national accelerator laboratory

NAL-Pub-73/91-THY
December 1973

Structure of the Pomeranchuk Singularity in Reggeon Field Theory

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ABSTRACT

Using the methods of the renormalization group we study the structure of Pomeranchukon Green's functions in a Reggeon calculus or Reggeon field theory model. We are able to determine the behavior of all Green's functions in the "infrared" limit of small Reggeon momenta and small Reggeon energy ($-E$ = angular momentum minus one). This behavior is governed by a zero of the classic Gell-Mann-Low variety which arises when the triple Pomeranchukon coupling is pure imaginary as suggested by Gribov's analysis of Feynman graphs in ordinary field theory. The form of the Pomeranchukon propagator dictates that the trajectory function be singular at $t=0$ and that a variety of scaling laws for the Green's functions be obeyed. By coupling particles into the theory, we find that total cross sections are predicted to rise as a small power of log s, which in the model is approximately $\sigma_T(s) \sim (\log s)^{1/6}$.

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Operated by Universities Research Association Inc. Under Contract with the United States Atomic Energy Commission

I. INTRODUCTION

By studying the nature of branch points in the angular momentum plane in Feynman graph models, Gribov has been able to abstract a Reggeon calculus or Reggeon field theory[1] which provides a powerful analytic tool for the discussion of the interplay between ℓ-plane poles and cuts. This field theory treats Reggeons as quasi-particles or elementary excitations in a space of one time and two spatial degrees of freedom. By choosing various forms of local interaction among the "free" field operators one can use the techniques of quantum field theory to study the physical or renormalized partial wave amplitudes in, say, elastic scattering processes.

These field theories have been analyzed at some length by Gribov, Midgal and Levin[2] in a long and occasionally difficult set of papers. In these articles a whole set of alternative renormalized solutions to the Reggeon field theories were presented using at various times perturbation theory, the full Schwinger-Dyson equations of the theory, or simple soluble static theories as methods of solution. Further study of such field theories have been carried out by Bronzan,[3] who provides a dynamical reason for the famous vanishing of the triple Pomeron vertex.[4] A recent summary of ideas in the Reggeon calculus and references to a variety of applications can be found in the work of Cardy and White,[5] who use Bronzan's observations to argue about the structure of $\sigma_T(s)$. A full scale review of the subject will be available shortly as well.[6]

In this and subsequent papers we will examine the structure of renormalized Green's functions in a variety of Reggeon field theories, some of substantial physical interest, using the method of the renormalization group to carry out our analysis.

We begin by briefly reviewing the motivation for a Reggeon field theory and discuss some of its limitations. After this we set up the field theory and establish the Feynman rules for calculating the Green's functions. Following this section we set up the renormalization group equations and study the constraints they place on the Green's functions and show how they enable one to determine the behavior of these quantities in the "infrared" limit of small Reggeon momentum and all angular momenta near one.

The particular theory we examine in this paper has a "bare" Reggeon with a linear trajectory passing through one at $t=0$

$$\alpha_0(t) = 1 + a_0' t \tag{1}$$

and a local triple Reggeon coupling only. In subsequent papers we shall discuss both more general "bare" trajectory functions and more elaborate coupling schemes. This example has quite enough physical interest and structure in itself to serve as a model for any further analyses of the type we present.

In his original study Gribov demonstrated that taking proper account of signature leads one to make the triple Pomeron coupling pure imaginary. We find that this observation is crucial in our renormalization group treatment. It allows us to find (when the number of spatial dimensions, D, is near four) a Gell-Mann-Low zero[7] of the relevant renormalization group functions which determine the infrared behavior ($\ell \approx 1$, $t \approx 0$) of the field theory. We shall show that this zero occurs in fact at a small value of a renormalized dimensionless coupling constant and allows one to make a kind of perturbation expansion in $\frac{\epsilon}{12} = \frac{4-D}{12}$. Thus, we are able to calculate in what *a priori* would appear to be a strong coupling problem.

We shall also present an analysis of the scaling structure that is dictated by the renormalization group for the Green's functions. When we couple the Reggeons to particles this implies that the elastic amplitude has the leading behavior

$$A(s,t) \approx s(\log s)^{\gamma} F[t(\log s)^{z}] \tag{2}$$

where the indices γ and z and the function F can be calculated as power series in $\epsilon/12$. (Note that $\epsilon/12$ is the natural perturbation parameter, which is not large when $\epsilon = 2$.) In the present theory we find to first order in $\epsilon/12$ that $\gamma \approx 1/6$ and $z \approx 13/12$ when D = 2. The total cross section arising from (2) is then

$$\sigma_T(s) \sim (\log s)^{1/6}, \tag{3}$$

with corrections down by approximately order $(\log s)^{-1/2}$.

Some of the scaling laws have been given by Gribov and Midgal with no indication how one might evaluate the indices γ and z, or the function F. Our presentation has the double attractiveness of being direct and of showing how one may indeed determine these quantities in perturbation theory. There are, not suprisingly at this stage, a host of unanswered questions within the framework of Reggeon field theories. Besides the enormous uncertainty of how to choose the Lagrangian, which problem plagues all field theorists, there is the additional tricky question of how to treat external particles and Reggeons[5,8,9] in inclusive and production amplitudes. The solution of this we defer for the present.

II. MOTIVATION FOR A REGGEON FIELD THEORY: REGGEON UNITARITY EQUATIONS

The cleanest indication that a Reggeon field theory might be a useful tool in the consideration of branch cuts in angular momentum comes from an examination of the formulae for discontinuities across ℓ-plane cuts. These discontinuity equations were derived in an heuristic manner by Gribov, Pomeranchuk and Ter-Martiroysyan some years ago[10] and have been formulated and discussed in more recent work found in Refs. 8 and 9.

The simplest example of such discontinuity formulae is given by the discontinuity across the two Reggeon cut in the partial wave amplitude $F(\ell, t)$ for the elastic scattering of two spinless particles. In Fig. 1 we show pictorially the t-channel exchange of two trajectories

$\alpha_1(t)$ and $\alpha_2(t)$ which gives rise to a cut in the ℓ plane. The analyses of Refs. 8, 9, and 10 tells us that

$$\mathrm{disc}_{\ell}\, F(\ell,t) = \int \frac{dt_1 dt_2\, \theta[-\Delta(t,t_1,t_2)]}{\sqrt{-\Delta(t,t_1,t_2)}}$$

$$\delta[\ell - \alpha_1(t_1) - \alpha_2(t_2) + 1]\; N_2(\ell + i\epsilon, t, t_1, t_2) N_2(\ell - i\epsilon, t, t_1, t_2) \tag{4}$$

where

$$\Delta(x,y,z) = (x+y-z)^2 - 4xy \tag{5}$$

and $N_2(\ell, t, t_1, t_2)$ is the partial wave amplitude for the "process": Particle 1 + Particle 2 $\rightarrow$ Reggeon $\alpha_1(t_1)$ + Reggeon $\alpha_2(t_2)$. We have suppressed signature labels and numerous factors of i's and π's in writing (4). If we switch to two dimensional vectors to re-express (4) [as pictured in Fig. 2], so that $t = -|\vec{q}|^2$ and $t_i = -|\vec{q}_i|^2$, we may write

$$\mathrm{disc}_E\, F(E,\vec{q}) = \int d^2q_1\, d^2q_2\; \delta(\vec{q}_1 + \vec{q}_2 - \vec{q})\, \delta(E - E_1 - E_2)$$

$$N_2(E + i\epsilon, \vec{q}, \vec{q}_1, \vec{q}_2) N_2(E - i\epsilon, \vec{q}, \vec{q}_1, \vec{q}_2), \tag{6}$$

where we have further chosen to write

$$E = 1 - \ell \text{ and } E_i = 1 - \alpha_i(-|\vec{q}_i|^2). \tag{7}$$

This formula suggests that the Reggeon is acting like a quasi-particle in two space and one time dimension with the "energy momentum relation"

$$E = 1 - \alpha(-|\vec{q}|^2). \tag{8}$$

Phase space is simply

$$d^2q_1 d^2q_2 \ \delta^2(\vec{q}_1 + \vec{q}_2 - \vec{q}) \, \delta[E - E_1(\vec{q}_1) - E_2(\vec{q}_2)] \, . \qquad (9)$$

Certain conservation laws are implied by the form of the discontinuity integral. First of all, two momentum is conserved. This is not a surprise since the two momentum degrees of freedom are precisely what is left after we integrate out two angular variables from four dimensional space-time to form the partial wave amplitude. Second, there is an energy conservation rule, but this is trickier. The energy of the two Reggeons $E_1 + E_2$ is constrained to be the net energy E emanating from the black box we have called N_2. It is important to note that E is <u>not</u> the sum of $1-\ell_i$ for the incoming particles. That, for external spinless particles, would restrict us to $E = 2$ ($\ell = -1$), which is not in the least implied by anyone's discontinuity relation. We must view the external particles on a rather different footing than the internal Reggeons and regard the blob N_2 as some kind of "external source" for energy.

The quasi-particle interpretation is made even firmer by the formula for the discontinuity across the n Reggeon intermediate state shown in Fig. 3

$$\text{disc}_E \, F(E, \vec{q}) = \int \prod_{j=1}^{n} d^2q_j \ \delta^2\Big(\sum_{j=1}^{n} \vec{q}_j - \vec{q}\Big) \ \delta\Big(\sum_j E_j - E\Big)$$

$$N_n(E+i\epsilon, \vec{q}, \vec{q}_1 \ldots \vec{q}_n) \, N_n(E-i\epsilon, \vec{q}, \vec{q}_1, \ldots \vec{q}_n). \qquad (10)$$

N_n is another source function for energy E.

One can write discontinuity equations for the Reggeon-particle partial wave amplitudes N_n, but it is expedient to skip this step and proceed directly to the four Reggeon partial wave amplitude M_4 shown in Fig. 4. This is a function of E_i and $\vec{k}_i$ for each Reggeon and the overall energy E. The discontinuity across the two Reggeon cut produced by Reggeons of energy ϵ_i and momentum $\vec{q}_i$ is depicted in Fig. 5:

$$\begin{aligned} &\mathrm{disc}_E\, M_4\,(E, E_i, \vec{k}_i) \\ &= \int d^2q_1\, d^2q_2\;\; \delta^{(2)}(\vec{k}_1 + \vec{k}_2 - \vec{q})\, \delta\,(E - \epsilon_1 - \epsilon_2) \\ &\quad M_4(E + i\epsilon, E_1, \vec{q}_1, E_2, \vec{k}_2; \epsilon_1, \vec{q}_1, \epsilon_2, \vec{q}_2) \\ &\quad \times M_4\,(E - i\epsilon, \epsilon_1, \vec{q}_1, \epsilon_2, \vec{q}_2, E_3, \vec{k}_3, E_4, \vec{k}_4)\;. \end{aligned} \tag{11}$$

In this formula E is conserved in the phase space integration but $E \neq E_1 + E_2 \neq E_3 + E_4$. Here $\epsilon_i = 1 - \alpha(-|\vec{q}_i|^2)$, as in Eqs. (6) and (10). Momentum, of course, is conserved throughout. In the language of potential scattering as used in Ref. 9, this equation is an off the energy shell unitarity formula as are Eqs. (6) and (10).

Now if we agree to deal exclusively with two-to-two particle processes, then all Reggeons appear inside internal integrations in which E is conserved. In that instance we may, in Eq. (11), set $E = E_1 + E_2 = E_3 + E_4$, that is go on the energy shell and encounter the usual form of the two particle unitarity relation. If we want to consider inclusive processes such as in Fig. 6 where the function N_2 appears, we shall have to enlarge our treatment to off energy shell processes. Even then all Reggeons appearing <u>inside</u> the black box of N_2 are <u>on shell</u>. (A. R. White informs us that the proper treatment of Regge cuts in inclusive processes may involve more than merely continuing our formulae off the energy shell.) So agreeing to consider only internal Reggeons in subsequent discussion, we write the discontinuity of M_4 across the n Reggeon cut as

$$\text{disc } M_4(E_i, \vec{k}_i) = \int \prod_{j=1}^{n} d^2 q_j \, \delta^2(\vec{k}_1 + \vec{k}_2 - \sum_{j=1}^{n} \vec{q}_j)$$

$$\delta\left[E_1 + E_2 - \sum_{j=1}^{n} \epsilon_j(\vec{q}_j)\right] M_{n+2}(E_1 + i\epsilon, \vec{k}_1, E_2 + i\epsilon, \vec{k}_2, \epsilon_1, \vec{q}_1 \ldots \epsilon_n, \vec{q}_n)$$

$$M_{n+2}(E_3 - i\epsilon, \vec{k}_3, E_4 - i\epsilon, \vec{k}_4, \epsilon_1, \vec{q}_1, \ldots \epsilon_n, \vec{q}_n)\Big|_{E_1 + E_2 = E_3 + E_4} \tag{12}$$

The restriction to $E = E_1 + E_2 = E_3 + E_4$, or going to the energy shell, puts us at $\ell = \alpha_1 + \alpha_2 - 1$ in M_4. By the analysis of Refs. 8-10, we see that this is the first nonsense point in the conventional partial wave amplitude.

There are two standard procedures for "solving" discontinuity equations like (11) and (12). The first is the S-matrix approach as considered at some length in Ref. 9. This is useful when one knows or can vigorously argue that only the two or possibly three Reggeon cuts are at all important to the problem at hand. This may well be the case when

$\alpha(0) < 1$ for all the Reggeons. The second is a field theory of the quasi-particles with $E(\vec{q}) = 1 - \alpha(-|\vec{q}|^2)$. This is useful when many cuts become important and is indispensable when $\alpha(0) \approx 1$ for any of the trajectories. In particular when one of the trajectories is the vacuum singularity with $\alpha(0) = 1$, then as a matter of principle, all cuts become important in the neighborhood of $t=0$ and one must either sum them all or indulge in generous foot shuffling to defend any other procedure.

In a langauge familiar to many readers we may describe the situation when $\alpha(0) = 1$ as an infrared or zero mass gap problem since the E, $\vec{q}$ relation is such that $E(0) = 0$. Field theory is notably more successful than S-matrix theory for dealing with infrared situations, and indeed, is just the tool required.

The field theoretic approach has its drawbacks, of course. One can know the functions like M_n in (12) only by solving the field theory. This is a formidable task in general and indeed one usually turns to perturbation theory in some coupling of local fields. There is also the unavoidable ambiguity of what free Lagrangian and what interaction Lagrangian one is supposed to choose. On top of that, one must face renormalization. The thing one is guaranteed to satisfy is the full set of unitarity equations to whatever order in perturbation theory his fortitude has lead him. This is the motivation for writing a Reggeon field theory, and we now turn our attention to that.

III. THE FIELD THEORY WITH A TRIPLE COUPLING

Our task is now to describe a field theory for Reggeons which have the energy-momentum relation $E = 1 - \alpha(-|\vec{q}|^2)$. Clearly to proceed in any but the most formal sense we must specify the $E, \vec{q}$ relation for the non-interacting field and then choose an interaction. Let us start by taking a linear trajectory

$$\alpha(-|\vec{q}|^2) = \alpha_0 - \alpha_0' q^2 \tag{13}$$

so

$$E = \alpha_0' \vec{q}^2 + (1 - \alpha_0) \tag{14}$$

This is reminiscent of a non-relativistic particle with mass $m = (2\alpha_0')^{-1}$ and energy gap $(1-\alpha_0)$. {Clearly other $E, \vec{q}$ relations lead to a whole variety of theories. Some of these will be explored in subsequent work.}

We associate with the quasi-particle a field $\psi(\vec{x}, t)$ in D space dimensions and one time dimension. The generalization to D space co-ordinates is a device which will prove very convenient in the following. Physics takes place at D=2.

The Lagrangian which gives (14) is

$$\mathscr{L}_0(\vec{x}, t) = \frac{i}{2}[\psi^+(\vec{x}, t) \overleftrightarrow{\frac{\partial}{\partial t}} \psi(\vec{x}, t)]$$

$$- \alpha_0 \nabla \psi^+(\vec{x}, t) \cdot \nabla \psi(\vec{x}, t) - \Delta_0 \psi^+(\vec{x}, t) \psi(\vec{x}, t) \tag{15}$$

where $\Delta_0 = 1 - \alpha_0$.

Varying the action

$$A_0 = \int d^D x\, dt \quad \mathscr{L}_0(\vec{x}, t) \tag{16}$$

with respect to ψ and ψ^+ gives

$$i \frac{\partial}{\partial t} \psi(\vec{x}, t) = -\alpha_0 \nabla^2 \psi(\vec{x}, t) + \Delta_0 \psi(\vec{x}, t) , \tag{17}$$

for the equations of motion. This clearly leads to

$$E = \alpha_0' \vec{k}^2 + \Delta_0 \tag{18}$$

for the non-interacting theory.

For an interaction we have a wide range of choices. The Feynman graph analysis[1, 2] indicates we may choose a bare coupling with any number of Reggeons. Only the coupling of 3 or 4 Reggeons is renormalizable in any conventional sense, so we will restrict ourselves to those couplings in order to have any control over the results. If the Lagrangian does not have a bare triple coupling, then no three Reggeon vertex function will ever appear in the theory. This is, however, of prime interest so we will study in this paper the field theory whose Lagrange function is

$$\mathscr{L}(\vec{x}, t) = \mathscr{L}_0(\vec{x}, t) - \frac{\lambda_0}{2} [\psi^+(\vec{x}, t)\psi(\vec{x}, t)^2 + \text{h. c.}] \tag{19}$$

and consider ψ^4 couplings in other publications.

It will be important in our subsequent discussions to record the canonical dimensions of the various quantities appearing in $\mathscr{L}$. We distinguish between dimensions of space and dimensions of time. Using the standard notation of [quantity] to indicate the dimensions of any quantity we note that

$$[x] = k^{-1}, \tag{20}$$

$$[t] = E^{-1} \tag{21}$$

and, of course,

$$[A = \int d^0x \; dt\mathscr{L}] = E^0 k^0. \tag{22}$$

This leads immediately to

$$[\psi] = k^{D/2}, \tag{23}$$

$$[\alpha_0'] = Ek^{-2}, \tag{24}$$

$$[\Delta_0] = E, \tag{25}$$

and

$$[\lambda_0] = Ek^{-D/2}. \tag{26}$$

The important point to observe is that the coupling λ_0 is not dimensionless. We shall shortly find a dimensionless coupling constant.

The quantities of interest to us are the Green s functions for n incoming and m outgoing Reggeons

$$G^{(n,m)}(\vec{x}_1, t_{x1}, \ldots \vec{x}_n, t_{xn}; \vec{y}_1, t_{y1}, \ldots \vec{y}_m, t_{ym})$$

$$= \langle 0 | T[\psi^+(\vec{y}_1, t_{y1}) \ldots \psi^+(\vec{y}_m, t_{ym}) \psi(\vec{x}_1, t_{x1}) \ldots \psi(\vec{x}_n, t_{xn})] | 0 \rangle \tag{27}$$

where the distinction between incoming and outgoing Reggeons is required because ψ is not hermitean and contains destruction operators only.

Indeed

$$\psi(\vec{x}, 0) = \int \frac{d^D q}{(2\pi)^{D/2}} e^{i\vec{q}\cdot\vec{x}} a(\vec{q}) \tag{28}$$

where $a(\vec{q})$ annihilates the vacuum

$$a(\vec{q})\,|0\rangle = 0, \tag{29}$$

and

$$[a(\vec{q}), a^{+}(\vec{q})] = \delta^D(\vec{q} - \vec{q}), \tag{30}$$

assures that

$$[\psi(\vec{x}, 0), \psi^{+}(\vec{y}, 0)] = \delta^D(\vec{x} - \vec{y}). \tag{31}$$

These commutation relations allow us to derive the Feynman rules for the computation in momentum space of the Green's functions

$$\delta\left(\sum_{i=1}^{n} E_i - \sum_{j=n+1}^{n+m} E_j\right) \delta^D\left(\sum_{i=1}^{n} \vec{q}_i - \sum_{j=n+1}^{n+m} q_j\right) G^{(n,m)}(E_i, \vec{k}_i)$$

$$= \int d^D x_1 dt_{x1} \ldots d^D y_m dt_{ym}\, e^{-i\vec{k}_1\cdot\vec{x}_1 + iE_1 t_{y1} \ldots + i\vec{k}_{n+m}\cdot\vec{y}_m - iE_{n+m} t_{ym}}$$

$$G^{(n,m)}(\vec{x}_1, t_{x1}, \ldots \vec{y}_m, t_{ym}). \tag{32}$$

We record these rules:

1. Draw all topologically distinct diagraphs (graphs with arrows indicating the directions of propagation of the Reggeons).

2. $\int d^D q\, dE q$ around each loop.

3. At each vertex put $-i\lambda_0/(2\pi)^{\frac{D+1}{2}}$.

4. For each Reggeon of momentum $\vec{k}$, energy E use the propagator

$$G_0^{(1,1)}(E,\vec{k}) = i/\left[E - \alpha_0'\vec{k}^2 - \Delta_0 + i\epsilon\right] . \tag{33}$$

5. For each two Reggeon loop with both momenta in the same direction, multiply by 1/2. See Fig. 7.

6. Conserve E and q at each vertex.

7. Because of the $i\epsilon$ prescription in 4, telling us that only the retarded propagator enters this theory, Reggeon loops in which all momenta go the same direction are zero. For example, Fig. 8 is zero.

As an example, the diagram in Fig. 7 gives the contribution to $G^{(1,1)}(E,\vec{k})$

$$\left[G_0^{(1,1)}(E,\vec{k})\right]^2 \left[\frac{-i\lambda_0}{(2\pi)^{\frac{D+1}{2}}}\right]^2 \frac{1}{2}\int d^Dq\, dE_q \frac{i}{E_q - \alpha_0'\vec{q}^2 - \Delta_0 + i\epsilon} \frac{i}{E - E_q - \alpha_0'(\vec{k}-\vec{q})^2 - \Delta_0 + i\epsilon} . \tag{34}$$

Using these rules for $D=2$, $\Delta_0 = 0$ and $\lambda_0 = ir_0$, r_0 real, one reproduces Gribov's Pomeranchukon interactions abstracted from hybrid Feynman diagrams.[1]

IV. RENORMALIZATION GROUP CONSTRAINTS ON REGGEON GREEN'S FUNCTIONS

The unrenormalized theory coming from the evaluation of $G^{(m,n)}$, to whatever order in perturbation theory in λ_0 one calculates, depends on the parameters λ_0, α_0', Δ_0 and possibly a cutoff Λ to control the ultraviolet behavior. The quantities appearing in the Lagrangian will be renormalized by the interaction and acquire new values λ, α', and Δ. We wish to consider a theory in which both the bare Pomeranchukon intercept $1 - \Delta_0$ and the renormalized intercept $1 - \Delta$ are one, so $\Delta_0 = \Delta = 0$. This requires a mass counterterm in $\mathscr{L}$ which is a function of λ_0 and α_0' to be determined in perturbation theory.

To define the renormalized theory we require a "subtraction" or renormalization point at which to define the renormalized quantities λ and α'. If Δ were not zero, it would provide a natural (but not a mandatory) normalization point. Since it is zero, we must seek another prescription. It is convenient to choose this point away from the various branch points in E which arise in perturbation theory at $E_n = \alpha_0' \vec{k}^2/n$. If we look at Fig. 9 where the branch point trajectories are shown, we can see that by selecting a normalization point in the fourth quadrant of the E, $\vec{k}^2$ plane we will stay off all perturbation theoretic cuts. For simplicity we will normalize at $E = -E_N < 0$ and $\vec{k}^2 = 0$. Any other choice entails a finite renormalization.

Our discussion will concentrate on the connected proper vertex functions $\Gamma^{(n,m)}$ defined by taking off the external legs of $G_c^{(n,m)}$:

$$\Gamma^{(n,m)}(E_1, \vec{k}_1, \ldots E_{n+m}, \vec{k}_{n+m}) = \prod_{j=1}^{n+m} G^{(1,1)}(E_j, \vec{k}_j)^{-1}$$

$$\times\, G_c^{(n,m)}(E_1, \vec{k}_1, \ldots\, E_{n+m}, \vec{k}_{n+m}) \,. \qquad (35)$$

This has the simple virtue of our not having to continually fret about the singularities associated with these external propagators, or with delta functions associated with disconnected contributions to $G^{(n,m)}$.

Now we place a set of conditions on the renormalized vertex functions $\Gamma_R^{(n,m)}$ which serve to define the renormalized quantities α' and λ. First we ask that the singularities of the inverse propagator $\Gamma_R^{(1,1)}(E,\vec{k}^2)$ occur at $E = 0 (\ell = 1)$ when $\vec{k}^2 = 0$ $(t = 0)$, so

$$\Gamma_R^{(1,1)}(E,\vec{k}^2)\Big|_{\substack{E=0\\ \vec{k}^2=0}} = 0. \tag{36}$$

This does not commit us to a <u>pole</u> in the renormalized propagator. It merely says that the singularity, whatever it may be, passes through $\ell = 1$, $t = 0$. This is the embodiment of our restriction that $\Delta = 0$.

Next we want the inverse propagator to look more or less like $[G_0^{(1,1)}]^{-1}$ and, of course, reduce to it when $\lambda_0 = 0$. This leads us to require

$$\frac{\partial i\Gamma_R^{(1,1)}}{\partial E}(E,\vec{k}^2)\Big|_{\substack{E=-E_n\\ \vec{k}^2=0}} = 1, \tag{37}$$

and

$$\frac{\partial}{\partial \vec{k}^2}\, i\,\Gamma_R^{(1,1)}(E,\vec{k}^2)\Big|_{\substack{E=-E_N\\ \vec{k}^2=0}} = -\alpha'(E_N) \quad . \tag{38}$$

Finally by noting that $\Gamma^{(1,2)}$ in lowest order perturbation theory is just $-i\lambda_0/(2\pi)^{\frac{D+1}{2}}$ we choose our final normalization condition to be (see Fig. 10).

$$\Gamma_R^{(1,2)}(E_1, \vec{k}_1, E_2, \vec{k}_2, E_3, \vec{k}_3)\Bigg|_{\substack{E_1=-E_N \\ E_2=E_3=-E_{N/2} \\ \vec{k}_i \cdot \vec{k}_j = 0}} = -\frac{i\lambda(E_N)}{(2\pi)^{\frac{D+1}{2}}} . \quad (39)$$

Before employing these conditions perhaps a word is in order about the spirit of this and all renormalization group investigations. What we are doing in essence is giving up the desire or ability to compute Δ, α' and λ from the given Lagrangian. Instead we are choosing their values by our normalization conditions, and then we shall parameterize all the other Reggeon Green's functions (presumably the full content of the theory is in them) in terms of these parameters. The parameters $\alpha'(E_N)$ and $\lambda(E_N)$ are not to be thought of as the trajectory slope or the renormalized triple Pomeranchukon coupling. The former, if indeed there is a trajectory, is determined by finding the trajectory and finding its slope. The latter is a function we have called $\Gamma_R^{(1,2)}$. It is true that as $\lambda_0 \to 0$, $\alpha'(E_N) \to \alpha'_0$ and $\lambda(E_N) \to \lambda_0$, but otherwise the parameters have no special significance.

The unrenormalized Green's functions depend on E_i, $\vec{k}_i$, α'_0, λ_0 and possibly a cutoff Λ

$$\Gamma^{(n,m)}(E_i, \vec{k}_i, \alpha'_0, \lambda_0, \Lambda) \quad (40)$$

The $\Gamma_R^{(n,m)}$ depend on E_i, $\vec{k}_i$, α', λ and E_N

$$\Gamma_R^{(n,m)}(E_i, k_i, \alpha', \lambda, E_N). \tag{41}$$

We choose to eliminate $\lambda(E_N)$ in favor of a dimensionless coupling

$$y(E_N) = \frac{\lambda(E_N)}{[\alpha'(E_N)]^{D/4}} E_N^{D/4-1} . \tag{42}$$

Note at this juncture the simplicity which transpires at D=4.[11]

The renormalization procedure consists of replacing $\psi(\vec{x},t)$, the unrenormalized field operator, by $\psi_R(\vec{x},t)$ which is related to ψ by

$$\psi_R(\vec{x},t) = Z^{-1/2}\psi(\vec{x},t) . \tag{43}$$

The proper vertex functions $\Gamma_R^{(n,m)}$ and $\Gamma^{(n,m)}$ are given then as

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, \alpha, y, E_N) = Z^{\frac{n+m}{2}} \Gamma^{(n,m)}(E_i, \vec{k}_i, \alpha_0', \lambda_0, \Lambda). \tag{44}$$

The so called renormalization group equations then follow from the straightforward observation that $\Gamma^{(n,m)}$, not knowing about E_N, cannot depend on it, so

$$E_N \frac{\partial}{\partial E_N} [\Gamma^{(n,m)}(E_i, k_i, \alpha_0', \lambda_0, \Lambda)] = 0. \tag{45}$$

Using the relation between Γ and Γ_R, this translates into

$$[E_N \frac{\partial}{\partial E_N} + \beta(y)\frac{\partial}{\partial y} + \zeta(y,\alpha')\frac{\partial}{\partial \alpha'} - \frac{(n+m)}{2}\gamma(y)]$$

$$\times \Gamma_R^{(n,m)}(E_i, \vec{k}_i, y, \alpha', E_N) = 0 , \tag{46}$$

where

$$\gamma(y) = E_N \frac{\partial}{\partial E_N} \log Z(\alpha_0', \lambda_0, \Lambda, E_N)\Bigg|_{\alpha_0', \lambda_0, \Lambda \text{ fixed}} , \qquad (47)$$

$$\beta(y) = E_N \frac{\partial}{\partial E_N} y(E_N)\Bigg|_{\alpha_0', \lambda_0, \Lambda \text{ fixed}} , \qquad (48)$$

and

$$\zeta(\alpha', y) = E_N \frac{\partial}{\partial E_N} \alpha'(E_N)\Bigg|_{\alpha_0', \lambda_0, \Lambda \text{ fixed}} . \qquad (49)$$

The functions β and γ are familiar from modern renormalization group analyses;[12] the function ζ is present because α' is renormalized in theories with bare linear Regge trajectories. By dimensional analysis, β and ζ/α' can depend only on y. If the E, $\vec{k}^2$ relation had been $E^2 = -\alpha_0 \vec{k}^2$, ζ would be zero. Although the functions β, γ, and ζ will be known only in perturbation theory, they serve as we will now see to determine non-perturbative properties of $\Gamma_R^{(n,m)}$.

The dimensional analysis of Sec. III tells us

$$[\Gamma_R^{(n,m)}] = E\, k^{D-(n+m)D/2} . \qquad (50)$$

This means we may write

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, y, \alpha', E_N)$$

$$= E_N \left[\frac{E_N}{\alpha'}\right]^{(2-n-m)D/4} \psi_{n,m}\left(\frac{E_i}{E_N}, \frac{\alpha'}{E_N} \vec{k}_i \cdot \vec{k}_j, y\right) . \qquad (51)$$

This observation tells us that

$$\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, y, \alpha', E_N) =$$

$$= \xi \frac{E_N}{\xi}\left[\frac{E_N/\xi}{\alpha/\xi}\right]^{(2-n-m)D/4} \psi_{n,m}\left(\frac{E_i}{E_N/\xi}, \frac{\alpha'/\xi}{E_N/\xi}\vec{k}_i\cdot\vec{k}_j, y\right) \tag{52}$$

$$= \xi\Gamma_R^{(n,m)}(E_i, \vec{k}_i, y, \frac{\alpha'}{\xi}, \frac{E_N}{\xi}) . \tag{53}$$

In the renormalization group equation we may eliminate $E_N\ \partial/\partial E_N$ in favor of $\xi(\partial/\partial\xi)$ using

$$\xi \frac{\partial}{\partial \xi}\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, y, \alpha', E_N) =$$

$$(1 - \alpha'\frac{\partial}{\partial\alpha} - E_N \frac{\partial}{\partial E_N})\,\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, y, \alpha', E_N), \tag{54}$$

so

$$\left\{\xi \frac{\partial}{\partial\xi} - \beta(y)\frac{\partial}{\partial y} + [\alpha' - \zeta(\alpha', y)\frac{\partial}{\partial\alpha'} + [\frac{(n+m)}{2}\gamma(y) - 1]\right\}$$

$$\times\ \Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, y, \alpha', E_N) = 0 . \tag{55}$$

The solution to this equation is standard[12] and is given by

$$\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, y, \alpha', E_N) =$$

$$\Gamma_R^{(n,m)}[E_i, \vec{k}_i, \tilde{y}(-t), \tilde{\alpha}'(-t), E_N]$$

$$\times\ \exp\int_{-t'}^{0} dt'\left\{1 - (\frac{n+m}{2})\ \gamma[\tilde{y}(t')]\right\} , \tag{56}$$

where

$$\frac{dy(t)}{dt} = -\beta\,[\tilde{y}(t)] \;, \tag{57}$$

$$\frac{d\alpha'(t)}{dt} = \alpha'(t) - \zeta\,[\,\alpha'(t), \tilde{y}(t)] \;, \tag{58}$$

and

$$t = \log \xi \,. \tag{59}$$

If we were to know β, ζ and γ, then the renormalization group constraint in (56) enables us to study $\Gamma_R^{(n,m)}\,(E_i, \vec{k}_i)$ as the E_i vary for fixed k_i. Equations similar to (55) can be derived for the response of $\Gamma_R^{(n,m)}$ when the E_i are fixed and the $\vec{k}_i$ vary, or when both the E_i and $\vec{k}_i$ vary. The Reggeon field theory is richer than relativistic field theory because of the absence of Lorentz invariance linking E and $\vec{k}$ dependence. In Eq. (56) we are interested in $E_i \approx 0$ or $\xi \to 0$ or $t \to -\infty$.

Alas, knowing β, ζ, and γ exactly is tantamount to having solved the full field theory. In that case, of course, the renormalization group is a rather redundant device. So we turn to perturbation theory to act as our guide. First, we wish to know Z. This we evaluate by computing $\Gamma^{(1,1)}(E, \vec{k}^2)$ to some order in λ_0 and using the normalization condition (37) to find

$$\frac{1}{Z(\alpha'_0, \lambda_0, \Lambda, E_N)} = \frac{\partial}{\partial E}\, i\Gamma^{(1,1)}(E, \vec{k}^2)\Bigg|_{\substack{E = -E_N \\ \vec{k}^2 = 0}} \;. \tag{60}$$

Knowing $\Gamma^{(1,1)}$ also allows us, via Eq. (38) to determine $\alpha'(E_N)$

$$\frac{-\alpha'(E_N)}{Z} = \frac{\partial}{\partial \vec{k}^2}\, i\,\Gamma^{(1,1)}(E, \vec{k}^2)\Bigg|_{\substack{E=-E_N \\ \vec{k}^2=0}} \qquad (61)$$

We will evaluate $\Gamma^{(1,1)}$ to the lowest nontrivial order in perturbation theory by considering the graphs in Fig. 11. This yields

$$i\,\Gamma^{(1,1)}(E, \vec{k}^2, \alpha'_0, \lambda_0) =$$

$$E - \alpha'_0 \vec{k}^2 + \frac{\lambda_0^2\, \pi^{D/2}\, \Gamma(1-D/2)}{2(2\pi)^D\,(2\alpha'_0)^{D/2}} \left(\frac{\alpha'_0 \vec{k}^2}{2} - E\right)^{D/2-1}, \qquad (62)$$

which gives

$$\frac{1}{Z} = 1 + \frac{\lambda_0^2\, \pi^{D/2}\, \Gamma(2-D/2)(E_N)^{D/2-2}}{2(2\pi)^D\,(2\,\alpha'_0)^{D/2}}, \qquad (63)$$

and

$$\gamma(y) = \left(\frac{\pi}{2}\right)^{D/2} \frac{\Gamma(3-D/2)}{2(2\pi)^D}\, y^2. \qquad (64)$$

In writing this expression we have not introduced a cutoff . Instead we use the simple device of keeping the dimension D of space a free parameter and letting it define a regularization procedure. This trick has been widely employed and discussed by 't Hooft and Veltman[13] in the context of relativistic field theories, especially gauge field theories. In all expressions where there is no singularity at D=2, for example γ, one may freely set D=2.

Similarly computing $\alpha'(E_N)$ we find for $\zeta(\alpha', y)$ to this order in perturbation theory

$$\zeta(\alpha', y) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{D/2} \frac{\Gamma(3 - D/2)}{2(2\pi)^D} \alpha' y^2 . \qquad (65)$$

The evaluation of $\lambda(E_N)$ and then $\beta(y)$ involves the computation of the graphs in Fig. 12 at the normalization point. Then

$$\frac{-i\lambda(E_N)}{(2\pi)^{(D+1)/2}} = Z^{3/2} \Gamma^{(1,2)}(E_i, \vec{k}_i)\Big|_{\text{normalization point}} . \qquad (66)$$

This yields for $\beta(y)$

$$\beta(y) = -\left(\frac{4-D}{4}\right) y - \left[\widetilde{K} + \frac{D}{4} K\right] y^3 , \qquad (67)$$

where K is the constant appearing in γ and ζ

$$K = \left(\frac{\pi}{2}\right)^{D/2} \frac{\Gamma(3 - D/2)}{4(2\pi)^D} , \qquad (68)$$

and

$$\widetilde{K} = K[\,8I(D) - 3\,] , \qquad (69)$$

and

$$I(D) = 2\,\frac{1-2^{1-(D/2)}}{(D/2)-1} \qquad (70)$$

Noting that $I(4) = 1$, $I(3) = 2(2-\sqrt{2}) > 3/8$ and $I(2) = \log 2 > 3/8$, we see that in the range $2 \leq D \leq 4$, the constants K and $\widetilde{K}$ are positive numbers. The function $\beta(y)$ has the form shown in Fig. 13.

The general analysis presented in Ref. 12 informs us at this juncture that the zero in $\beta(y)$ at $y=0$ governs the behavior of $\Gamma(\xi E)$ as $\xi \to \infty$ because

$$\left.\frac{d\beta}{dy}\right|_{y=0} = -\left(\frac{4-D}{4}\right) < 0 \tag{71}$$

for $D < 4$. The asymptotic behavior as $\xi \to 0$ which is our interest here is governed by a zero, absent in (67), where $\frac{d\beta}{dy} > 0$.

The astute reader will have observed that until now our entire analysis has been carried out with λ_0 and λ and y real. The Gribov study of Feynman graphs tells us that in fact λ_0 is pure imaginary: $\lambda_0 = i\, r_0$, $\lambda = i\, r$ and $y = i\, g$. This means

$$\beta(y) = i \left\{ -\frac{(4-D)}{4} g + [\widetilde{K} + \frac{D}{4} K]\, g^3 \right\} \tag{72}$$

$$= i\, \beta(g). \tag{73}$$

The term $\beta(y)\partial/\partial y$ becomes $\beta(g)\,\partial/\partial g$ in the renormalization group equation (55). A function $\widetilde{g}(t)$ replaces $\widetilde{y}(t)$ in all other formulae where

$$\frac{d\widetilde{g}(t)}{dt} = -\beta\,[\widetilde{g}(t)]. \tag{74}$$

Now the important change brought about by this alteration in λ is that $\beta(g)$, shown in Fig. 13, has a zero where $d\beta/dg > 0$. The general analysis tells us that this zero at

$$g_1 = \left[\frac{(4-D)}{4\widetilde{K} + DK}\right]^{\frac{1}{2}} \tag{75}$$

governs the infrared or $E \to 0$ behavior of $\Gamma_R^{(n,m)}(E_i, \ldots)$.

Since the dimension D is at our disposal we see that for D near 4 the zero (a classic Gell-Mann Low zero[7]) in $\beta(g)$ is for small

$\{0[(4-D)^{\frac{1}{2}}]\}$ renormalized dimensionless coupling g. The importance of this observation comes when we look at Eq. (56), the solution to the renormalization group equations. We want to know for $\xi \to 0$, $g(-t)$ as $t = \log \xi \to -\infty$. Now

$$\frac{d\tilde{g}(t)}{dt} = +\left(\frac{4-D}{4}\right)\tilde{g} - \mathscr{K}\tilde{g}^3 , \tag{76}$$

with $\mathscr{K} > 0$. This has the solution

$$\tilde{g}(t) = e^{\left(\frac{4-D}{4}\right)t}\left\{\frac{1}{g^2} + \frac{4\mathscr{K}}{4-D}\left[e^{\frac{4-D}{2}t} - 1\right]\right\}^{-1/2} , \tag{77}$$

using the boundary condition $\tilde{g}(0) = g$, so

$$\lim_{t\to -\infty} \tilde{g}(-t) = g_1 .$$

If $D \approx 4$, or better g_1, is small, then the infrared behavior of $\Gamma_R^{(n,m)}$ is governed by small renormalized coupling and one may hope to determine to excellent accuracy the full $\Gamma^{(n,m)}$ by doing perturbation theory on the right hand side of (56). This whole scheme is entirely self-contained when we recall that we only know the crucial functions β, γ, and ζ for small g anyway.

What is suggested then is an expansion in the parameter $\epsilon = 4-D$ of all Green's functions. Away from points of known non-analyticity in the $\Gamma^{(n,m)}$ this ought to be a meaningful procedure. The justifiably nervous reader will observe now that for D=2, which is where physics lives, $\epsilon = 2$ and in most ways of reasoning that is not a small number.

However, examining the functions γ and ζ as an expansion in ϵ we see that near $g = g_1$

$$\gamma \sim -\epsilon/12\,, \tag{79}$$

and

$$\frac{\zeta}{\alpha'} \sim -\epsilon/24, \tag{80}$$

which is much more to one's taste in expansion parameters.

V. GENERAL CONSEQUENCES OF A ZERO IN $\beta(g)$

In the previous section we discovered, albeit in perturbation theory, that when the bare triple Pomeron coupling was taken to be purely imaginary a zero appeared in the crucial function $\beta(g)$ at $g = g_1$ where $d\beta/dg > 0$. Here we would like to explore more general consequences of such a zero in β which may or may not occur at small $g=g_1$.

First, suppose that the renormalized coupling g chooses to lie exactly at g_1 where $\beta(g_1) = 0$. Then turning to the solution of the renormalization group equations we find

$$\frac{d\tilde{g}(t)}{dt} = 0, \qquad \tilde{g}(t) = g_1, \tag{81}$$

and

$$\frac{1}{\alpha'(t)}\,\frac{d\,\tilde{\alpha}'(t)}{dt} = 1 - \frac{\zeta(\tilde{\alpha}',\tilde{g})}{\tilde{\alpha}'} = z(g_1)\,, \tag{82}$$

so

$$\tilde{\alpha}'(-t) = \alpha'\xi^{-z(g_1)} \tag{83}$$

In perturbation theory $z(g_1) > 1$.

The renormalization group equations tell us that

$$\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, g_1, \alpha', E_N)$$
$$= \Gamma_R^{(n,m)}(E_i, k_i, g_1, \alpha' \xi^{-z(g_1)}, E_N)$$
$$\times \xi^{1-\left(\frac{n+m}{2}\right)\gamma(g_1)} \tag{84}$$
$$= \xi^{1-\left(\frac{n+m}{2}\right)\gamma(g_1) + z(g_1)\frac{D}{4}(2-n-m)} E_N \left[\frac{E_N}{\alpha'}\right]^{\frac{D}{4}(2-n-m)} \times$$
$$\psi_{n,m}\left(\frac{E_i}{E_N}, \xi^{-z(g_1)} \frac{\vec{k}_i \cdot \vec{k}_j \alpha'}{E_N}, g_1\right) , \tag{85}$$

using the dimensional analysis of the last section. This result implies that $\Gamma(\xi E_i, \ldots)$ has a very scaling property

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, g_1, \alpha', E_N) =$$
$$E_N \left[\frac{E_N}{\alpha'}\right]^{\frac{D}{4}(2-n-m)} \left(-\frac{E}{E_N}\right)^{1-\frac{(n+m)}{2}\gamma(g_1)+z(g_1)\frac{D}{4}(2-n-m)}$$
$$\times \phi_{n,m}\left[\frac{E_i}{E}, \left(-\frac{E}{E_N}\right)^{-z(g_1)} \frac{\vec{k}_i \cdot \vec{k}_j}{E_N} \alpha', g_1\right], \tag{86}$$

where $\phi_{n,m}$ is a function not determined by this analysis and $E = \sum_{i=1}^{n} E_n$ is convenient energy to use for the scaling. What is new in Eq. (86) is the fact that $\phi_{n,m}$ depends on $\frac{E}{E_N}$ and $\alpha' \vec{k}_i \cdot \vec{k}_j$ only in the product form indicated.

This scaling result for $\Gamma^{(n,m)}$ can be found in the work of Gribov and Migdal who discuss it for n+m = 2 and 3 at D=2 in their Schwinger-Dyson equation analysis. The indices γ and z are undetermined by them.

At least in principle we know how to proceed to find γ and z here. The Schwinger-Dyson equations do provide a set of terrifically nonlinear equations which yield up to the $\phi_{n,m}$.

There is an immediate consequence of the scaling equation which is of some importance. Consider $\Gamma_R^{(1,1)}$ which takes the form

$$\Gamma_R^{(1,1)}(E, \vec{k}^2, g_1, \alpha', E_N) = E_N \left[\frac{-E}{E_N}\right]^{1-\gamma(g_1)} \times \phi_{1,1}\left[\left(\frac{E}{E_N}\right)^{-z(g_1)} \frac{\vec{k}^2 \alpha'}{E_N}, g_1\right] . \qquad (87)$$

If $\Gamma_R^{(1,1)}$ has a zero which moves with $\vec{k}^2$, then that trajectory must have the structure

$$E(\vec{k}^2, g_1) = -E_N \left(\frac{\vec{k}^2 \alpha'}{E_N}\right)^{1/z(g_1)} f(g_1) , \qquad (88)$$

where $f(g_1)$ is just some function of g_1. Clearly the trajectory function $1 - E = \alpha$

$$\alpha(\vec{k}^2) = 1 + E_N \left(\frac{\vec{k}^2 \alpha'}{E_N}\right)^{1/z(g_1)} f(g_1) \qquad (89)$$

is not analytic at $\vec{k}^2 = 0$ in general. Our perturbation theory analysis indicates that $z(g_1) > 1$ so that the slope of $\alpha(\vec{k}^2)$ at $\vec{k}^2 = 0$ is infinite. The situation described here is what the renormalization program tells us to expect on quite general grounds. It means that the "weak coupling" Pomeranchukon favored by Gribov is suspect. On the other hand, we shall later see that the renormalization group also suggests a Pomeranchukon which is somewhat different from the Gribov-Migdal "strong coupling" Pomeranchukon.

We have assumed thus far in this section that the renormalized coupling g chose to sit precisely at a zero of $\beta(g)$. Now we relax this and imagine that $\beta(g)$ has a zero at g_1 with positive slope and that g lies either above or below g_1 but between g_1 and the next zero of $\beta(g)$. We approximate $\beta(g)$ as

$$\beta(g) = \beta_0(g - g_1), \quad \beta_0 > 0. \tag{90}$$

We may solve for g(t) now

$$\frac{d\tilde{g}(t)}{dt} = \beta_0 \; [\tilde{g}(t) - g_1] \tag{91}$$

to find

$$\tilde{g}(t) = g_1 + \xi^{-\beta_0} (g - g_1) . \tag{92}$$

Similarly we may solve for $\alpha'(t)$ for $t \to \infty$

$$\tilde{\alpha}'(t) = \alpha' C_\alpha \exp \; [z(g_1)t + 0(e^{-\beta_0 t})] \;, \tag{93}$$

where

$$C_\alpha = \exp \sum_{n=1}^{\infty} z_n \, (g-g_1)^n / \beta_0 n, \tag{94}$$

and we have written

$$z(g) = z(g_1) + \sum_{n=1}^{\infty} z_n (g-g_1)^n. \tag{95}$$

For the term in the expression for $\Gamma_R^{(n,m)}(\xi E_i, \ldots)$ reading $\exp \int_{-t}^{0} dt' \left\{ 1 - \frac{n+m}{2} \gamma\, [g(t')] \right\}$ we find for $t \to -\infty$, that is $\xi \to 0$

$$C_\gamma \exp \left\{ t[1 - (\frac{n+m}{2})\gamma(g_1)] + 0(\xi^{\beta_0}) \right\} \quad , \tag{96}$$

where

$$C_\gamma = \exp\left(\frac{n+m}{2}\right) \sum_{n=1}^{\infty} \gamma_n (g-g_1)^n / \beta_0 n, \tag{97}$$

and the γ_n are the coefficients of $\gamma(g)$ in an expansion about g_1

$$\gamma(g) = \gamma(g_1) + \sum_{n=1}^{\infty} \gamma_n (g-g_1)^n . \tag{98}$$

Using these results in the dimensional analysis above we find that for small E_i and fixed $\vec{k}_i$

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, g, \alpha', E_N) =$$

$$C_\gamma E_N \left[\frac{E_N}{C_\alpha \alpha'}\right]^{(2-n-m)D/4} \left(-\frac{E}{E_N}\right)^{1+z(g_1)\frac{D}{4}(2-n-m)-\left(\frac{n+m}{2}\right)\gamma(g_1)}$$

$$\times\ \phi_{n,m}\left[\frac{E_i}{E}, \left(-\frac{E}{E_N}\right)^{-z(g_1)} \frac{\vec{k}_i \cdot \vec{k}_j}{E_N} C_\alpha \alpha', g_1\right] . \tag{99}$$

In other words the scaling result is essentially the same with two dimensionless functions of g and g_1, C_α and C_γ, which rescale α' and $\phi_{n,m}$ respectively. We have also derived the renormalization group differential equations when the $\vec{k}_i$ are scaled to zero with the E_i fixed, and when both are scaled together. We find that Eq. (99) continues to hold in both these limits, and is therefore true when either the E_i or $\vec{k}_i$ (or both) are small.

From our scaling formulae we can make an interesting observation on the renoramalized triple Pomeranchukon vertex $\Gamma_R^{(1,2)}$. Suppose $\Gamma_R^{(1,1)}$ yields up a trajectory $E \propto (\vec{k}^2)^{1/z}$. Set the E_i in $\Gamma_R^{(1,2)}$ on these trajectories and then let $\vec{k}_i \to \eta^{\frac{1}{2}} \vec{k}_i$ and consider the limit as $\eta \to 0$. That is consider the limit of $\Gamma_R^{(1,2)}$ as the $E_i \to 0$ and the $\vec{k}_i$ go to zero staying on the Regge trajectories. Then $\Gamma_R^{(1,2)}$ behaves as $[\eta^{1/z}]^{1-(D/4)z-3/2\gamma}$ which, in perturbation theory, vanishes as $\eta \to 0$. So the renormalized triple Pomeranchukon vertex vanishes in general as a non-integer power of its arguments. All discussions of the triple Pomeranchukon vertex which have an analytic vanishing as the arguments go to zero would seem to require re-examination.

Our general scaling results permit us to calculate the scaling function $\phi_{n,m}$ as power series in ϵ. This will complete the program discussed in Sec. IV and lead to some important points. As an example, we consider $\phi_{1,1}(\rho, \epsilon)$, where

$$\rho = \left(-\frac{E}{E_N}\right)^{-z(\epsilon)} \frac{\vec{k}_i \cdot \vec{k}_i}{E_N} C_\alpha \alpha', \tag{100}$$

and we now regard the dependence on g_1 as a dependence on ϵ. We suppose $\phi_{1,1}$ can be expanded as a power series in its second argument.

$$\phi_{1,1}(\rho, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \phi_{1,1}^{(n)}(\rho) \tag{101}$$

Using this we write the right side of Eq. (87) as a power series in ϵ. The first two terms are

$$\Gamma_R^{(1,1)}[E, \vec{k}^2, g_1(\epsilon), \alpha', E_N] = -E\left(-\frac{E}{E_N}\right)^{-\gamma(0)}$$

$$\times \left\{ \phi_{1,1}^{(0)}(\rho_0) + \epsilon \left[\phi_{1,1}^{(1)}(\rho_0) - \gamma'(0) \ln\left(-\frac{E}{E_N}\right) \phi_{1,1}^{(0)}(\rho_0) \right.\right.$$

$$- \rho_0 z'(0) \ell n\left(- \frac{E}{E_N}\right) \frac{d}{d\rho_0} \phi^{(0)}_{1,1}(\rho_0)]\Bigg\} \quad , \tag{102}$$

where $\rho_0 = \left(- \frac{E}{E_N}\right)^{-z(0)} \frac{\alpha' \vec{k}^2}{E_N}$. We use $\gamma(\epsilon) = - \frac{\epsilon}{12}$, $z(\epsilon) = 1 + \frac{\epsilon}{24}$, and the renormalized second order inverse propagator

$$i \Gamma^{(1,1)}_R (E, \vec{k}^2, g_1(\epsilon), \alpha', E_N) = E - \alpha' \vec{k}^2$$

$$- \frac{\epsilon}{12} [\frac{\alpha' \vec{k}^2}{2} - E] [\ell n(\frac{\alpha' \vec{k}^2 - 2E}{2E_N}) - 1] + 0(\epsilon^2) \tag{103}$$

to find

$$i\phi^{(0)}_{1,1} = - 1 - \rho, \tag{104}$$

$$i\phi^{(1)}_{1,1}(\rho) = - \frac{1}{12} [1 + \frac{1}{2} \rho] [\ell n(1 + \frac{1}{2} \rho) - 1] \ . \tag{105}$$

These results strongly suggest the presence of a Pomeranchukon pole near

$$\rho = -1 + \frac{\epsilon}{24} (1 + \ell n 2) , \tag{106}$$

$$\alpha'(t) = 1 + E_N \left[\frac{C_\alpha \alpha' t}{1 - \frac{\epsilon}{24} (1 + \ell n 2)} \right]^{\frac{1}{1 + \epsilon/24}} \tag{107}$$

When t is negative, there are poles at both $\alpha(t)$ and its complex conjugate.

Two final points are worth noting. When $\vec{k}^2 = 0$, we find

$$i \Gamma^{(1,1)}_R (E, 0, g, \alpha', E_N) = (1 - \frac{\epsilon}{12}) E (- \frac{E}{E_N})^{\epsilon/12} \tag{108}$$

In the next section we shall see that $\Gamma^{(1,1)}_R$ makes the leading contribution to the total cross section. From Eq. (108) it follows that this contribution is positive, so the renormalization group avoids the problems which lead to a negative cross section in the Gribov-Migdal strong coupling theory.

The second point is that $\phi_{1,1}(\rho, \epsilon)$ should behave like const. $\times$ $(\rho)^{[1-\gamma(\epsilon)]/z(\epsilon)}$ for large ρ so that $\Gamma_R^{(1,1)}$ is finite for general $\vec{k}^2$ and $E=0$, as it is in perturbation theory. This means that $\phi_{1,1}^{(n)}(\rho)$ has the leading behavior for large ρ

$$\phi_{1,1}^{(n)}(\rho) = \frac{-\rho(\ell n\, \rho)^n}{n!\ \ (24)^n} \tag{109}$$

This behavior is verified for $\phi_{1,1}^{(0)}$ and $\phi_{1,1}^{(1)}$. We observe that the natural expansion parameter in Eq. (101) is $(\epsilon\ \ell n\ \rho)/24$. Thus, while Eq. (99) is valid for finite $\vec{k}^2$ when E is small, any finite number of terms in Eq. (101) is not useful in this limit. This is not surprising since we have only had to calculate the two Pomeranchukon cut to obtain the first two terms.

VI. USE OF THE SCALING RELATIONS ON REGGEON GREEN'S FUNCTIONS

We would like to apply our scaling formulae on the Reggeon Green's functions to study the asymptotic behavior of total cross sections and some properties of elastic cross sections. We proceed by assuming that there are some given particle Reggeon couplings N_j which take two particles into j Reggeons as in Fig. 15. Further, we assume N_j is just a constant independent of $E = \sum_{k=1}^{j} E_j$.

With this we can write the contribution to the partial wave amplitude $F(E,\vec{q})$ coming from n Pomeranchukons being emitted from the left vertex by N_n interacting in all possible ways via $G^{(n,m)}$ and producing m Pomeranchukons to be reabsorbed on the right by N_m; see Fig. 16. The analytic expression for this is

$$I_{n,m}(E,\vec{q}) = N_n^R N_m^R \int d^Dk_1 \ldots d^Dk_{n+m}\, dE_1 \ldots dE_{n+m}$$

$$\delta(E_1 + \ldots + E_n - E)\ \delta^D(\vec{k}_1 + \ldots + \vec{k}_n - \vec{q})\ \delta(E_{n+1} + \ldots + E_{n+m} - E)$$

$$\delta^D(\vec{k}_{n+1} + \ldots + \vec{k}_{n+m} - \vec{q})\ G_R^{(n,m)}(E_1, \vec{k}_1, \ldots E_{n+m}, \vec{k}_{n+m})\ . \tag{110}$$

Now using the scaling properties of $\Gamma_R^{(n,m)}$ derived above we find the integral $I_{n,m}$ can be scaled to yield

$$I_{n,m}(E,\vec{q}) = E^{-1+\gamma(g_1)}\ E^{(n+m-2)\left[\frac{\gamma(g_1)}{2} + \frac{D}{4} z(g_1)\right]}$$

$$\times F_{n,m}(|\vec{q}|^2 / E^{z(g_1)})\ . \tag{111}$$

This result involves no approximation when $g = g_1$, the zero of $\beta(g)$. When $g \neq g_1$, we proceed somewhat differently. The function $I_{n,m}$ can also be written as an integral over unrenormalized Green s functions $G^{(n,m)}$, with external unrenormalized vertices $N_n = N_n^R (Z)^{n/2}$. The equality of these expressions allows us to repeat the steps of Sec. IV: a differential equation analogous to Eq. (55) can be derived, with its solution again yielding Eq. (111) for either E or $\vec{q}^{\,2}$ small. The reason this rederivation of Eq. (111) is more than an academic exercise is that the energies and momenta in the integral are not necessarily small when E or $\vec{q}$ are. Finally, we note that disconnected parts of $G^{(n,m)}$ give rise to no problems in the integral.

From the Sommerfeld-Watson transform we learn that the elastic amplitude given via contributions like $I_{n,m}$ is

$$T_{e\ell}(s,t) = s(\log s)^{-\gamma(g_1)} \sum_{n,m=1}^{\infty} (\log s)^{-(n+m-2)[\frac{\gamma(g_1)}{2} + \frac{D}{4} z(g_1)]}$$

$$\times \tilde{F}_{n,m} [t(\log s)^{z(g_1)}] \quad . \tag{112}$$

In perturbation theory for D=2 we learned above that

$$\gamma(g_1) \approx -1/6 \ , \tag{113}$$

and

$$z(g_1) \approx 1 + 1/12 = 13/12 \ . \tag{114}$$

To a good approximation then we may write for large s, fixed t

$$T_{e\ell}(s,t) \sim s(\log s)^{1/6} \Big\{ \tilde{F}_{1,1}[t(\log s)^{13/12}]$$

$$+ (\log s)^{-1/2} \tilde{F}_{1,2}[t(\log s)^{13/12}] + 0[(\log s)^{-1}] \Big\} \quad . \tag{115}$$

From this expression we have an approximate expansion of $\sigma_T(s)$

$$\sigma_T(s) \sim (\log s)^{1/6} [A + B/(\log s)^{1/2} + \ldots] \ , \tag{116}$$

in which, if we take the simple model we have made quite seriously, <u>the leading term,</u> A, <u>factorizes</u> (See Fig. 17).

VII. DISCUSSION

In this paper we have considered in detail the implications of the renormalization group for the most simple physically interesting interacting Reggeon field theory. We chose a Lagrangian

$$\mathscr{L}(\vec{x},t) = \frac{i}{2}\,\psi^{+}(\vec{x},t)\overleftrightarrow{\partial_t}\,\psi(\vec{x},t) - \alpha_0'\nabla\psi^{+}\cdot\nabla\psi - \frac{\lambda_0}{2}\,[\,\psi^{+}(\vec{x},t)\,\psi(\vec{x},t)^2 + \text{h.c.}\,]\,, \tag{117}$$

which represents a bare quasi-particle with an energy momentum relation $E = \alpha_0'\vec{k}^2$ interacting with a triple coupling. Our major observations about this theory are (1) when $\lambda_0 = i\,r_0$, r_0 real, as suggested by Gribov's[1] treatment of signature, then the renormalization group equations for the renormalized vertex functions $\Gamma_R(E_i, \vec{k}_i)$ have an infrared stable zero of order $\sqrt{4-D}$, where D is the number of space degrees of freedom. When $D \approx 4$, this suggests a perturbation theory (akin to the ϵ-expression[14] of statistical mechanics; indeed, suggested by it) around the four dimensional theory. (2) When such an infrared stable zero (Gell-Mann Low zero[7]) is present, the Reggeon Green's functions obey the scaling properties summarized in Eq. (99). (3) In a model of the couplings to particles using the values of the renormalization group functions found in perturbation theory we find, for example that

$$\sigma_T(s) \sim A\,(\log s)^{1/6}\left\{1 + 0[\log s)^{-1/2}]\right\} \tag{118}$$

where A factorizes. (4) We have pointed out the radical difference between the renormalization group results and the "weak coupling" Pomeranchukon of Gribov. Typical instances of this difference are the cusp with infinite slope in the Pomeranchukon pole trajectory at $t=0$, and the fractional power vanishing of the triple Pomeranchukon vertex function. On the other hand, we also disagree sharply with the Gribov-Migdal strong coupling solution. At $\vec{k}^2=0$, we have $\Gamma_R^{(1,1)}$ vanishing faster than linearly in E, whereas Gribov and Migdal have $\Gamma_R^{(1,1)}$ vanishing less rapidly than linearly. We might point out that a strong coupling solution also seems to have difficulties when the triple Regge coupling is real. Figure (13) indicates there is no Gell-Mann Low zero near the origin for $D<4$; and for $D>4$ the theory is nonrenormalizable. Thus any zero, if it exists, cannot be calculated by searching for a scale invariant dimensionality.

The fact that Green's functions like the inverse propagator vanish more rapidly than linearly in E or $\vec{k}^2$ raises a subtle question. Gribov has stressed that the high energy and momentum parts of Reggeon graphs are really arbitrary because the bare vertices and trajectories have dependence on energy and frequency which has been suppressed here. If we arbitrarily modify the high energy and momentum tails of graphs, it would seem that linear terms in E and $\vec{k}^2$ would appear in Green's functions like $\Gamma_R^{(1,1)}$. This would imply that the results we have found depend on a special treatment of the high momentum and energy parts of graphs, and are unstable against small modifications of graphs. It is difficult to characterize such modifications systematically, and to treat them by the renormalization group because the vertices and propagators have new energy and momentum scales in them. However, we have checked the effect of keeping a finite cutoff Λ in the graphs, and find the behavior cited in Eq. (99) still holds. Perhaps, then, the additive argument we have given is invalid because the renormalization group is multiplicative. Further studies are planned.

Any number of future investigations are suggested by the analysis we have carried out. The most straightforward set of investigations would include: (a) alter the $E, \vec{k}^2$ relationship of the "bare" or non-interacting theory, (b) change the nature of the interaction [the ψ^4 theory corresponding to the ψ^3 theory in this paper has been studied by the authors; the results will be presented elsewhere], (c) try to bootstrap the renormalization group functions γ and ζ, which govern the structure of $T_{elastic}(s,t)$, by studying the Schwinger-Dyson equations of the field theories. Clearly the interesting possibilities are legion.

At our present stage of understanding of the Reggeon field theories it would perhaps be hasty to point directly at the most physically significant possibilities. One can argue with some confidence, both on the basis of the present work and that in Ref. 2, that the major alterations of the conclusions established here will be in details, no doubt interesting, involving the renormalization group functions.

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[11] A word to the renormalization group connoisseur: because time in the present non-relativistic analogy has dimensions akin to $(\text{space})^2$, D=4 here corresponds to six dimensional space-time in a relativistic theory. It is known that a ϕ^3 has a dimensionless coupling in six dimensions; see, G. Mack, in Lecture Notes in Physics: Strong Interaction Physics, ed. W. Rühl and A. Vancura (Springer Verlag, Berlin, 1973), p. 300.

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FIGURE CAPTIONS

Fig. 1 A representation of the discontinuity across the two Reggeon cut in the particle partial wave amplitude $F(\ell, t)$ arising from the t-channel exchange of Reggeons α_1 and α_2. N_2 is the Reggeon-particle partial wave amplitude. The x's on the wavy Reggeon lines indicate that a discontinuity has been taken.

Fig. 2 The expression of Fig. 1 with kinematics expressed in terms of two dimensional vectors $\vec{p}_i \cdot t = -|\vec{q}|^2$ in this picture.

Fig. 3 A picture of the discontinuity across the n Reggeon cut contribution to $F(\ell, t)$.

Fig. 4 The Reggeon-Reggeon four point amplitude.

Fig. 5 The two Reggeon discontinuity in $E = 1-\ell$ of the four Reggeon amplitude. Since $E \neq E_1 + E_2$ in this expression, it is an off-shell unitarity relation.

Fig. 6 A single particle inclusive cross section as $s \to \infty$, t, M^2 fixed. One encounters the "off-shell" N_2 here.

Fig. 7 The lowest order Feynman graph correction to the Reggeon propagator $G^{(1,1)}$.

Fig. 8 A possible Feynman graph contribution to the Reggeon propagator $G^{(1,1)}$. Because of the $i\epsilon$ prescription which gives only retarded propagation, this graph vanishes.

Fig. 9 The trajectories of poles and perturbation theory branch points in the $E, \vec{k}^2$ plane. The normalization point $E = -E_N, \vec{k}^2 = 0$ is chosen out of harm's way.

Fig. 10 The renormalized triple Reggeon vertex function.

Fig. 11. The lowest order perturbation theory graphs evaluated to determine Z and α'.

Fig. 12 The perturbation theory graphs needed to determine the renormalized coupling $\lambda(E_N)$ and the function β.

Fig. 13 The crucial function $\beta(y)$ computed from the graphs in Fig. 12 using a real coupling constant.

Fig. 14 The function $\beta(g)$ when the bare triple Pomeron coupling is chosen to be pure imaginary. The zero at g_1 is proportional to $(4-D)^{1/2}$.

Fig. 15 A j Reggeon-two particle vertex used in the model for coupling particles into the Pomeron interactions.

Fig. 16 The n+m Reggeon contribution to the particle partial wave amplitude. The center is the renormalized (n, m) Reggeon Green's function.

Fig. 17 The leading behavior of $T_{e\ell}(s,t)$ and $\sigma_T(s)$ comes from this contribution in the simple model discussed in the text. This yields a factorized contribution to $\sigma_T(s)$ which behaves as $(\log s)^{1/6}$.

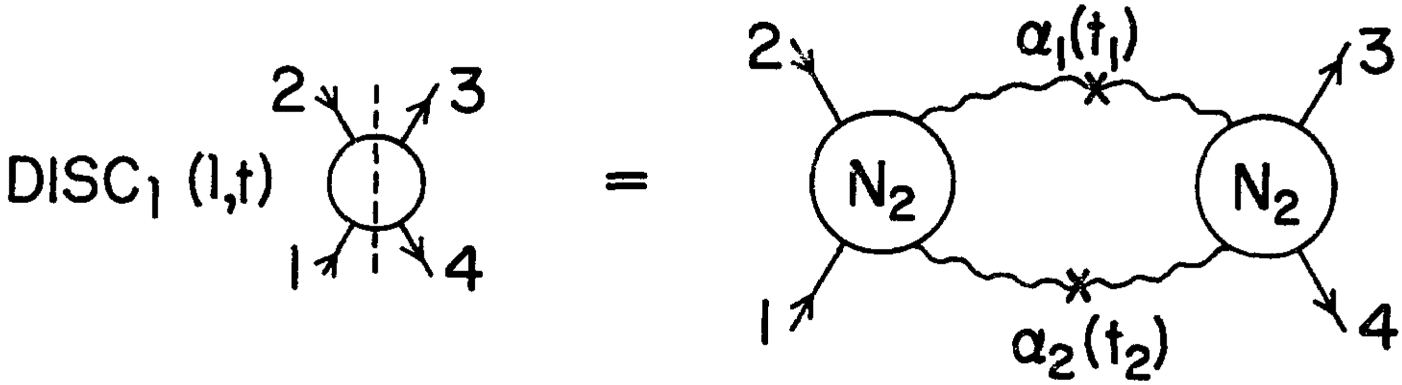


FIG. 1

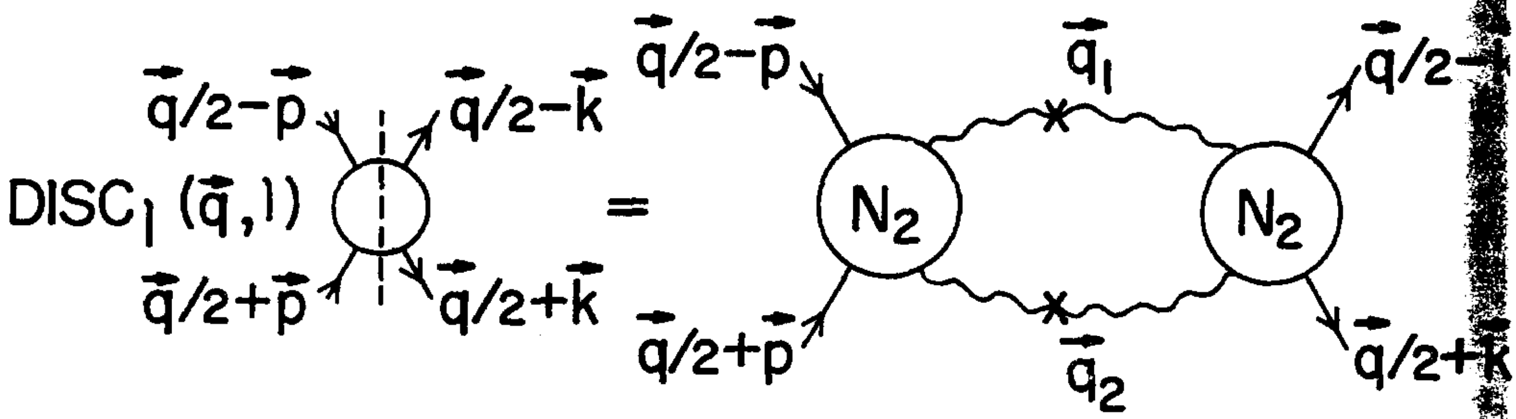


FIG. 2

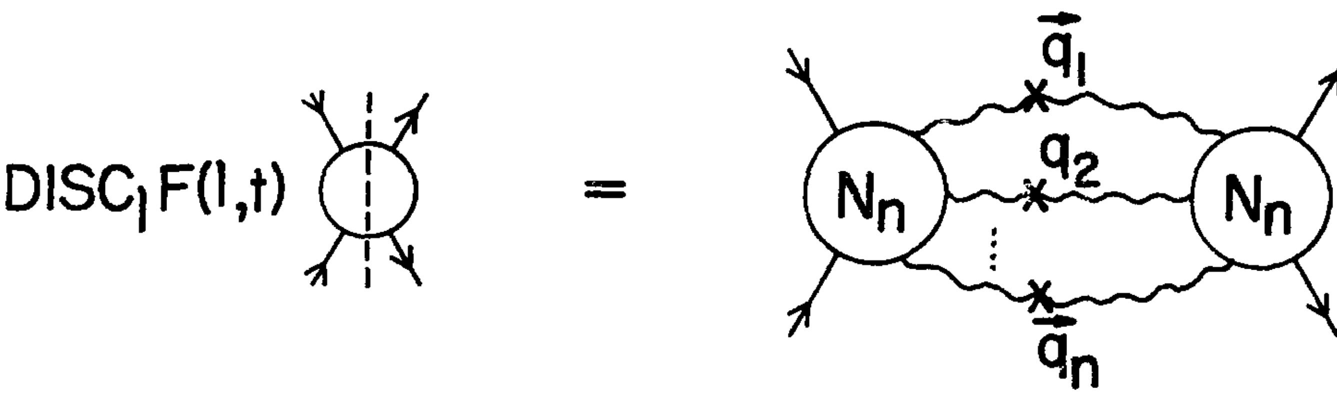


FIG. 3

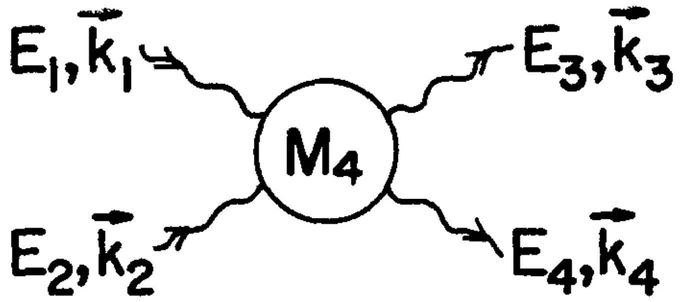
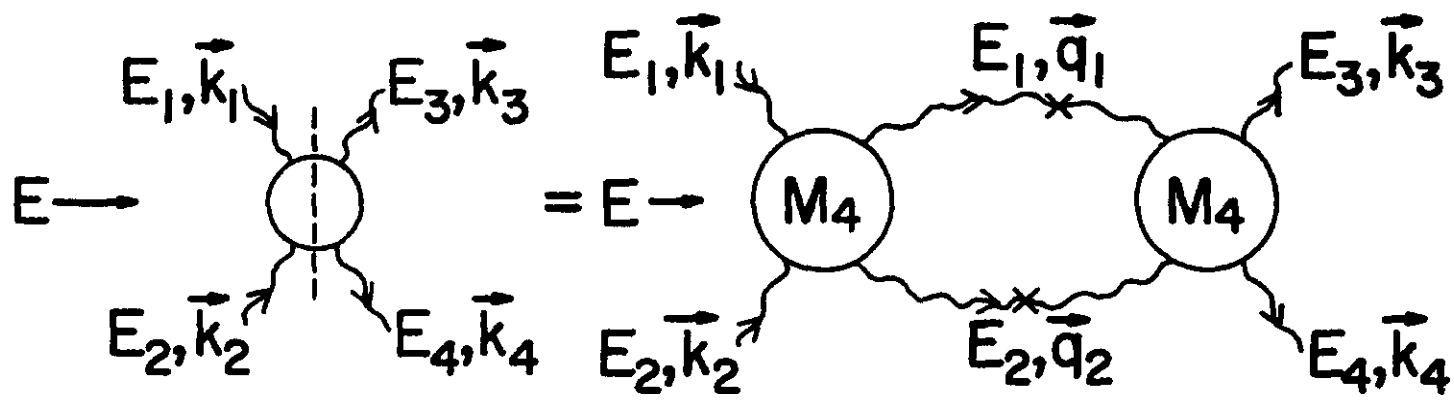


FIG. 4



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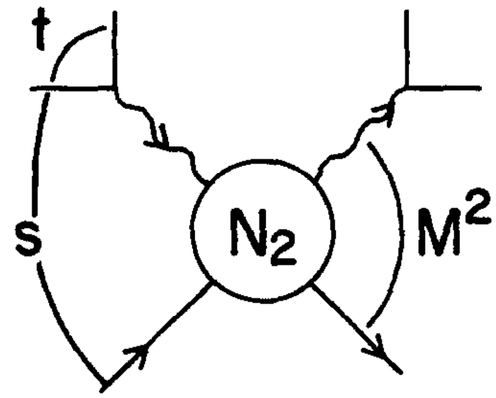

FIG. 6

FIG. 7

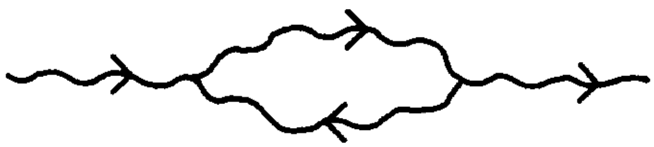

FIG. 8

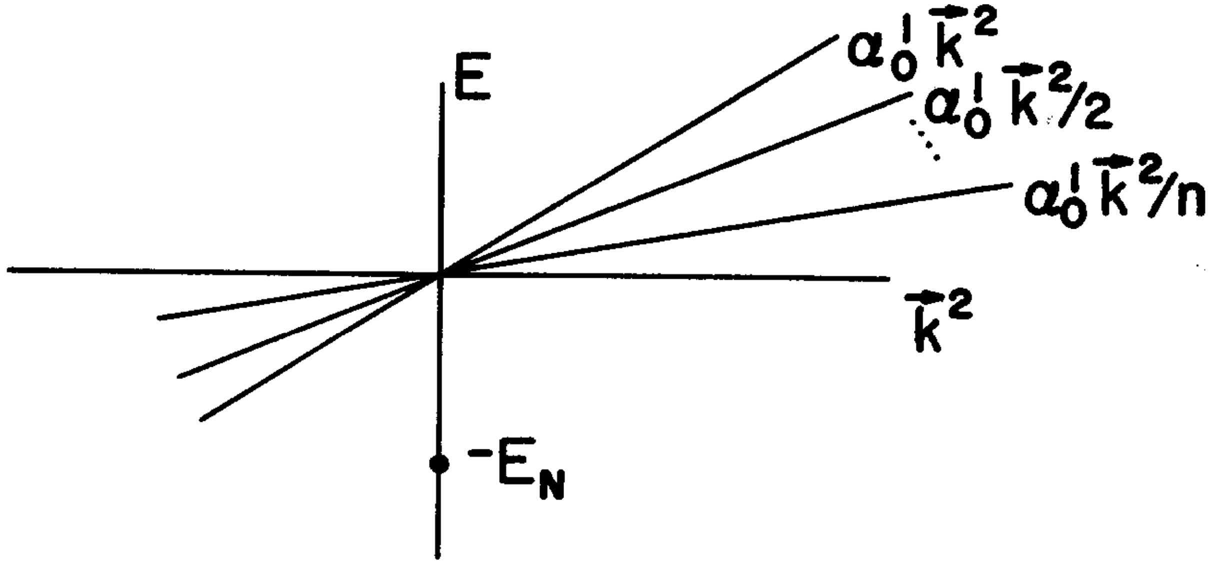

FIG. 9

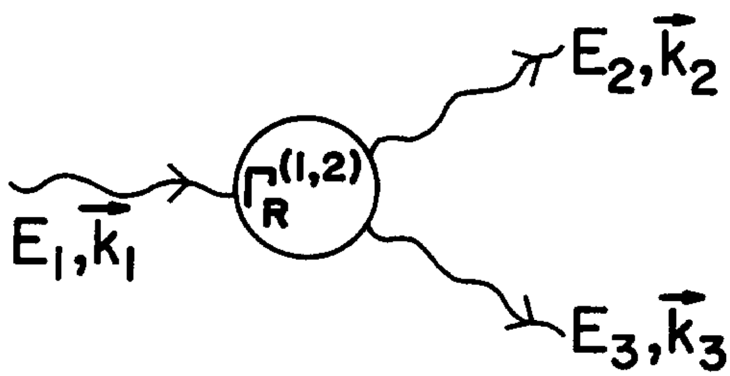

FIG. 10

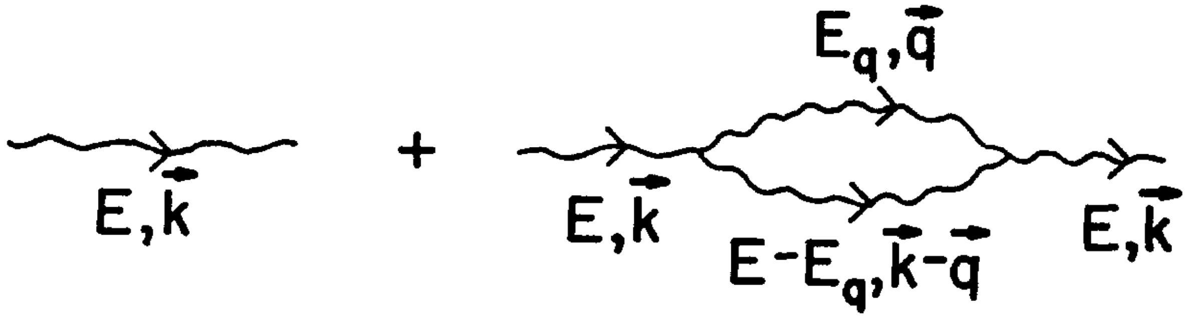

FIG. 11

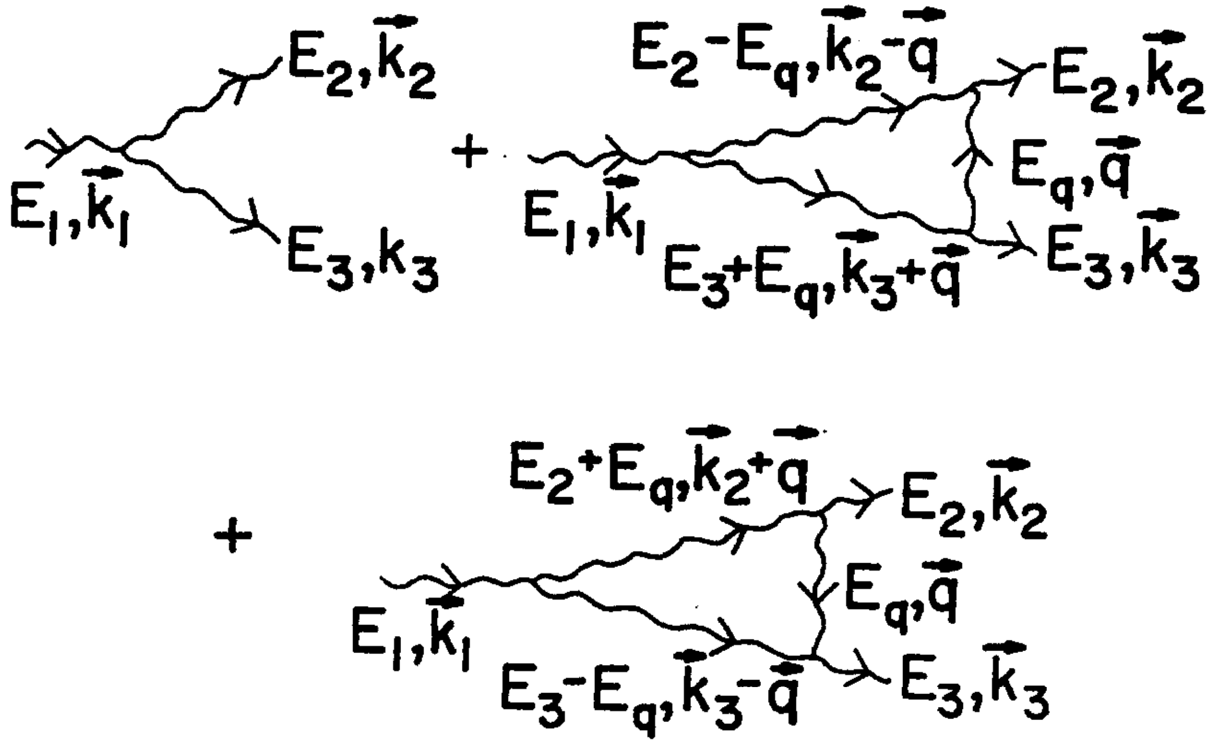

FIG. 12

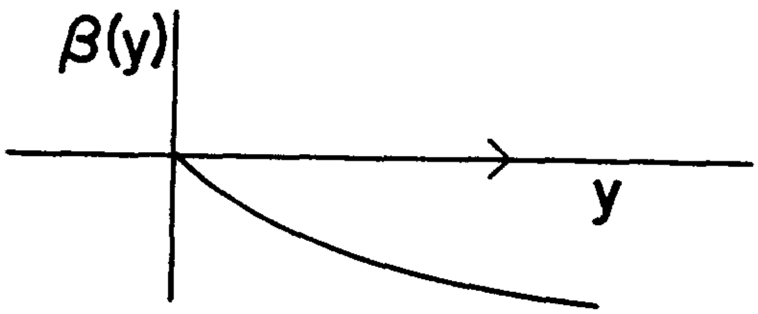

FIG. 13

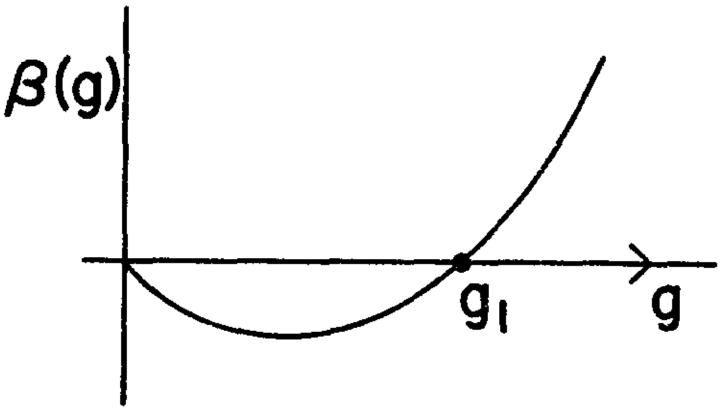

FIG. 14

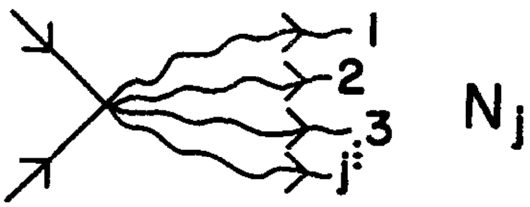

FIG. 15

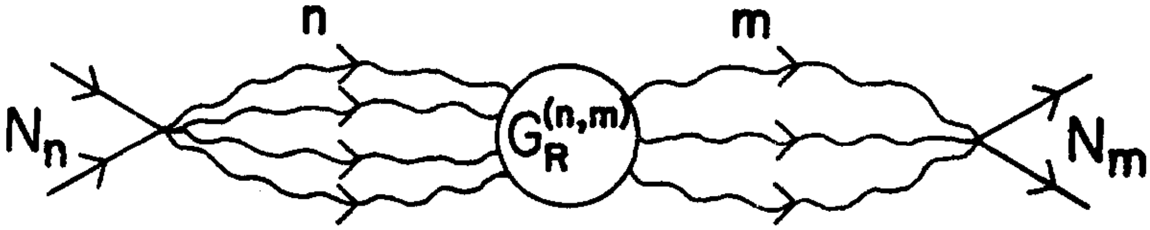

FIG. 16

FIG. 17