+(k_3^2-m^2) \langle p \,|\, T(s^+) \,|\, k_1k_2k_3 \rangle \langle k_1k_2k_3 |$$

$$\times\, |\, T(s^-) \,|\, q \rangle \ . \tag{13}$$

The first term represents the situation where the composite propagates as a stable particle in the intermediate state (elastic bound state scattering); the second term represents the situation where the composite breaks up in the intermediate state and the same or a different composite is formed in the final state. The term $\langle p \,|\, T(s) \,|\, k_1k_2k_3 \rangle$ is the production (two-body →three body) amplitude. It has the form (Fig. 8)

$$\langle p \,|\, T(s) \,|\, k_1k_2k_3 \rangle = \frac{1}{\sqrt{3!}} \sum_{n=1}^{3} \langle p \,|\, T(s) \,|\, k_n \rangle S(q_{kn})\, v(p_n^2) \tag{14}$$

where $p_1^2 = (k_2-k_3)^2$, etc.; v is the vertex for the dissociation of the composite, and S is a propagator which is related to τ in a way to be demonstrated below. With the definition for $\langle p \,|\, T(s) \,|\, k_1k_2k_3 \rangle$ given in Eq. 14 , Eq. 13 becomes

$$\langle p \,|\, T(s^+) \,|\, q \rangle - \langle p \,|\, T(s^-) \,|\, q \rangle$$

$$= \frac{i}{(2\pi)^2}\int d^4k\, \delta^+(q_k - m_R^2)\, \delta^+(k^2-m^2) \langle p \,|\, T(s^+) \,|\, k \rangle \langle k \,|\, T(s^-) \,|\, q \rangle$$

$$+\frac{i}{(2\pi)^5}\int d^3k_1\, d^3k_2\, d^3k_3\, \delta^{(4)}(P-k_1-k_2-k_3)\, \delta^+(k_1^2-m^2)\, \delta^+(k_2^2-m^2)$$

$$\times\ \delta^+(k_3^2-m^2)\,\frac{1}{3!}\sum_{n,m=1}^{3} <p\,|\,T(s^+)\,|\,k_n>\,S(\sigma_{kn}^+)\ v(p_n^2)\ v(p_m^2)$$

$$\times\ S(\sigma_{km}^-) < k_m\,|\,T(s^-)\,|\,q> \quad . \tag{15}$$

The first term in Eq. 15 obviously contributes to the discontinuity of τ . In the second term $m = n$ contributes to the discontinuity of τ , being the direct term corresponding to cutting the propagator bubble (Fig. 9). The terms $m \neq n$ correspond to the exchange of a particle between bound states and will contribute to the discontinuity of $B(s)$ (Fig. 10).

For $m \neq n$, the second term in Eq. 15 may be written

$$\frac{i}{(2\pi)^5}\int d^4k\, d^4k'\, d^4k''\ \delta^{(4)}(P-k-k'-k'')\ \delta^+(k^2-m^2)\ \delta^+(k'^2-m^2)$$

$$\times\ \delta^+(k''^2-m^2)\,\frac{1}{3!}\ 6 <p\,|\,T(s^+)\,|\,k>\ S(\sigma_k^+)\ v[\,(k''-k')^2]\ v[\,(k''-k)^2]$$

$$\times\ S(\sigma_{k'}^-) < k'\,|\,T(s^-)\,|q> \tag{16}$$

$$= \frac{i}{(2\pi)^5}\int d^4k\, d^4k'\ \delta^+(k^2-m^2)\ \delta^+(k'^2-m^2)\ \delta^+[\,(P-k-k')^2-m^2\,]$$

$$\times <p|\,T(s^+)\,|\,k>\ S(\sigma_k^+)\ v[\,(P-k-2k')^2]\ \ v[\,(P-2k-k')^2\,]$$

$$\times\ S(\sigma_{k'}^-) < k'\,|\,T(s^-)\,|q> \quad . \tag{17}$$

Comparing Eq. 17 with the second term of Eq. 10 , we obtain the relation

$$\tau(q_k^+)[<k \mid B(s^+) \mid k'> - <k \mid B(s^-) \mid k'>]\ \tau(q_{k'}^-)$$

$$= iv[\,P-k-2k'\,)^2]\ S(\sigma_k^+)(2\pi)^3\,\delta^+(k^2-m^2)\delta^+(k'^2-m^2)$$

$$\times\,\delta^+[\,(P-k-k')^2-m^2\,]\ S(q_{k'}^-)v\,[\,(P-2k-k')^2\,] \quad . \tag{18}$$

For $m = n$, writing $p_{12} = \frac{1}{2}(k_1-k_2)$, the second term in Eq. 15 becomes

$$\frac{i}{(2\pi)^5}\int d^4k\,\frac{\delta^+(k^2-m^2)}{2(2\pi)^3} <p \mid T(s^+) \mid k> S(\sigma_k^+)\,S(q_k^-)<k \mid T(s^-) \mid q>$$

$$\times\int d^4p_{12}(2\pi)^2\ v^2(p_{12}{}^2)\,\delta^+(k_1{}^2-m^2)\ \delta^+(k_2{}^2-m^2) \quad . \tag{19}$$

Equations 19, 15 and 10 yield

$$\tau(q_k^+) - \tau(q_k^-) = i(2\pi)^2\,\delta^+(k^2-m^2)\delta^+(q_k{}^2-m_R{}^2)$$

$$+\frac{i\delta^+(k^2-m^2)}{2(2\pi)^3}\ S(q_k^+)S(q_k^-)\ \int d^4p_{12}v^2(p_{12}{}^2)\delta^+(p_1{}^2-m^2)\delta^+(p_2{}^2-m^2) \tag{20}$$

which together with Eq. 18 suggests the identification

$$\tau(q_k) = (2\pi)\delta^+(k^2-m^2)S(q_k) \quad . \tag{21}$$

Finally we have

$$<k \mid B(s^+) \mid k'> - <k \mid B(s^-) \mid k'>$$

$$= iv[\,(P-k-2k')^2\,]\ 2\pi\ \delta^+[\,(P-k-k')^2-m^2\,]\ v[\,(P-2k-k')^2\,] \tag{22}$$

with the constraints $k^2 = m^2$, $k'^2 = m^2$. For the composite particle propagator we get

$$S(q_k^+) - S(q_k^-) = 2\pi\, i\, \delta^+(q_k^2 - m_R^2)$$
$$+ \frac{S(q_k^+)S(q_k^-)}{2(2\pi)^4}\, i \int d^4p_{12}\, v^2(4p_{12}^2)(2\pi)^2 \delta^+(p_1^2 - m^2)\delta^+(p_2^2 - m^2) \quad . \qquad (23)$$

The potential term B is obtained from a dispersion relation in s (no cut contribution from the vertex functions) . We easily find in the s-channel center of mass

$$\langle k \,|\, B(s) \,|\, k' \rangle = \frac{v[(P-k-2k')^2]\{\omega_k + \omega_{k'} + \omega_{k+k'}\}\, v[(P-2k-k')^2]}{\omega_{k+k'}[(\omega_k + \omega_{k'} + \omega_{k+k'})^2 - s]} \qquad (24)$$

where $\omega_k = (\underset{\sim}{k}^2 + m^2)^{\frac{1}{2}}$.

The composite particle propagator is found with greater difficulty and its calculation is done in the Appendix. Its form for a spinless composite formed from two spinless, equal mass particles (with vertex functions set, for simplicity, equal to coupling constants) is

$$S(\sigma) = -\frac{1}{D(\sigma)} \qquad (A-1)$$

$$D(\sigma) = \sigma - m_R^2 + \frac{g^2}{32\pi^2}\left(\frac{4m^2 - \sigma}{\sigma}\right)^{\frac{1}{2}} \qquad (A-9)$$

$m_R 2$ = mass of the composite.

So our final equation is

$$\langle p \,|\, T(s) \,|\, q \rangle = \langle p \,|\, B(s) \,|\, q \rangle + \frac{1}{(2\pi)^4} \int d^4k \langle p \,|\, B(s) \,|\, k \rangle\, \tau(q_k) \langle k \,|\, T(s) \,|\, q \rangle \qquad (25)$$

with $\langle p | B(s) | q \rangle$ given by Eq. 24 and $\tau(q_k) = (2\pi)\delta^+(k^2 - m^2)S(q_K)$

with $S(q_k)$ given by Eqs. A-1 and A-2.

III. DERIVATION OF THE "MULTIPERIPHERAL" EQUATION

We now formulate our cross channel multiperipheral equation by using a spectral representation in momentum transfer for the composite-particle amplitude. [6] We begin with the equation derived in Sec. II

$$T(p,q,s) = B(p,q,s) + \frac{1}{(2\pi)^4} \int d^4k\, B(p,k,s)\, \tau(q_k)\, T(k,q,s) \quad . \tag{26}$$

Noting that $\tau(\sigma_k) = 2\pi\ \delta^+(k^2-m^2)\ S(q_k)$, we have

$$T(p,q,s) = B(p,q,s) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} \frac{1}{D(q_k)}\ B(p,k,s)T(k,q,s) \quad . \tag{27}$$

Making the angular integration explicit, we find

$$T(p,q,s) = B(p,q,s) + \frac{1}{(2\pi)^3} \int \frac{k^2 dk}{2\omega_k D(q_k)}\, d\Omega_{\hat{k}}\ B(p,k,s)T(k,q,s) \tag{28}$$

Again

$$B(p,q,s) = 2\pi g^2 \int_{s_0}^{\infty} \frac{ds'}{s'-s}\ \delta_+[\,(P'-p-q)^2-m^2\,] \tag{22}$$

which leads to Eq. 28 being written as

$$T(p,q,s) = B(p,q,s) + \frac{g^2}{(2\pi)^2} \int \frac{k^2 dk}{2\omega_k D(q_k)}\ d\Omega_{\hat{k}} \int_{s_0}^{\infty} \frac{ds'}{s'-s}$$

$$\times\ \delta_+[\,(P'-p-k)^2-m^2]\ T(k,q,s) \quad . \tag{29}$$

Implicit in the dependence of T on momentum variables is a dependence on momentum transfer t, so we may write $T(p,q,s) = T(p,q,s,t)$. We now use a spectral representation in momentum transfer for $T(p,q,s,t)$ [6] ,

$$T(p,q,s,t) = B(p,q,s,t) + \frac{1}{\pi}\int_{t_0}^{\infty} \frac{A(p,q,s,t')}{t'-t}\,dt' \quad . \tag{30}$$

This is consistent with Eq. 29 if $A(p,q,s,t)$ tends to zero for $t \to \infty$. Otherwise, subtractions have to be made; this is inconsequential because we are interested in A , which is the discontinuity of T across the cut in t .

Introducing Eq. 30 into Eq. 29 we obtain

$$\begin{aligned}
&\frac{1}{\pi}\int \frac{dt'}{t'-t} A(p,q,s,t') \\
&= \frac{g^2}{(2\pi)^2}\int \frac{k^2 dk\, d\Omega_{\hat{k}}}{2\omega_k D(q_k)} \int_{s_0}^{\infty} \frac{ds'}{s'-s}\, \delta_+[\,(P'-p-k)^2 - m^2\,]\, B(k,q,s,t_2) \\
&+ \frac{g^2}{(2\pi)^2}\int \frac{k^2 dk\, d\Omega_{\hat{k}}}{2\omega_k D(q_k)} \int_{s_0}^{\infty} \frac{ds'}{s'-s}\, \delta_+[\,(P'-p-k)^2 - m^2\,]\, \frac{1}{\pi}\int \frac{dt_2'}{t_2'-t_2} A(q,k,s,t_2') .
\end{aligned} \tag{31}$$

The δ - function may be written (since the s' integration is performed in the s' center of mass).

$$\begin{aligned}
\delta\,[\,(P'-p-k)^2 - m^2\,] &= \delta[\,(\sqrt{s'} - \omega_p - \omega_k)^2 - (p^2+k^2+m^2) - 2pk z_1\,] \\
&= \frac{1}{2pk}\,\delta\left[\frac{(\sqrt{s'} - \omega_p - \omega_k)^2 - (p^2+k^2+m^2)}{2pk} - z_1\right] .
\end{aligned} \tag{32}$$

We use the relations

$$d\Omega_{\hat{k}} = dz_1 d\phi \quad , \tag{33}$$

$$z_2 = zz_1 + \sqrt{1-z^2}\,\sqrt{1-z_1^2}\cos\phi \quad , \tag{34}$$

$$z_2 - z_2' = zz_1 + \sqrt{1-z^2}\,\sqrt{1-z_1^2}\cos\phi - z_2'$$

$$= a + b\cos\phi \quad . \tag{35}$$

With

$$a = zz_1 - z_2' \tag{36}$$

and

$$b = \sqrt{1-z^2}\,\sqrt{1-z_1^2} \quad , \tag{37}$$

we find

$$\int \frac{d\phi}{a+b\cos\phi} = \frac{2\pi}{(z_2'+z^2+z_1^2-2zz_1z_2'-1)^{\frac{1}{2}}} \quad . \tag{38}$$

With this we obtain for $A(p,q,s,t)$, the discontinuity of T across the t cut:

$$A(p,q,s,t) = \mathrm{disc}_t\ \beta(p,q,s,t)$$
$$+\frac{g^2}{\pi}\int \frac{k^2 dk}{2\omega_k D(q_k)}\ \frac{1}{2pk\ 2kq}\int_0^t dt_2'\int_{s_0}^{\infty}\frac{ds'}{s'-s}$$
$$\times \frac{A(k,q,s,t_2')}{K^{\frac{1}{2}}(z,\bar{z},z_2')} \tag{39}$$

where

$$K(z,\bar{z},z_2') = [\, z^2+\bar{z}^2+z_2'^2-1-2z\bar{z}z_2'\,] \tag{40}$$

and z_2' is linearly related to t_2' [7] . Also

$$\bar{z} = \frac{(\sqrt{s'} - \omega_p - \omega_k)^2 - (p^2+k^2+m^2)}{2pk} \tag{41}$$

The expression $\mathrm{disc}_t\, \beta(p,q,s,t)$ represents the discontinuity of the expression in Eq. 39 with the term B.

At this point we take the limit $t \to \infty$, $t_2' \to \infty$. If we assume in doing this that $A(k,q,s,t_2')$ converges sufficiently fast when $k \to \infty$, we can neglect p, k, q and masses compared to t and t_2'. We then obtain

$$A(p,q,s,t) = \frac{g^2}{2\pi} \int_0^t dt_2' \int_{s_0}^{\infty} \frac{ds'}{s'-s} \int_{k_0}^{\infty} \frac{k^2 dk}{2\omega_k D(q_k)} \times \frac{A(k,q,s,t_2')}{2k^2 t \left[\frac{p^2}{k^2} x^2 - \frac{2p}{k} x\bar{z} + 1\right]^{\frac{1}{2}}} \tag{42}$$

with $x = \frac{t_2'}{t}$; $k_0' = px(\bar{z} \pm \sqrt{\bar{z}^2 - 1})$. (43)

Due to the dilatation invariance of the equation under the transformation $t_2' \to ct_2'$ and $t \to ct$, $A(p,q,s,t)$ has the form $t^{\alpha}\phi(p,q,s)$ and we obtain $\phi(p,q,s) =$

$$\phi(p,q,s) = \frac{g^2}{2\pi} \int_0^1 dx\, x^{\alpha} \int_{s_0}^{\infty} \frac{ds'}{s'-s} \int_{px\left(\bar{z}+\sqrt{\bar{z}^2-1}\right)}^{\infty} \frac{dk}{2\omega_k D(q_k)} \times \frac{\phi(k,q,s)}{\left[1+\frac{p^2}{k^2} x^2 - \frac{2p}{k} x\bar{z}\right]^{\frac{1}{2}}} \quad . \tag{44}$$

The integral Eq. 44 is Fredholm, with kernel

$$K(\alpha,p,k,q) = \frac{1}{2\omega_k D(q_k)} \int_0^1 dx x^{\alpha} \int_{s_0}^{\infty} \frac{ds'}{s'-s}$$

$$\times \frac{\theta\left[1+\frac{p^2}{k^2}x^2-\frac{2p}{k}x\bar{z}\right]}{\left[1+\frac{p^2}{k^2}x^2-\frac{2p}{k}\ x\bar{z}\right]^{\frac{1}{2}}} \tag{45}$$

If $y = \frac{p}{k}x$ the x integration becomes

$$\frac{k}{p}^{\alpha+1}\int_0^{\bar{y}} \frac{y^\alpha dy}{[1+y^2-2y\bar{z}]^{\frac{1}{2}}}$$

where

$$\bar{y} = \text{one of the zeros of the denominator} = \bar{z} \pm \sqrt{\bar{z}^2-1} \quad . \tag{46}$$

Consider the integral

$$\int_0^{\bar{y}} \frac{y^\alpha dy}{[1+y^2-2y\bar{z}]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}\int_0^{\bar{y}} \frac{y^{\alpha-\frac{1}{2}}dy}{\left[\frac{1+y^2}{2}-\bar{z}\right]^{\frac{1}{2}}} \quad ; \tag{47}$$

using

$$y = e^{-t} \qquad \bar{z} = \cosh \beta \tag{48}$$

and

$$\frac{1}{\sqrt{2}}\int_b^\infty \frac{e^{-(\alpha+\frac{1}{2})t}\,dt}{[\cos t - \cos \beta]} = \frac{1}{\sqrt{2}}\,Q_\alpha[\cos \beta] \tag{49}$$

$$(\beta > 0,\ \alpha > -1) \quad ,$$

We find

$$\left(\frac{k}{p}\right)^{\alpha+1}\frac{1}{\sqrt{2}}\,Q_\alpha(\bar{z}) = \int_0^1 \frac{x^\alpha dx}{\left[1+\frac{p^2}{k^2}x^2-2\,\frac{p}{k}\,x\bar{z}\right]^{\frac{1}{2}}} \quad . \tag{50}$$

Defining

$$f(p,q,s) = p^{\alpha+1}\,\phi(p,q,s) \quad , \tag{51}$$

we obtain

$$f(p,q,s) = \frac{1}{\pi}\frac{g^2}{2\sqrt{2}}\int_0^\infty \frac{dk}{2\omega_k D(q_k)}\int_{s_0}^\infty \frac{ds'}{s'-s}\, Q_\alpha\left[\frac{(\sqrt{s'}-\omega_p-\omega_k)^2-(p^2+k^2+m^2)}{2pk}\right]$$

$$\times\ f(k,q,s) \quad . \tag{52}$$

The kernel of the integral equation is now:

$$K(\alpha,p,k,s) = \frac{1}{2\sqrt{2\pi}}\,\frac{g^2}{2\omega_k D(q_k)}\int_{s_0}^\infty \frac{ds'}{s'-s}\, Q_\alpha\left[\frac{(\sqrt{s'}-\omega_p-\omega_k)^2-(p^2+k^2+m^2)}{2pk}\right] \tag{53}$$

Equation 52 is an eigenvalue equation of the form

$$f = K_\alpha f \tag{54}$$

This is the same form as the homogeneous equation for a bound state in the Bethe Salpeter problem. The condition for a solution is that

$$\det(1-K_\alpha) = 0 \tag{55}$$

If we call

$$\det(1-K_\alpha) = D(\alpha, g^2, s) \quad , \tag{56}$$

to solve Eq. 52 we have the Fredholm eigenvalue condition

$$D(\alpha, g^2, s) = 0 \quad . \tag{57}$$

with D given by [6]

$$D(\alpha, g^2, s) = 1 + \sum_{n=1}^\infty \frac{(-g^2/2\sqrt{2}\pi)^n}{n!}\int dk_1\cdots dk_n$$

$$\times \begin{vmatrix} K(\alpha, k_1, k_1; s) & K(\alpha, k_1, k_n; s) \\ K(\alpha, k_n, k_1; s) & K(\alpha, k_n, k_n; s) \end{vmatrix} . \tag{58}$$

This gives an implicit relation for the trajectory function α ;

$$\alpha = \alpha(g^2, s) \quad . \tag{59}$$

IV. ALTERNATIVE DERIVATION OF THE "MULTIPERIPHERAL" EQUATION

In this section we derive the cross channel equation in terms of invariant variables. Again, we begin with the equation derived in Sec. II.

$$T(p, q, s) = B(p, q, s) + \frac{1}{(2\pi)^4} \int d^4k\, B(p, k, s)\, \tau(q_k) T(k, q, s) \tag{26}$$

or

$$T(p, q, s) = B(p, q, s) + \frac{1}{(2\pi)^3} \int d^4k\, \frac{B(p, k, s)}{D(q_k)}\, \delta_+(k^2-m^2) T(k, q, s) \quad . \tag{60}$$

To proceed we must compute the kernel resulting from the Jacobian of the transformation from momentum variables to invariant variables. Writing $B(p, k, s)$ in terms of its dispersion integral, the second term in Eq. 60 becomes

$$\frac{g^2}{(2\pi)^2} \int \frac{d^4k}{D(q_k)} \int \frac{ds'}{s'-s}\, \delta_+[\,(P'-p-k)^2-m^2\,]\, \delta_+(k^2-m^2)\; T(k, q, s) \quad .$$

If we call $m^2 = -u_2'$ and define two new invariants u_1' and t' as follows

$$\int du_1' \, dt' \, \delta[(P-k)^2+u_1'] \, \delta[(p-q-k)^2-t'] = 1 \quad , \tag{61}$$

the Jacobian of the transformation to invariant variables is then

$$\int d^4k \; \delta_+[(P-p-k)^2-m^2] \, \delta[(P-k)^2+u_1'] \, \delta[k^2+u_2'] \, \delta[(P-q-k)^2-t']$$

$$= \frac{\theta(H)}{\sqrt{H}} \quad , \tag{62}$$

where

$$H = \left[-\left\{\frac{(u_1'-u_2')}{2} - \frac{(\eta_1-\eta_2)}{2}\, x\right\}^2 \right.$$

$$\left. - \, s(1-x)\left\{\frac{u_1'+u_2'-(\eta_1+\eta_2)x}{2} - \frac{\mu^2 x}{1-x} + \frac{s}{4}(1-x)\right\}\right] \tag{63}$$

and

$$x = \frac{t'}{\tau} \;, \quad t = (q-p)^2$$

$$\tau = (P'-P-q-p)^2$$

$$\eta_1 = -[P'-(P/2+p)]^2 \tag{64}$$

$$\eta_2 = -[P'-P/2-p)]^2 \quad .$$

If, as in Sec. III, we use a spectral representation for $T(p,q,s,t) = T(p,q,s)$

$$T(p,q,s,t) = B(p,q,s,t) + \frac{1}{\pi}\int_{t_0}^{\infty} \frac{A(p,q,s,t')}{t'-t}\, dt' \tag{30}$$

and take the absorptive part of T in t, we obtain

$$A(u_1, u_2, s, t) = \text{disc}_t\, \beta(u_1, u_2, s, t)$$

$$+ \frac{g^2}{(\pi)} \int dt' \int \frac{du_1'}{D(-u_1')} \int \frac{ds'}{s'-s} \; \frac{\theta(H)}{\sqrt{H}} \; \frac{1}{\tau} \; A(u_1', u_2', s, t') \quad , \tag{65}$$

For high t, we obtain $(\tau \to t)$
$\lim_{t \to \infty}$

$$A(u_1, u_2, s, t) = \frac{g^2}{\pi} \int_0^t \frac{dt'}{t} \int \frac{du_1'}{D(-u_1')} \int \frac{ds'}{s'-s} \; \frac{A(u_1', u_2', s, t')}{\sqrt{H}}$$

$$= \frac{g^2}{(\pi)} \int_0^1 dx \int_{u_{1\,\min}'}^{\infty} \frac{du_1'}{D(-u_1')} \int \frac{ds'}{s'-s} \; \frac{A(u_1', u_2', s, tx)}{\sqrt{H}} \; \theta(H) \quad . \tag{66}$$

Due to the dilatation invariance of the kernel under the transformation $t \to ct$, $t' \to ct'$, we may write $A(s, t, u_1, u_2) = t^{\alpha(s)} \phi(s, u_1, u_2)$; $\alpha(s) = \ell$; the integral equation becomes

$$\phi(s, u_1, u_2) = \frac{g^2}{\pi} \int_0^1 dx x^{\ell} \int_{u_{1\,\min}'}^{\infty} \frac{du_1'}{D(-u_1')} \int \frac{ds'}{s'-s} \; \frac{\phi(s, u_1', u_2')}{\sqrt{H}} \; \theta(H) \tag{67}$$

Again the Jacobian may be written

$$H = \frac{1}{4} \left(-\xi^2 - \xi_1^2 - \xi_2^2 + 2\xi_1 \xi_2 + 2\xi\xi_1 + 2\xi\xi_2 \right) \quad , \tag{68}$$

where

$$\xi = -s(1-x)$$

$$\xi_1 = u_1' - \eta_1 x - \frac{\mu^2 x}{1-x}$$
$$\xi_2 = u_2' - \eta_2 x - \frac{\mu^2 x}{1-x} \tag{69}$$

or

$$H = \frac{x^2}{4}\left[(-(\eta_1-\eta_2)^2 - 2s(\eta_1+\eta_2) - s^2 \right]$$

$$-\frac{2x}{4}\left[-(u_1'-u_2')(\eta_1-\eta_2) - s(\eta_1+\eta_2) - s(u_1'+u_2') - 2\mu^2 s - s^2 \right]$$

$$+\frac{1}{4}\left[-(u_1'-u_2')^2 - 2s(u_1'+u_2') - s^2 \right]$$

$$= -\alpha x^2 + 2\beta x - \alpha' \tag{70}$$

where

$$\alpha = (\eta_1-\eta_2)^2 + 2s(\eta_1+\eta_2) + s^2$$

$$= \lambda(s, -\eta_1, -\eta_2)$$

$$\alpha' = (u_1'-u_2')^2 + 2s(u_1'+u_2') + s^2 \tag{71}$$

$$= \lambda(s, -u_1', -u_2')$$

$$\beta = (u_1'-u_2')(\eta_1-\eta_2) + s(\eta_1+\eta_2) + s(u_1'+u_2') + 2\mu^2 s + s^2 \quad .$$

If we use the integral representation of the Legendre function of the second kind $Q_\ell(z)$ [8]

$$Q_\ell\left[-\frac{\beta}{\sqrt{\alpha\alpha'}}\right] = \sqrt{-\alpha}\left(\sqrt{\frac{\alpha}{\alpha'}}\right)^\ell \int_0^1 x^\ell \frac{dx\ \theta[-\alpha x^2+2\beta x-\alpha']}{[-\alpha x^2+2\beta x-\alpha']^{\frac{1}{2}}} \tag{72}$$

Valid when $\mathrm{Re}\ \ell > -1$, Eq. 72 becomes

$$\phi(s,u_1,u_2) = \frac{g^2}{\pi}\int\frac{du_1'}{D(-u_1')}\int\frac{ds'}{s'-s}\left(\sqrt{\frac{\alpha'}{\alpha}}\right)^\ell \frac{1}{\sqrt{-\alpha}}\, Q_\ell\left[\frac{-\beta}{\sqrt{\alpha\alpha'}}\right] \times \phi(s,u_1',u_2') \quad . \tag{73}$$

Remembering that $u_1 = -m^2$, $u_2 = -M^2$, $u_2' = -m^2$, we have

$$\phi(s,-m^2,-M^2) = \frac{g^2}{\pi}\int\frac{du_1'}{D(-u_1')}\int\frac{ds'}{s'-s}\left(\sqrt{\frac{\alpha'}{\alpha}}\right)^\ell \frac{1}{\sqrt{-\alpha}}\, Q_\ell\left[\frac{-\beta}{\sqrt{\alpha\alpha'}}\right] \quad . \tag{74}$$

The kernel of the equation (which is a Fredholm equation) is

$$\frac{g^2}{\pi}\frac{1}{D(-u_1')}\int\frac{ds'}{s'-s}\frac{1}{\sqrt{-\alpha}}\left(\sqrt{\frac{\alpha'}{\alpha}}\right)^\ell Q_\ell\left[\frac{-\beta}{\sqrt{\alpha\alpha'}}\right] \quad . \tag{75}$$

V. DISCUSSION

We began with a linear, relativistic, three-body Bethe-Salpeter equation describing scattering in the s channel between a spinless particle and a spinless composite. This equation has the virtue of satisfying two and three body unitarity exactly in the s-channel. The mechanism for this is the particular form chosen for the Born term and the composite-particle propagator, using the Blankenbecler-Sugar prescription. [4] .

We perform a continuation of our three body Bethe-Salpeter equation to the cross channel by using a spectral representation in momentum transfer (t) for the composite particle scattering amplitude. By taking the t discontinuity of the equation and going to high t , we obtain a homogeneous integral equation for the absorptive part, which is reminiscent of multiperipheral [9, 10] models for high energy in the t channel. In particular, the topology of the equation is similar to that of multiperipheral models and we obtain a Reggeized solution

$$t^{\alpha(s)} \; \phi(p, q, s) \quad .$$

This not surprising since a connection between the Bethe-Salpeter equation in the ladder approximation and the ϕ^3 multiperipheral model has been discussed by several authors [10, 6] .

When the $t^{\alpha(s)}$ is factored out, we are left with an eigenvalue equation for $\alpha(s)$, and $\phi(p, q, s)$. However, this integral equation has two peculiarities due to the particular form of two and three body unitarity chosen: 1) an integration involving s (the s' integration from the Born exchange term), so that the Born term has the correct three particle cut in s , 2) a more complicated analytic structure in $k\ (u_1')$ due to the form of the composite particle propagator, so that the propagator has the correct two and three particle cuts in s .

The original composite-particle equation and the amplitude constructed from it can be used to investigate the following things. First, we can extract the behavior of the leading Regge singularity via a determinental solution of the equation and study the effect of the extra unitarity on the small and large s behavior of the output trajectory. This has been done and will appear in a subsequent paper. Second, we can attach a propagator and vertex function for the decay of the final state composite to form a production amplitude (two to three scattering). Finally, by attaching a propagator and a vertex function for the formation of the initial state composite, we can construct an amplitude describing three body scattering. This amplitude can be used to calculate, via Mueller's Optical Theorem, a single particle inclusive distribution. We are investigating whether a triple Regge limit exists for this problem.

The solution of our "multiperipheral" equation may be used to construct an amplitude describing the production of four particles at high energy (via an isobar mechanism). We can then study the effect of the extra cross channel unitarity on the high energy behavior, the behavior in rapidity and final state correlations of the produced particles. If we Reggeize the composites we may study an amplitude with Reggeon - Reggeon - particle vertices where one of the Reggeons carries the effect of more unitarity into the vertex than the other (since Reggeizing the composites involves just two body unitarity in keeping with the structure of the original model). There is also a possibility that two Reggeon cut discontinuities may be studied using this amplitude.

A presumption in all of this has been that a path of analytic continuation exists from the s-channel to the t-channel. Such a continuation could possibly be affected by the presence of anomalous thresholds. We are investigating this by an s-channel partial wave analysis of the original composite-particle Bethe-Salpeter equation.

Authoress wishes to thank Professor J. E. Young (of Physics Department, M. I. T.) for suggesting this line of investigation.

APPENDIX: PROPAGATOR CALCULATION

Beginning with the expression in Eq. 23 for the composite particle propagator, $S(\sigma)$, Aaron, Amado and Young derive the following equation for the inverse propagator

$$S(\sigma) = -\frac{1}{D(\sigma)} \tag{A-1}$$

$$D(\sigma) = \sigma - m_R^2 \left[1 + \frac{(\sigma - m_R^2)}{2(2\pi)^3} \int \frac{d^3k\, g^2}{\omega_k (\sigma - \bar{\sigma})(\bar{\sigma} - m_R^2)^2} \right] \tag{A-2}$$

where

$$\bar{\sigma} = 4(k^2 + m^2) = 4\omega_k^2 \tag{A-3}$$

for a composite, of mass m_R^2, formed from equal mass spinless particles of mass m^2.

Replacing the vertex function with the coupling constant g, we have

$$D(\sigma) = (\sigma - m_R^2) \left[1 + \frac{(\sigma - m_R^2)}{2(2\pi)^3} \int \frac{d^3k\, g^2}{\omega_k (\sigma - \bar{\sigma})(\bar{\sigma} - m_R^2)^2} \right] \tag{A-4}$$

Concentrating on the integral

$$\int \frac{d^3k\, g^2}{\omega_k (\sigma - \bar{\sigma})(\bar{\sigma} - m_R^2)^2} = \int \frac{k^2 d\,|k|\, d\Omega\, g^2}{(\sigma - 4\omega^2)\omega\,(m_R^2 - 4\omega^2)^2} \quad , \tag{A-5}$$

we use

$$\omega^2 = k^2+m^2 \; ; \; k = (\omega^2-m^2)^{\frac{1}{2}} \tag{A-6}$$

$$d|\underset{\sim}{k}| = \frac{\omega\, dk}{|\underset{\sim}{k}|}$$

to cast the integral in the form

$$\frac{\pi g^2}{16} \int_m^{\infty} \frac{d\omega\, (\omega^2-m^2)^{\frac{1}{2}}}{(a^2-\omega^2)(b^2-\omega^2)^2} \tag{A-7}$$

where

$$\begin{aligned} a^2 &= \sigma/4 \\ b^2 &= m_R^2/4 \qquad . \end{aligned} \tag{A-8}$$

Using contour integration and residue calculus we obtain for the inverse propagator

$$\begin{aligned} D(\sigma) &= \sigma - m_R^2 + \frac{ig^2}{32\pi^2} \frac{(\sigma-4m^2)^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} \\ &= \sigma - m_R^2 + \frac{g^2}{32\pi^2}\left(\frac{4m^2-\sigma}{\sigma}\right)^{\frac{1}{2}} \end{aligned} \tag{A-9}$$

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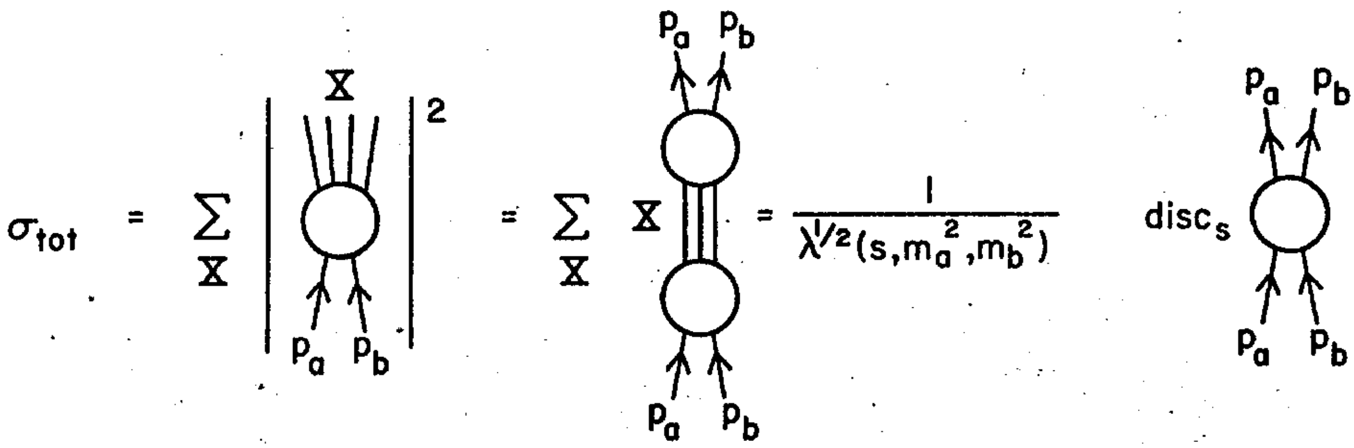
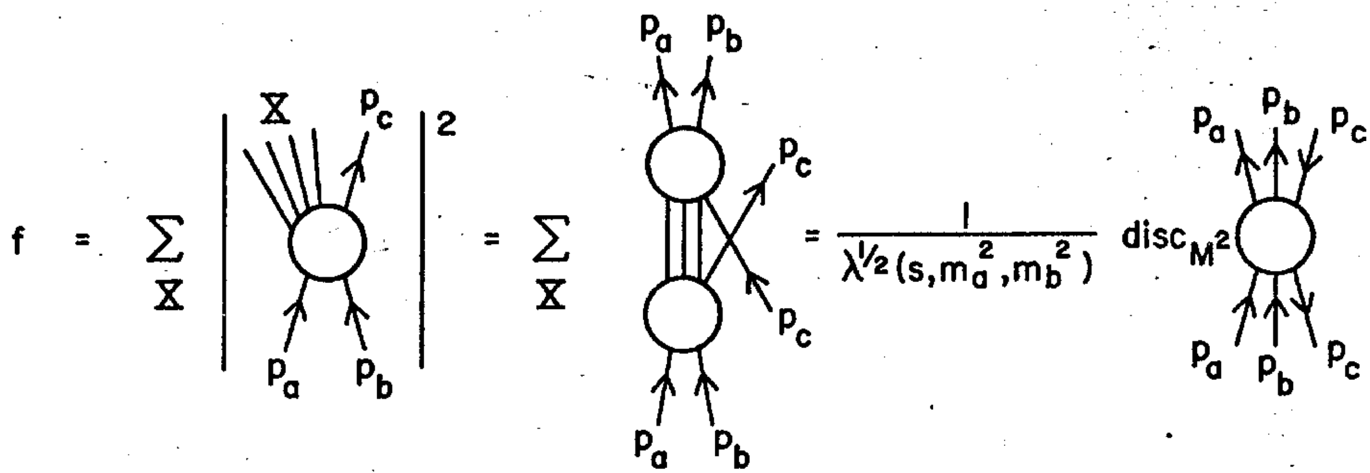
FIGURE CAPTIONS

Fig. 1	The Optical Theorem
Fig. 2	Mueller's Optical Theorem
Fig. 3	Amplitude for three-to-three scattering
Fig. 4	Composite-Particle Bethe Salpeter Equation
Fig. 5	Composite-Particle Equation with Momentum Labels
Fig. 6	The discontinuity of T(s)
Fig. 7	Two and three body unitarity
Fig. 8	The production amplitude
Fig. 9	The direct term in the unitarity relation
Fig. 10	The exchange term in the unitarity relation



$$s=(p_a+p_b)^2$$

FIG. 1



$$M^2=(p_a+p_b-p_c)^2$$

FIG. 2

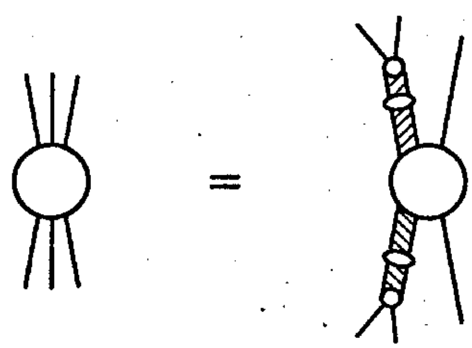


FIG. 3

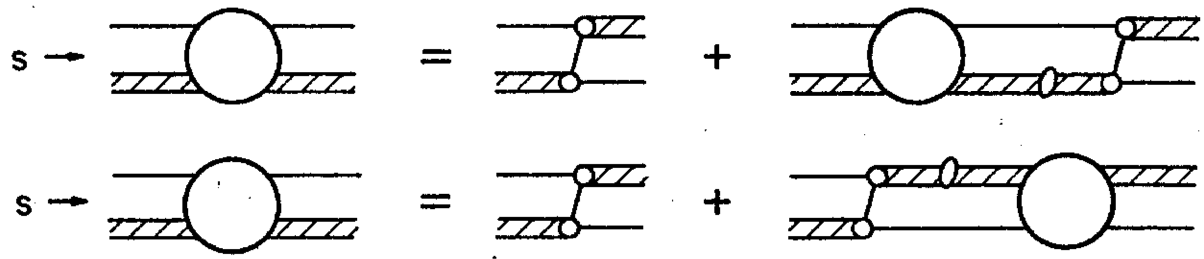


FIG. 4

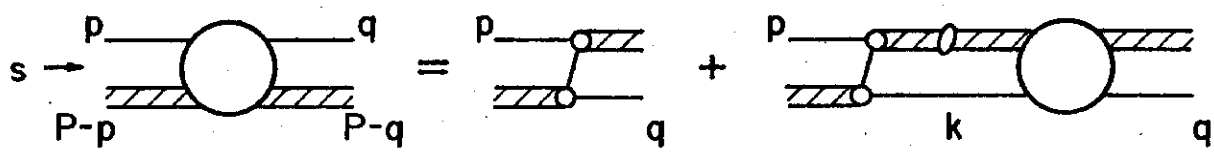


FIG. 5

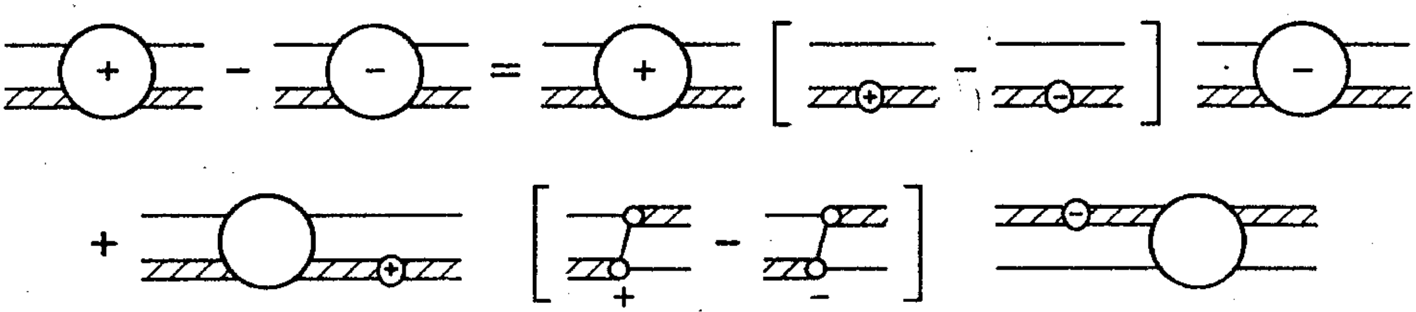


FIG. 6

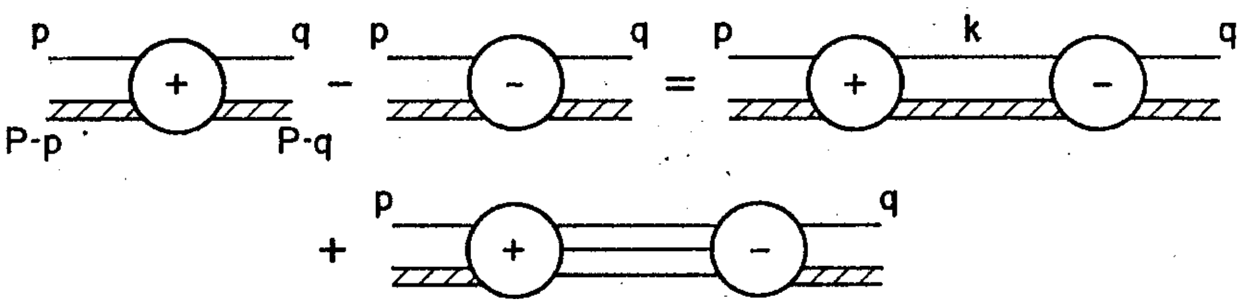


FIG. 7

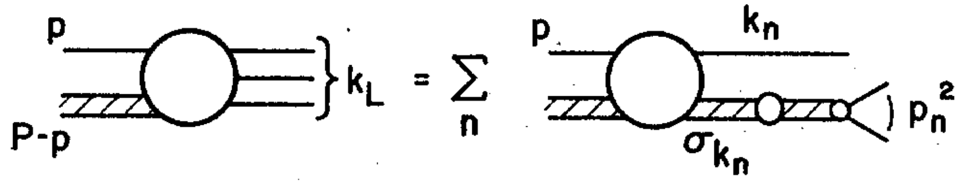


FIG. 8

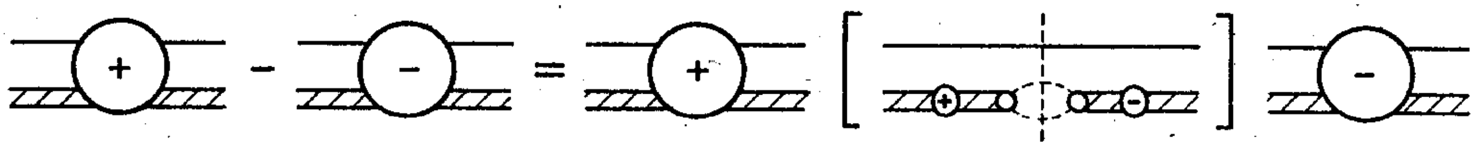


FIG. 9

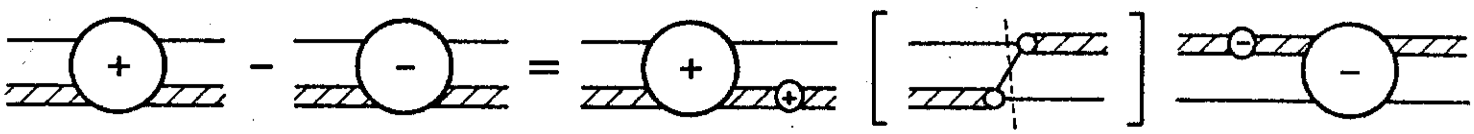


FIG. 10

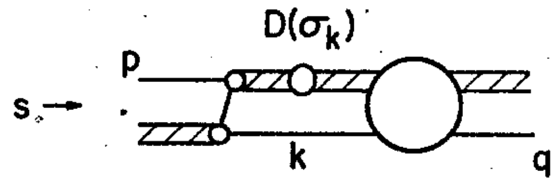


FIG. 11