NAL-Pub-73/86-THY

Nuclear Charge Exchange Corrections to Leptonic Pion Production in the (3, 3)-Resonance Region

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ABSTRACT

We discuss nuclear charge exchange corrections to leptonic pion production in the region of the (3, 3) resonance, both from a phenomenological viewpoint and from the evaluation of a detailed model for pion multiple scattering in the target nucleus. Using our analysis, we estimate the nuclear corrections needed to extract the ratio

$$R = [\sigma(\nu_\mu + n \rightarrow \nu_\mu + n + \pi^0) + \sigma(\nu_\mu + p \rightarrow \nu_\mu + p + \pi^0)] / 2\sigma(\nu_\mu + n \rightarrow \mu^- + p + \pi^0)$$

from neutral current search experiments using ${}_{13}Al^{27}$ and other nuclei as targets.

*Operated by Universities Research Association, Inc., under contract with the U. S. Atomic Energy Commission.

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1. INTRODUCTION

Although weak interaction experiments on hydrogen and deuterium targets are most readily interpreted theoretically, experimental considerations necessitate the use of complex nuclear targets in many of the current generation of accelerator neutrino experiments. As a result, in such experiments, corrections for nuclear effects must be made in order to extract free nucleon cross sections from the experimental data. Our aim in the present paper is to analyze these corrections in a particularly simple case: that of leptonic single pion production in the region of the (3, 3) resonance. This reaction has gained prominence recently because measurement of the ratio

$$R = \frac{\sigma(\nu_\mu + n \rightarrow \nu_\mu + n + \pi^0) + \sigma(\nu_\mu + p \rightarrow \nu_\mu + p + \pi^0)}{2\sigma(\nu_\mu + n \rightarrow \mu^- + p + \pi^0)} \qquad (1)$$

appears to be one of the better ways of searching for hadronic weak neutral currents.[1] Experiments measuring R use aluminum (and in some cases also carbon) as target materials, so the experimentally measured quantity is actually (T', T'' denote unobserved final target states)

$$R'(T) = \frac{\sigma(\nu_\mu + T \rightarrow \nu_\mu + T' + \pi^0)}{2\sigma(\nu_\mu + T \rightarrow \mu^- + T'' + \pi^0)}, \qquad T = {}_{13}Al^{27},\ {}_6C^{12}. \quad (2)$$

As Perkins has emphasized,[2] nuclear charge exchange effects can cause substantial differences between R and R', which makes reliable estimation of these effects an important ingredient in correctly interpreting the experiments. Fortunately, pion production in the (3, 3) region is also a particularly favorable case for theoretical analysis, primarily because the elasticity of the (3, 3) resonance implies that nuclear effects will not bring multipion or other more complex hadronic channels into play.

Our discussion is organized as follows. In Section 2 we introduce our basic phenomenological assumption, that leptonic pion production on a nuclear target may be represented as a two-step compound process, in which pions are first produced from constituent nucleons with the free lepton-nucleon cross section (apart from a Pauli principle reduction factor), and subsequently undergo a nuclear interaction independent of the identity of the leptons involved in the production step. This assumption allows us to isolate nuclear effects (pion charge exchange and absorption) in a 3×3 "charge exchange matrix." We analyze the structure of the charge exchange matrix in the particularly simple case of an isotopically neutral target nucleus. [For ${}_{13}Al^{27}$, with a neutron excess of 1 and a corresponding isospin of 1/2, the approximation of isotopic neutrality should be quite adequate. For ${}_{6}C^{12}$, of course, no approximation is involved.] Our main phenomenological result is that parameters of the charge exchange matrix, which can be measured in high rate pion electroproduction experiments, can be used to calculate the nuclear corrections to the weak pion production process, and in particular to give the connection between R' and R. In Section 3 we develop a detailed multiple scattering model for the charge exchange matrix. Our model is quite similar to a successful calculation by Sternheim and Silbar[3] of pion production in the (3, 3) resonance region induced by protons incident on nuclear targets, and uses the nuclear pion absorption cross section which they determine (as well as the experimental pion-nucleon charge exchange cross section) as inputs. The principal difference between our calculation and that of Sternheim and Silbar (apart from obvious changes stemming from the fact that they deal with a strongly absorbed, rather than a weakly interacting projectile) is that we use an improved approximation to the multiple scattering problem, based on a one-dimensional scattering solution introduced by Fermi in the early days of neutron physics.[4] Using our model for the charge exchange matrix, and a theoretical calculation of free nucleon electro- and weak-pion production which has been described elsewhere,[5] we present detailed predictions for R' in the Weinberg weak interaction theory and some of its variants. We also use the production model to test averaging approximations implicit in the phenomenological discussion of Section 2. In Section 4 we summarize briefly our conclusions. Three appendices are devoted to mathematical details. In Appendix A we formulate and solve the one-dimensional scattering problem which forms the basis for the approximate solution of the three-dimensional multiple scattering problem actually encountered in our model. To calibrate this approximation, in Appendix B we compare the approximate solution with the exact multiple scattering solution for the simple case of isotropic scattering centers uniformly distributed within a sphere. Finally, in Appendix C we collect miscellaneous formulas for cross sections and for Pauli exclusion factors which are needed in the text.

2. PHENOMENOLOGY

A. Kinematics[6]

We consider the leptonic pion production reaction

$$\ell(k_1) + T \rightarrow \ell'(k_2) + T' + \pi^{(\pm 0)}, \tag{3}$$

with k_1 and k_2 respectively the four-momenta of the initial and final lepton ℓ and ℓ', with T a nuclear target initially at rest in the laboratory, and with T' an unobserved final nuclear state. Let $k^2 = (k_1 - k_2)^2$ be the leptonic invariant four-momentum transfer squared, and let $k_0^L \equiv k_{10}^L - k_{20}^L$ be the laboratory leptonic energy transfer to the hadrons. Corresponding to the three pionic charge states in Eq. (3) we have three doubly differential cross sections with respect to the variables k^2 and k_0^L, which we denote by

$$\frac{d^2\sigma(\ell\ell' T; \pm 0)}{d\,k^2\, d\,k_0^L} . \tag{4}$$

When the target T is a single nucleon N (of mass M_N), below the two-pion production threshold the recoil target T' must also be a single nucleon, and we can specify the kinematics more precisely. We write in this case

$$\ell(k_1) + N(p_1) \rightarrow \ell'(k_2) + N'(p_2) + \pi^{(\pm 0)}(q), \tag{5}$$

with the hadron four-momenta indicated in parentheses. We denote the final pion-nucleon isobar mass by W,

$$W^2 = (p_2 + q)^2 . \tag{6a}$$

This variable is evidently related to the leptonic energy transfer k_0^L by

$$W^2 = M_N^2 - |k^2| + 2M_N k_0^L . \tag{6b}$$

B. Factorization Assumption

We now introduce a factorization assumption which is basic to all of our subsequent arguments. We assume that leptonic pion production on a nuclear target may be regarded as a two step compound process. In the first step of this process pions are produced from constituent nucleons of the target nucleus with the free lepton-nucleon cross section. In the second step the produced pions undergo a nuclear interaction, dependent on properties of the target nucleus and on the kinematic variables k_0^L and (possibly) k^2, but independent of the identities of the leptons involved in the first step. Since we are considering only excitation energies below the two-pion production threshold, the nuclear interaction in the second step involves only two types of processes, (i) scattering of the pion, and (ii) absorption of the pion in two-nucleon or more complex nuclear processes. In particular, the two pion production channel cannot come into play, and hence the factorization assumption allows us to write a simple matrix relation between the cross sections for leptonic pion production on nuclear and on free nucleon targets. We have

$$\begin{pmatrix} \dfrac{d^2\sigma(\ell\ell'\,{}_ZT^A;+)}{d\,k^2\,d\,k_0^L} \\[2ex] \dfrac{d^2\sigma(\ell\ell'\,{}_ZT^A;0)}{d^2\,k^2\,\,dk_0^L} \\[2ex] \dfrac{d^2\sigma(\ell\ell'\,{}_ZT^A;-)}{dk^2\,dk_0^L} \end{pmatrix} = [\,M({}_ZT^A;k^2k_0^L)] \begin{pmatrix} \dfrac{d^2\sigma(\ell\ell'N_T;+)_F}{d\,k^2\,d\,k_0^L} \\[2ex] \dfrac{d^2\sigma(\ell\ell'N_T;0)_F}{dk^2\,dk_0^L} \\[2ex] \dfrac{d^2\sigma(\ell\ell'N_T;-)_F}{dk^2\,dk_0^L} \end{pmatrix}, \qquad (7)$$

with

$$\frac{d^2\sigma\,(\ell\ell'\,N_T;\pm 0)_F}{dk^2\,dk_0^L} = Z\,\frac{d^2\sigma\,(\ell\ell'\,p;\pm 0)_F}{dk^2\,dk_0^L} + (A-Z)\,\frac{d^2\sigma\,(\ell\ell'\,n;\pm 0)_F}{dk^2\,dk_0^L} \tag{8}$$

an appropriately weighted linear combination of free proton and free neutron cross sections. The subscript "F" indicates that these cross sections are to be averaged over the Fermi motion of the individual target nucleon in the nucleus, which substantially alters the shape (but not the integrated area) of the (3, 3) resonance[7] when plotted versus the excitation energy k_0^L. In writing Eq. (7) we have obviously used rotational symmetry, which implies that when the angular variables of the pion emerging from the nucleus are integrated over, no dependence remains on the angles characterizing the initial production of the pion. The matrix M appearing in Eq. (7) is a 3×3 "charge exchange matrix" which is independent of the nature of the initial and final leptons ℓ and ℓ'. In addition to including pion scattering and absorption effects, M also takes into account the reduction of leptonic pion production in a nucleus resulting from the Pauli exclusion principle.[8] We will keep this effect in M in the ensuing phenomenological discussion, but when we make our multiple scattering model in Section 3 will separate it off as an explicit multiplicative factor.

C. Structure of M

Up to this point Eq. (7) applies to all nuclei, even those with a large neutron excess. In order to simplify the subsequent discussion we now restrict ourselves to the case of isotopically neutral targets, with the neutron excess and isotopic spin equal to zero.[9] As noted in the Introduction, this approximation is reasonable for the targets of greatest experimental interest. With this restriction, the pion charge structure of the matrix M_{fi} (f, i = $\pm$, 0) is that of the inclusive reaction

$$\pi_i + T\ (I = 0) \rightarrow \pi_f + T'\ \text{(unobserved)}, \tag{9a}$$

or equivalently, of the forward scattering process

$$\pi_i + \bar{\pi}_f + T(I = 0) \rightarrow \pi_i + \bar{\pi}_f + T\ (I = 0). \tag{9b}$$

Since the isospin of the system $\pi_i + \bar{\pi}_f$ can be either 0, 1, 2, we conclude that the matrix M_{fi} involves three independent parameters. We introduce them by writing

$$M_{fi} = A(c-d)\ \psi_f \cdot \psi_i \psi_f^* \cdot \psi_i^* + Ad\,\psi_i \cdot \psi_i^* \psi_f \cdot \psi_f^* + A(1-c-2d)\psi_f \cdot \psi_i^* \psi_i \cdot \psi_f^*, \tag{10}$$

with ψ_i and ψ_f the isospin wave functions of π_i and π_f respectively. Substituting

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \tag{11}$$

and writing out Eq. (10) component by component, we get the basic form

$$\begin{bmatrix} M_{++} & M_{+0} & M_{+-} \\ M_{0+} & M_{00} & M_{0-} \\ M_{-+} & M_{-0} & M_{--} \end{bmatrix} = A \begin{bmatrix} 1-c-d & d & c \\ d & 1-2d & d \\ c & d & 1-c-d \end{bmatrix}. \tag{12}$$

It is useful to consider the form which the above equations take when no distinction is made between π^+ and π^-, but only between charged and neutral pions. Eq. (7) is then replaced by

$$\begin{pmatrix} \dfrac{d^2\sigma(\ell\ell'\,{}_Z T^A;+)}{dk^2\,dk_0^L} + \dfrac{d^2\sigma(\ell\ell'\,{}_Z T^A;-)}{dk^2\,dk_0^L} \\ \dfrac{d^2\sigma(\ell\ell'\,{}_Z T^A;0)}{dk^2\,dk_0^L} \end{pmatrix} = [N({}_Z T^A;k^2 k_0^L)] \begin{pmatrix} \dfrac{d^2\sigma(\ell\ell' N_T;+)_F}{dk^2\,dk_0^L} + \dfrac{d^2\sigma(\ell\ell' N_T;-)_F}{dk^2 dk_0^L} \\ \dfrac{d^2\sigma(\ell\ell' N_T;0)_F}{dk^2\,dk_0^L} \end{pmatrix}, \tag{13}$$

with N a 2×2 matrix. When the target T is isotopically neutral, the matrix N can be expressed in terms of the parameters of Eq. (12), giving

$$\begin{bmatrix} N_{ch\,ch} & N_{ch\,0} \\ N_{0\,ch} & N_{00} \end{bmatrix} = A \begin{bmatrix} 1-d & 2d \\ d & 1-2d \end{bmatrix} . \qquad (14)$$

The dependence on the parameter c has dropped out, leaving only two parameters which determine the nuclear corrections in this case.

D. Applications

Eqs. (13) and (14) can be applied in two ways. First, they can be used to generate a specific theoretical prediction for the ratio R' of Eq. (2), by integrating with respect to k^2 and k_0^L [with the latter integration extending only over the (3, 3)-resonance region] to give

$$\sigma(\nu_\mu + T \rightarrow \nu_\mu + T' + \pi^0) = \int dk^2 dk_0^L\, A(T;k^2 k_0^L) \Bigg\{ d(T;k^2 k_0^L) \left[\frac{d^2\sigma(\nu_\mu \nu_\mu N_T;+)_F}{dk^2\, dk_0^L} + \frac{d^2\sigma(\nu_\mu \nu_\mu N_T;-)_F}{dk^2\, dk_0^L} \right] + [1-2d(T;k^2 k_0^L)] \frac{d^2\sigma(\nu_\mu \nu_\mu N_T;0)_F}{kd^2\, dk_0^L} \Bigg\} ;$$

$$\sigma(\nu_\mu + T \rightarrow \mu^- + T' + \pi^0) = \int dk^2 dk_0^L\, A(T;k^2 k_0^L) \Bigg\{ d(T;k^2 k_0^L) \left[\frac{d^2\sigma(\nu_\mu \mu^- N_T;+)_F}{dk^2\, dk_0^L} + \frac{d^2\sigma(\nu_\mu \mu^- N_T;-)_F}{dk^2\, dk_0^L} \right] + [1-2d(T;k^2 k_0^L)] \frac{d^2\sigma(\nu_\mu \mu^- N_T;0)_F}{dk^2\, dk_0^L} \Bigg\} . \qquad (15)$$

In the next section we will evaluate Eq. (15) (and thus R') using our multiple scattering model for M and the weak pion production calculation of Ref. 5 as inputs.[10]

The second way of applying Eqs. (13) and (14) is to use them in a purely empirical fashion to extract the charge exchange matrix parameters $A(T;k^2 k_0^L)$ and $d(T;k^2 k_0^L)$ from a comparison of pion electroproduction on free nucleons with pion electroproduction on a nuclear target T. Specifically, we find from Eqs. (13) and (14) that

$$d(T;k^2k_0^L) = \frac{r(eeT)-r(eeN_T)}{[2-r(eeN_T)]\,[1+r(eeT)]} \tag{16}$$

with

$$r(eeT) = \left[\frac{d^2\sigma(eeT;+)}{dk^2\,dk_0^L}+\frac{d^2\sigma(eeT;-)}{dk^2\,dk_0^L}\right]\left[\frac{d^2\sigma(eeT;0)}{dk^2\,dk_0^L}\right]^{-1},$$

$$r(eeN_T) = \left[\frac{d^2\sigma(eeN_T;+)_F}{dk^2\,dk_0^L}+\frac{d^2\sigma(eeN_T;-)_F}{dk^2\,dk_0^L}\right]\left[\frac{d^2\sigma(eeN_T;0)_F}{dk^2\,dk_0^L}\right]^{-1} \tag{17}$$

the electroproduction charged pion to neutral pion ratios on targets T and N_T, and

$$A(T;k^2k_0^L) = \left[\frac{d^2\sigma(eeT;+)}{dk^2\,dk_0^L}+\frac{d^2\sigma(eeT;0)}{dk^2\,dk_0^L}+\frac{d^2\sigma(eeT;-)}{dk^2\,dk_0^L}\right] \times\left[\frac{d^2\sigma(eeN_T;+)_F}{dk^2\,dk_0^L}+\frac{d^2\sigma(eeN_T;0)_F}{dk^2\,dk_0^L}+\frac{d^2\sigma(eeN_T;-)_F}{dk^2\,dk_0^L}\right]. \tag{18}$$

Once $d(T;k^2k_0^L)$ and $A(T;k^2k_0^L)$ have been extracted from electroproduction data by use of Eqs. (16)-(18), they can be substituted into Eqs. (13) and (14) and used to calculate the nuclear corrections to <u>weak</u> pion production on the same target T. Note that Eqs. (16) and (17) for d are independent of the absolute normalization of the electron cross sections, and that A, which does depend on absolute normalization, appears as a simple multiplicative factor in both numerator and denominator of Eq. (2) for R'. Hence the relation between R and R' given by our empirical procedure is also independent of the absolute normalization of the electron cross sections used to extract A and d.

In many applications it is convenient to deal, not with the doubly differential cross sections of Eq. (4), but rather with these cross sections integrated in excitation energy over the (3, 3) resonance region,

$$\frac{d^2\sigma(\ell\ell' T; \pm 0)}{dk^2} = \int_{\substack{(3,3)\,\text{resonance}\\ \text{region}}} dk_0^L \; \frac{d^2\sigma(\ell\ell' T; \pm 0)}{dk^2\, dk_0^L} . \tag{19}$$

In order to write simple formulas directly in terms of these integrated cross sections we note that, to a good first approximation, the k_0^L dependence of the doubly differential cross sections is governed by the dominant (3, 3) channel, and hence is independent of the identities of ℓ and ℓ' and of the pionic charge. This near-identity of excitation energy dependence allows us to make the averaging approximation of replacing Eqs. (7), (12), (13) and (14) by equations of identical form written directly in terms of the cross sections of Eq. (19),

$$\begin{pmatrix} \dfrac{d\sigma(\ell\ell' T;+)}{dk^2} \\[2ex] \dfrac{d\sigma(\ell\ell' T;0)}{dk^2} \\[2ex] \dfrac{d\sigma(\ell\ell' T;-)}{dk^2} \end{pmatrix} = [\bar{M}(T;k^2)] \begin{pmatrix} \dfrac{d\sigma(\ell\ell' N_T;+)}{dk^2} \\[2ex] \dfrac{d\sigma(\ell\ell' N_T;0)}{dk^2} \\[2ex] \dfrac{d\sigma(\ell\ell' N_T;-)}{dk^2} \end{pmatrix} , \tag{20}$$

$$\begin{pmatrix} \dfrac{d\sigma(\ell\ell' T;+)}{dk^2} + \dfrac{d\sigma(\ell\ell' T;-)}{dk^2} \\[2ex] \dfrac{d\sigma(\ell\ell' T;0)}{dk^2} \end{pmatrix} = [\bar{N}(T, k^2)] \begin{pmatrix} \dfrac{d\sigma(\ell\ell' N_T;+)}{dk^2} + \dfrac{d\sigma(\ell\ell' N_T;-)}{dk^2} \\[2ex] \dfrac{d\sigma(\ell\ell' N_T;0)}{dk^2} \end{pmatrix} ,$$

with $\bar{M}$ and $\bar{N}$ the lepton-independent matrices

$$[\bar{M}(T;k^2)] = \bar{A} \begin{bmatrix} 1-\bar{c}-\bar{d} & \bar{d} & \bar{c} \\ \bar{d} & 1-2\bar{d} & \bar{d} \\ \bar{c} & \bar{d} & 1-\bar{c}-\bar{d} \end{bmatrix} \tag{21}$$

$$[\,\bar{N}(T,k^2)\,] = \bar{A}\begin{bmatrix} 1-\bar{d} & 2\bar{d} \\ \bar{d} & 1-2\bar{d} \end{bmatrix} \quad .$$

Because the excitation energy k_0^L has been integrated over,[7] we can use free nucleon cross sections on nucleon targets at rest in the right-hand side of Eq. (20); hence we have omitted the subscript F which indicated smearing of the production cross section over nucleon Fermi motion. (Any residual effects of nucleon Fermi motion on the excitation-energy-integrated pion production cross sections will, in this formulation, be absorbed in the phenomenological matrices $\bar{M}$ and $\bar{N}$.) The formulas for extracting $\bar{A}(T;k^2)$ and $\bar{d}(T;k^2)$ from electroproduction data are identical in form to Eqs. (16)-(18),

$$\bar{d}(T;k^2) = \frac{\bar{r}(eeT)-\bar{r}(eeN_T)}{[\,2-\bar{r}(eeN_T)\,]\,[\,1+\bar{r}(eeT)\,]} \quad ,$$

$$\bar{r}(eeT) = \left[\frac{d\sigma\,(eeT;+)}{dk^2} + \frac{d\sigma\,(eeT;-)}{dk^2}\right]\left[\frac{d\sigma\,(eeT;0)}{dk^2}\right]^{-1} , \qquad (22a)$$

$$\bar{r}(eeN_T) = \left[\frac{d\sigma\,(eeN_T;+)}{dk^2} + \frac{d\sigma\,(eeN_T;-)}{dk^2}\right]\left[\frac{d\sigma\,(eeN_T;0)}{dk^2}\right]^{-1} ;$$

$$\bar{A}(T;k^2) = \left[\frac{d\sigma\,(eeT;+)}{dk^2} + \frac{d\sigma\,(eeT;0)}{dk^2} + \frac{d\sigma\,(eeT;-)}{dk^2}\right] \times \left[\frac{d\sigma\,(eeN_T;+)}{dk^2} + \frac{d\sigma\,(eeN_T;0)}{dk^2} + \frac{d\sigma\,(eeN_T;-)}{dk^2}\right]^{-1} . \qquad (22b)$$

In terms of $\bar{d}$ and $\bar{A}$, the expression for R' analagous to Eqs.(2) and (15) is

$$R'(T) = \frac{\int dk^2 \bar{A}(T;k^2)\{\bar{d}(T;k^2)[\frac{d\sigma(\nu_\mu \nu_\mu N_T;+)}{dk^2} + \frac{d\sigma(\nu_\mu \nu_\mu N_T;-)}{dk^2}] + [1-2\bar{d}(T;k^2)]\frac{d\sigma(\nu_\mu \nu_\mu N_T;0)}{dk^2}\}}{2\int dk^2 \bar{A}(T;k^2)\{\bar{d}(T;k^2)[\frac{d\sigma(\nu_\mu \mu^- N_T;+)}{dk^2} + \frac{d\sigma(\nu_\mu \mu^- N_T;-)}{dk^2}] + [1-2\bar{d}(T;k^2)]\frac{d\sigma(\nu_\mu \mu^- N_T;0)}{dk^2}\}} \tag{23}$$

<u>Eqs. (20)-(23) are in a form convenient for direct comparison with experimental data, and constitute our principle phenomenological result.</u>

We continue by introducing one further averaging approximation. To the extent that $\bar{d}(T;k^2)$ and $\bar{A}(T;k^2)$ are slowly varying functions of k^2 (and this is suggested by the numerical work of Section 3) we can replace them by average values $\bar{\bar{d}}(T)$ and $\bar{\bar{A}}(T)$ in the integrals of Eq. (23). The parameter $\bar{\bar{A}}(T)$ then cancels between numerator and denominator and the integration over k^2 can be explicitly carried out. We are left with a simple formula relating R' to R,

$$R'(T) = R\,\frac{\bar{\bar{d}}(T)\bar{\bar{r}}(\nu_\mu \nu_\mu N_T) + 1-2\bar{\bar{d}}(T)}{\bar{\bar{d}}(T)\bar{\bar{r}}(\nu_\mu \mu^- N_T) + 1-2\bar{\bar{d}}(T)} \tag{24}$$

with

$$\bar{\bar{r}}(\nu_\mu \nu_\mu N_T) = \frac{\sigma(\nu_\mu \nu_\mu N_T;+) + \sigma(\nu_\mu \nu_\mu N_T;-)}{\sigma(\nu_\mu \nu_\mu N_T;\,0)}\,,$$
$$\bar{\bar{r}}(\nu\,\mu^- N_T) = \frac{\sigma(\nu_\mu \mu^- N_T;+) + \sigma(\nu_\mu \mu^- N_T;-)}{\sigma(\nu_\mu \mu^- N_T;\,0)} \tag{25}$$

the charged pion to neutral pion ratios produced on an average nucleon target by neutral and charged weak currents, respectively. In the approximation of Eq. (24), nuclear charge exchange effects are isolated in the single parameter $\bar{\bar{d}}(T)$. This description is particularly useful for giving a simple comparison of the charge exchange corrections expected for different nuclear targets T.

E. Discussion

We conclude by pointing out an experimental problem which will limit the direct applicability of the phenomenological results of Eqs. (16)-(23). In all of the above equations, we have assumed that the angular variables of the produced pion are unobserved, which corresponds to an experimental situation in which the acceptance for produced pions is 4π steradians. However, in realistic experiments observing the weak- and electro-production of pions, the pion acceptance will, in general, be rather small. Since the pion angular distributions <u>do</u> depend on the leptons involved in the production process,[11] the introduction of acceptance restrictions will tend to spoil the simple relation between nuclear charge exchange corrections to weak- and electro-pion production which we have developed above. There are two possible ways of dealing with this problem. One would be to simply go ahead and apply Eqs. (21)-(23) to the limited-acceptance case, interpreting the cross sections on T and N_T as being limited to the actual pion acceptance. If both the value of $\bar{d}$ extracted from electroproduction,[12] and the pion charge ratios observed in weak production, were found to be only weakly acceptance-dependent, one would have an <u>a posteriori</u> justification for applying the phenomenological recipe of Eq. (23) to the acceptance-limited case. An alternative procedure would be to develop a detailed model for the charge exchange parameters d, c and A, and then to numerically fold these charge exchange corrections into experimental or theoretical cross sections for pion production on a free nucleon target, taking acceptance limitations into account. Although, in this approach, one would forgo the possibility of direct phenomenological application of electroproduction data, a comparison of the theory with electroproduction experiments on nuclear targets would still be essential to test (and possibly revise) the charge exchange model. Once validated in this way, the charge exchange parameters could be substituted into Eqs. (15) and (23) to generate predictions for weak production experiments. The question of constructing a suitable model for the charge-exchange parameters will be pursued further in the next section.

3. MULTIPLE SCATTERING MODEL

We proceed in this section to develop a detailed multiple scattering model for nuclear charge exchange corrections. Our motivations are, first, to get an estimate of the magnitude of charge exchange corrections to be expected for various target nuclei, and second, as discussed above, to facilitate comparison with experiment in the realistic case in which there are pion acceptance limitations.

A. Formulation of the Model

Our model closely resembles (with differences which we explain below) a successful semiclassical treatment of $\pi^{\pm}$ production in proton-nculeus collisions which has been given by Sternheim and Silbar.[3] The ingredients of the model are as follows:

(1) We regard the target nucleus as a collection of independent nucleons, distributed spatially according to the density profile determined by electron scattering experiments. For aluminum and lighter nuclei, it is convenient to parametrize the nucleon density in the so-called "harmonic well" form

$$\rho(r) = \rho(0)\, e^{-r^2/R^2}\left[1 + c\frac{r^2}{R^2} + c_1\left(\frac{r^2}{R^2}\right)^2\right], \tag{26}$$

with the values of the various parameters given in Table I.

(2) In discussing pion multiple scattering within the target nucleus, we regard the nucleons as fixed within the nucleus, thus neglecting Fermi motion and nucleon recoil effects. [A numerical estimate of the importance of these effects will be made in Section 3B(ii) below.] This approximation allows us to characterize interactions of the pion with the constituent nucleons by a unique center of mass energy W, related to the lepton energy transfer k_0^L by Eq. (6b). Through all stages of the multiple scattering we approximate the target nucleus to be isotopically neutral, composed of equal numbers of protons and neutrons.

(3) Interactions of pions in the nucleus are treated in the approximation of complete incoherence, involving the use of pion-nucleon cross sections rather than scattering amplitudes in the multiple scattering calculation. In the region of the (3, 3) resonance, pion production and more complex hadron production channels are closed, and so there are only two relevant cross sections. The first is the cross section per nucleon $\sigma_{ABS}(W)$ for pion absorption via various nuclear processes; for this quantity we use the best fit value obtained by Sternheim and Silbar in

their study of pion production by protons,

$$\sigma_{ABS}(W) = \begin{cases} 0 & T_\pi < .788\ M_\pi , \\ 22\text{mb}\ \dfrac{T_\pi - .788 M_\pi}{2.077 M_\pi} & T_\pi > .788\ M_\pi , \end{cases} \tag{27}$$

$$T_\pi = \frac{W^2 - (M_N + M_\pi)^2}{2M_N} \ .$$

To allow for the considerable uncertainties in this expression for σ_{ABS}, we examine numerically the effect on the charge exchange corrections of multiplying Eq. (27) by factors of 1/2 or 2. The second cross section needed is the usual elastic cross section for pion-nucleon scattering. Since, in the (3, 3) region the $I = 1/2$ pion-nucleon cross section is very small, we neglect it entirely and regard all pion-nucleon scattering as proceeding through the $I = 3/2$ channel. The elastic cross section is then simply proportional to the cross section

$$\sigma_{\pi^+ p}(W) \ , \tag{28}$$

for which a simple parameterization is given in Appendix C. In order to solve the pion multiple scattering problem, we actually need the differential cross section for elastic scattering; in the approximation of (3, 3) dominance, this is given by

$$\frac{d\sigma_{elastic}}{d\Omega} \propto \sigma_{\pi^+ p}(W)\ (1 + 3\cos^2\phi), \tag{29}$$

with ϕ the pion scattering angle.

(4) When a pion is produced by leptons incident on a nucleus or undergoes subsequent rescatterings, with small momentum transfer to the nuclear system, the corresponding production or scattering cross section is

reduced by the Pauli exclusion principle.[7] We take this effect into account, within the framework of the independent nucleon picture, by multiplying the lepto-pion production cross section and the pion nucleon rescattering cross section by respective reduction factors $g(W, k^2)$ and $h(W, \phi)$. Formulas for these factors are given in Appendix C. Neutrino quasielastic scattering experiments at small momentum transfer k^2 provide some empirical evidence for the presence of the production factor g. The argument for including h is less compelling, since we are using a semiclassical picture, with fixed constituent nucleons, for treating the pion multiple scattering in the nuclear medium, and in a semiclassical picture there are no Pauli effects. To take this objection[13] into account, in the numerical work below we also calculate results for the case in which h is replaced by unity.

(5) The approximation of keeping only $I = 3/2$ pion-nucleon scattering allows us to reduce the problem of calculating the charge-exchange matrix M to a one-component scattering problem. To see this we let

$$\psi_i = \begin{pmatrix} n_i(\pi^+) \\ n_i(\pi^0) \\ n_i(\pi^-) \end{pmatrix} \tag{30}$$

denote the pion charge multiplicities initially present in a beam of pions, at a fixed isobar energy W. A simple isospin Clebsch analysis then shows that when the pion beam undergoes a single scattering on an equal mixture of protons and neutrons through the $I = 3/2$ channel, the effect is to replace ψ by $Q\psi$, with Q the matrix

$$Q = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{5}{6} \end{pmatrix} . \tag{31}$$

Obviously, the natural way to describe a multiple scattering process, in which Q acts on ψ repeatedly, is to decompose ψ into a sum of eigenvectors of Q. These eigenvectors, with their corresponding eigenvalues λ, are

$$
\begin{aligned}
q_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & \lambda_1 &= 1 \,, \\
q_2 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} & \lambda_2 &= \frac{5}{6} \,, \\
q_3 &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} & \lambda_3 &= \frac{1}{2} \,,
\end{aligned}
\tag{32}
$$

and the decomposition reads

$$
\begin{aligned}
\psi &= \sum_{k=1}^{3} C_k\, q_k \,, \\
C_1 &= \frac{1}{3}\left[n_i(\pi^+) + n_i(\pi^0) + n_i(\pi^-)\right] , \\
C_2 &= \frac{1}{2}\left[n_i(\pi^+) - n_i(\pi^-)\right] , \\
C_3 &= -\frac{1}{3}\, n_i(\pi^0) + \frac{1}{6}\left[n_i(\pi^+) + n_i(\pi^-)\right] .
\end{aligned}
\tag{33}
$$

The effect of a multiple scattering process on Eq. (33) will be to lead to a final pion multiplicity state ψ_f, related to ψ_i by

$$
\psi_f = \sum_{k=1}^{3} f(\lambda_k)\, C_k\, q_k \,, \tag{34}
$$

with $f(\lambda)$ a function of the eigenvalue λ which contains all geometric and dynamical information concerning nuclear parameters, magnitudes of cross sections, etc. Taking now ψ_i to be the initial distribution of lepto-produced pions in target T,

$$
\psi_i = g(W, k^2) \begin{pmatrix} \dfrac{d^2\sigma\,(\ell\ell' N_T;+)}{dk^2\, dk_0^L} \\[2ex] \dfrac{d^2\sigma\,(\ell\ell' N_T;0)}{dk^2\, dk_0^L} \\[2ex] \dfrac{d^2\sigma\,(\ell\ell' N_T;-)}{dk^2\, dk_0^L} \end{pmatrix} \tag{35}
$$

and ψ_f to be the distribution of exiting pions,

$$\psi_f = \begin{pmatrix} \dfrac{d^2\sigma\,(\ell\ell'\,T;+)}{dk^2\,dk_0^L} \\ \dfrac{d^2\sigma\,(\ell\ell'\,T;0)}{dk^2\,dk_0^L} \\ \dfrac{d^2\sigma\,(\ell\ell'\,T;-)}{dk^2\,dk_0^L} \end{pmatrix} , \qquad (36)$$

we find that the connection of Eq. (34) takes the form of Eqs. (7) and (12), with

$$\begin{aligned} A &= g(W, k^2)\, a, \\ a &= f(1); \\ c &= \frac{1}{3} - \frac{1}{2} f(\tfrac{5}{6})/f(1) + \frac{1}{6} f(\tfrac{1}{2})/f(1) \;; \\ d &= \frac{1}{3}\,[\,1 - f(\tfrac{1}{2})/f(1)\,] \;. \end{aligned} \qquad (37)$$

(6) We turn finally to the function $f(\lambda)$, which contains the dynamical details of pion multiple scattering in the nucleus. The precise statement of the problem defining $f(\lambda)$ is as follows: We introduce an initial distribution of monoenergetic pions into a nucleus, with the pion density proportional to the nuclear density [given by Eq. (26)] . The pions multiple scatter, with absorption cross section given by Eq. (27) and with elastic scattering cross section given by Eq. (29). At each elastic scattering the pion number is multiplied by a factor λ. The function $f(\lambda)$ is then defined as the expected number of pions eventually emerging from the nuclear medium, normalized to unit integrated initial pion density.

To get a simple (and, it turns out, surprisingly accurate) approximation to $f(\lambda)$, we replace the actual angular distribution [Eq. (29) times $h(W, \cos\phi)$] by a modified elastic scattering distribution, in which all forward hemisphere scattering ($0 \leq \phi \leq \pi/2$) is projected onto the forward direction ($\phi = 0$), and all backward hemisphere scattering ($\pi/2 \leq \phi \leq \pi$) is projected onto the backward direction ($\phi = \pi$). In this approximation, once a pion is produced in the nucleus, it scatters back and forth along its initial line of motion until it either is absorbed or it leaves the nucleus. Since both the initial pion distribution and the interaction probabilities are proportional to nucleon density, the nucleon density profile along the line can be scaled out of the problem by an appropriate change in length variable. Thus, for each line passing through the nucleus the expected fraction of pions which exit is independent of the density profile along the line, but depends only on the integrated density along the line (the so-called "optical thickness"), which we denote by L. Once we have solved for the one-dimensional exit fraction $f(\lambda, L)$, we need only average over the distribution of optical thickness in the nucleus to get an expression for $f(\lambda)$.

To put these remarks in quantitative form, let us take the central nucleon density $\rho(0)$ as the "standard density" relative to which densities elsewhere in the nucleus are measured. For given impact parameter b relative to the center of the nucleus, the optical thickness is then given by

$$L(b) = \int_{-\infty}^{\infty} dz\, e^{-(z^2+b^2)/R^2}\left[1 + c\left(\frac{z^2+b^2}{R^2}\right) + c_1\left(\frac{z^2+b^2}{R^2}\right)^2\right] \qquad (38)$$

$$= R\,\pi^{1/2} e^{-b^2/R^2}\left\{1 + c\left(\frac{1}{2} + \frac{b^2}{R^2}\right) + c_1\left[\frac{3}{4} + \frac{b^2}{R^2} + \left(\frac{b^2}{R^2}\right)^2\right]\right\} .$$

Averaging over impact parameters, the relation between $f(\lambda)$ and $f(\lambda, L)$ is given by

$$f(\lambda) = \frac{\int_0^\infty bdb\ L(b)\ f(\lambda, L(b))}{\int_0^\infty bdb\ L(b)} . \tag{39}$$

The one-dimensional problem defining $f(\lambda, L)$ is formulated precisely as follows: We consider a uniform one-dimensional medium of length L, in which pions are uniformly initially produced moving (say) to the right. The pions propagate in the medium with inverse interaction length κ, given in terms of the nucleon density and the absorption and scattering cross sections by

$$\kappa = \rho(0)\sigma_{TOT},$$
$$\sigma_{TOT} = \sigma_{ABS}(W) + \frac{1}{3}\sigma_{\pi^+p}(W)[h_+(W) + h_-(W)] , \tag{40}$$

The factors h_+ and h_- describe the forward- and backward- hemisphere projections of the Pauli reduction factor $h(W, \phi)$,

$$h_+ = \frac{1}{2}\int_0^{\pi/2} \sin\phi\, d\phi\, (1+3\cos^2\phi)\, h(W,\phi) ,$$
$$h_- = \frac{1}{2}\int_{\pi/2}^{\pi} \sin\phi\, d\phi\, (1+3\cos^2\phi)\, h(W,\phi) \tag{41}$$

and are explicitly calculated in Appendix C. At each interaction the pions are forward-scattered with probability μ_+ and back-scattered with probability μ_- (and, of course, absorbed with probability $1 - \mu_+ - \mu_-$), with

$$\mu_{\pm} = \frac{1}{3}\sigma_{\pi^+p}(W)\, h_{\pm}(W)/\sigma_{TOT}, \tag{42}$$

and concomitantly with each scattering, the pion number is multiplied by a factor λ. The desired quantity $f(\lambda, L)$ is the expected number of pions eventually emerging from the medium, normalized to unit integrated initial pion density. An explicit expression for f is calculated in Appendix A

[See Eq. (A.12) and Eqs. (A.25) - (A.27)] , as well as expressions for f_+ and f_-, the expected fractions of pions eventually emerging with and without a net reversal of direction of motion along the line. In Appendix B we compare the approximate solution for $f(\lambda)$ given by Eq. (39) with the exact solution in the simple geometry of a uniform sphere composed of material which scatters isotropically, and find very satisfactory agreement. Since the actual angular distribution of interest to us, $1 + 3 \cos^2 \phi$, is already peaked in the backward and forward directions,[14] our approximation should be at least as accurate for this case as it is for handling isotropic scattering.

This completes the specification of our multiple scattering model. As we have already noted, it closely resembles the calculation of Sternheim and Silbar, and the reader is referred to Ref. 3 for an excellent, detailed analysis of the approximations and physical assumptions which are involved. The aspects in which our model differs from that of Ref. 3 are: (1) We take into account the diffuseness of the nuclear edge, rather than treating the nucleon distribution as a uniform sphere; (2) We take Pauli exclusion effects into account in a crude way; and (3) We use an improved approximation for solving the pion multiple-scattering problem. Instead of using the back-forward approximation described above, Sternheim and Silbar use the considerably less accurate approximation of treating all scattering as purely forward scattering. A comparison of their approximation with the exact solution, in the case of a uniform sphere composed of material which scatters isotropically, is given in Appendix B.

B. Numerical Calculations

We turn now to numerical calculations, in which we combine our model for nuclear charge-exchange corrections with the theory of pion electro- and weak production from free nucleon targets developed in Ref. 5. For the hadronic weak neutral current, we adopt the Weinberg-model form[15]

$$J_\lambda^{\text{neutral}} = J_\lambda^{V3} + J_\lambda^{A3} - 2\sin^2\theta_W J_\lambda^{\text{em}}; \qquad (43)$$

we will say a few words below about variants of this model in which Eq. (43) contains an additional isoscalar current. We assume throughout an incident lab neutrino energy $k_{10}^L = 1$ GeV and a nucleon elastic form factor[2]

$$g_A(k^2) = \frac{1.24}{\left[1 + \frac{k^2}{(0.9\ \text{GeV}/c)^2}\right]^2}, \qquad (44)$$

and take integrations over the (3, 3)-resonance region to extend from the pion production threshold up to a maximum isobar mass of W = 1.47 GeV. In our calculations on aluminum, we weight the free nucleon production cross sections according to the actual neutron/proton ratio in aluminum [i.e., we take $N_T = 13p + 14n$], but as emphasized above, we adopt the approximation of isotopic neutrality in calculating charge exchange corrections.

(i) Calculation of R' from Eq. (15) (with Fermi motion neglected)

In Table II we present results for the ratio R' on an aluminum target, calculated by using Eq. (15) to fold the W-dependent charge exchange matrix into the production cross sections from a free nucleon target at rest [i.e., we neglect the Fermi-motion average symbolized by the subscript "F" in Eq. (15)] . In the second column we tabulate

$$R(N_T) = \frac{\sigma\,(\nu_\mu + N_T \rightarrow \nu_\mu + N'_T + \pi^0)}{2\sigma\,(\nu_\mu + N_T \rightarrow \mu^- + N''_T + \pi^0)} , \tag{45}$$

which is the ratio predicted by the production model when no charge-exchange corrections are made. In the third through seventh columns we tabulate values of the charge-exchange-corrected ratio R' obtained under various alternative assumptions. The column labeled "No variations" is the result obtained from the multiple scattering model of Sec. 3A above; the next three columns show how this result changes when the Pauli factors h in Eqs. (40) and (42) are replaced by unity, or when the absorption cross section of Eq. (27) is modified. The predictions for R' are evidently quite insensitive to these variations. The seventh column gives the result for R' when all isoscalar multiples are omitted. Since the isoscalar multipoles only contribute quadratically to R',[16] this column gives a lower bound on R' for any variant of the Weinberg theory in which the hadronic neutral current differs from Eq. (43) by purely isoscalar terms. In the final column we have used our production and charge exchange calculations to generate simulated pion weak and electroproduction cross sections on aluminum, which are then used to evaluate the lower bound on R' derived by Albright et. al.[17] in the isoscalar-target approximation,

$$R'\,({}_{13}Al^{27}) \geq \frac{1}{4}\,\{[\,\bar{\bar{r}}\,(\nu_\mu \mu^-\,{}_{13}Al^{27}) - 1]^{\frac{1}{2}} - 2\sin^2\theta_W \left[\frac{V^0_{em}}{\sigma\,(\nu_\mu \mu^-\,{}_{13}Al^{27};0)}\right]^{\frac{1}{2}}\}^2 ,$$

$$\bar{\bar{r}}\,(\nu_\mu \mu^-\,{}_{13}Al^{27}) = \frac{\sigma\,(\nu_\mu \mu^-\,{}_{13}Al^{27};+) + \sigma\,(\nu_\mu \mu^-\,{}_{13}Al^{27};-)}{\sigma\,(\nu_\mu \mu^-\,{}_{13}Al^{27};0)} , \tag{46}$$

$$V^0_{em} = \frac{G^2 \cos^2 \theta_C}{\pi} \frac{1}{4\pi \alpha^2} \int (k^2)^2 \, dk^2 \, \frac{d\sigma\,(ee_{13}Al^{27};0)}{dk^2} .$$

We see that the bound of Eq. (46) provides a satisfactory estimate of R' for small values of $\sin^2\theta_W$.

We turn next to Table III, where we have tabulated charged pion to neutral pion production ratios for the usual charged weak current. The first column gives the standard 5:1 prediction for an isotopically neutral target, assuming complete $I = 3/2$ dominance. When $I = 1/2$ multipoles are taken into account,[18] the prediction is lowered to 3.67:1, as shown in the second column. Finally, in the third column we give the prediction of 2.63:1 which results when Eq. (15), and its analog for charged pions, are used to fold in charge-exchange corrections for aluminum.[19] It would obviously be very desirable to try to check this prediction for $\bar{\bar{r}}$ simultaneously with the experimental determination of R'.

(ii) Averaging approximations, comparison of different nuclei and estimate of nucleon motion effects.

We conclude with a test of the averaging approximations introduced in Sec. 2 and a discussion of related topics. To study Eqs. (20)-(23), we fold together the electroproduction and charge exchange models, as in Eq. (15), to give simulated data for pion electroproduction cross sections on aluminum. Substituting this data into Eqs. (21) and (22) then gives the values for $\bar{d}$ and $\bar{A}$ tabulated in Table IV. The charge exchange parameters obtained this way are seen to be nearly independent of the incident electron energy k^L_{10}, and are slowly varying functions of

k^2 except in the region $k^2 \leq .3$, where Pauli exclusion effects and $I = 1/2$ multipoles arising from the pion exchange graph become important. Substituting the 2 GeV/c values of $\bar{d}$ and $\bar{A}$ into Eq. (23), and continuing to use our production model for the neutrino cross sections, gives the values of R' tabulated in the second column of Table V. In the third column we transcribe from Table II the values of R' obtained directly from Eq. (15); the good agreement indicates that the averaging approximation is working.

We turn next to the "double averaged" approximation of Eqs. (24) and (25). We define the double-barred charge exchange parameters by averaging the charge exchange matrix over the leading W-dependent part of the production cross section as obtained in the static approximation,[20]

$$\begin{aligned}
\bar{\bar{f}}(\lambda) &= \int dW q(W)^{-1} \sigma_{(3,3)}(W) f(\lambda) \Big/ \int dW q(W)^{-1} \sigma_{(3,3)}(W) \,, \\
\bar{\bar{a}} &= \bar{\bar{f}}(1), \\
\bar{\bar{c}} &= \frac{1}{3} - \frac{1}{2}\bar{\bar{f}}(\tfrac{5}{6})/\bar{\bar{f}}(1) + \frac{1}{6}\bar{\bar{f}}(\tfrac{1}{2})/\bar{\bar{f}}(1) \,, \\
\bar{\bar{d}} &= \frac{1}{3}\left[1 - \bar{\bar{f}}(\tfrac{1}{2})/\bar{\bar{f}}(1)\right] .
\end{aligned} \qquad (47)$$

Expressions for the resonant pion-nucleon scattering cross section $\sigma_{(3,3)}(W)$ and the pion momentum $q(W)$ are given in Appendix C. Evaluating Eq. (47) for aluminum gives $\bar{\bar{d}}({}_{13}Al^{27}) = .162$, which, when substituted into Eq. (24) along with the charged to neutral ratios tabulated in the second and third columns of Table VI, gives the predictions for R' tabulated in the fourth column. These agree well with the corresponding values of R' obtained directly from Eq. (15). As another test of the "double averaged" approximation, we consider the formula giving the charge exchange corrections to the charged to neutral ratio $\bar{\bar{r}}$,

$$\bar{\bar{r}}'(\nu_\mu \mu^- {}_{13}Al^{27}) = \frac{2\bar{\bar{d}}({}_{13}Al^{27}) + [1-\bar{\bar{d}}({}_{13}Al^{27})]\ \bar{\bar{r}}(\nu_\mu \mu^- N_T)}{1-2\bar{\bar{d}}({}_{13}Al^{27})+\bar{\bar{d}}({}_{13}Al^{27})\ \bar{\bar{r}}(\nu_\mu \mu^- N_T)} \quad . \tag{48}$$

Substituting $\bar{\bar{r}} = 3.67$, $\bar{\bar{d}} = .162$ into Eq. (48) gives $\bar{\bar{r}}' = 2.68$, as tabulated in the final column of Table III. This again is in close agreement with the value of $\bar{\bar{r}}'$ obtained directly from Eq. (15).

As we remarked in Section 2, the "double averaged" approximation provides a convenient format for comparing charge exchange effects in different nuclei. In Table VII we have tabulated the charge exchange parameters $\bar{\bar{a}}$, $\bar{\bar{c}}$ and $\bar{\bar{d}}$ for a range of light and medium weight nuclei up to aluminum. The key point to notice is that the parameter $\bar{\bar{d}}$ is slowly varying, indicating that charge exchange effects in different medium weight targets, such as, for example, freon (CF_3 Br) and aluminum, should be quite similar.

Finally, we apply the "double averaged" approximation to estimate the effect on our numerical results of including nucleon Fermi motion and nucleon recoil. Obviously, to include nucleon motion in a realistic way one would have to go outside the framework of the one-speed scattering theory used above, since once the nucleons are not regarded as fixed the pion changes energy in each collision. Rather than attempting to follow these energy changes in detail (which would require an elaborate numerical calculation), we adopt a simple approximation which can be treated by the methods used above. We observe that in the (3, 3) resonance region typical nucleon recoil momenta are of the same order as the nucleon

Fermi momentum ($\sim 1.6\ M_\pi/c$); hence a rough estimate of nucleon recoil and Fermi motion effects should be given by the simple randomizing approximation of regarding the pion energy as a constant throughout its motion in the nucleus, but replacing the pion production and charge exchange scattering cross sections by corresponding cross sections which are smeared over nucleon Fermi motion. Evaluating Eq. (47) using these smeared cross sections gives $\bar{\bar{d}}({}_{13}Al^{27}) = .142$, as compared with the value of .162 which results when nucleon motion is neglected. We see that the change in $\bar{\bar{d}}$ is relatively small, and is in the direction of reducing the size of charge exchange effects; we expect these qualitative features to survive in a more careful treatment of nucleon motion effects. In Table VIII we summarize the values of $\bar{\bar{d}}({}_{13}Al^{27})$ obtained in our original model and when various modifications are made.

C. Pion Angular Distributions

Up to this point we have only discussed charge exchange corrections to cross sections in which the pion angular variables have been integrated out. Our model, however, makes specific predictions for angular distributions as well, and although they are much more subject to error than the integrated predictions,[21] they are essential for describing experimental situations in which the pion acceptance is limited. To describe the angular distribution predictions, we let the column vector

$$d\sigma\,(N_T\hat{q}) = \begin{bmatrix} d\sigma\,(N_T\hat{q};+) \\ d\sigma\,(N_T\hat{q};0) \\ d\sigma\,(N_T\hat{q};-) \end{bmatrix} \tag{49}$$

denote the free-nucleon-target pion production cross section, with the pion emerging in direction $\hat{q}$. In the back-forward scattering approximation, after undergoing nuclear interactions the pion can either emerge in direction $\hat{q}$ or can emerge with reversed direction $-\hat{q}$.
In Appendix A, in addition to calculating the total expected fraction of emerging pions $f(\lambda, L)$, we also calculate the expected fractions $f_+(\lambda, L)$, $f_-(\lambda, L)$ which emerge respectively with, or without a net change in direction. Using these to define a forward charge exchange matrix M_+ and a backward matrix M_- in analogy with Eqs. (12), (37) and (39),

$$[M_\pm] = A_\pm \begin{bmatrix} 1-c_\pm-d_\pm & d_\pm & c_\pm \\ d_\pm & 1-2d_\pm & d_\pm \\ c_\pm & d_\pm & 1-c_\pm-d_\pm \end{bmatrix} ;$$

$$\begin{aligned} A_\pm &= g(W,k^2)\, a_\pm, \\ a_\pm &= f_\pm(1); \\ c_\pm &= \frac{1}{3} - \frac{1}{2} f_\pm(\tfrac{5}{6})/f_\pm(1) + \frac{1}{6} f_\pm(\tfrac{1}{2})/f_\pm(1); \\ d_\pm &= \frac{1}{3}\left[1 - f_\pm(\tfrac{1}{2})/f_\pm(1)\right]; \\ f_\pm(\lambda) &= \frac{\int_0^\infty bdb\, L(b)\, f_\pm(\lambda, L(b))}{\int_0^\infty bdb\, L(b)} , \end{aligned} \tag{50}$$

we get for the charge-exchange-corrected pion angular distribution

$$d\sigma(T\hat{q}) = [M_+]\, d\sigma(N_T\hat{q}) + [M_-]\, d\sigma(N_T\, -\hat{q}). \tag{51}$$

Since

$$[M_+] + [M_-] = [M] , \tag{52}$$

Eq. (51) implies that

$$d\sigma(T\hat{q}) + d\sigma(T\ -\hat{q}) = [\,M\,]\ [\,d\sigma(N_T\hat{q}) + d\sigma(N_T\ -\hat{q})\,]\ , \quad (53)$$

and so Eq. (51) reduces to our previous result for charge exchange corrections when integrated over pion angle.

4. CONCLUSIONS

We briefly summarize the results of the preceding sections, with particular emphasis on their implications for further experimental and theoretical work.

(1) Our model calculations confirm the suggestion of Perkins[2] that charge exchange corrections to weak pion production are a substantial effect, even for relatively light nuclear targets. To improve our understanding of these corrections it is important to do the analagous pion electroproduction experiments on nuclear targets, both to implement the phenomenological procedures of Section 2 and to test the predictions of the detailed multiple scattering model of Section 3. In the context of the multiple scattering model these electroproduction experiments have an independent nuclear physics interest, since they will permit a determination of the pion absorption cross section $\sigma_{ABS}(W)$ entering into the Sternheim-Silbar[3] calculation, independent of assumptions about the magnitude of proton absorption in nuclear matter.

(2) Again, in the context of the multiple scattering model, it is important to repeat the calculations of Sternheim and Silbar using the improved scattering approximation developed in Section 2 and Appendix A (as

extended[9] to the case of a neutron excess). This will permit the extraction of an optimized pion absorption cross section $\sigma_{ABS}(W)$ appropriate to the precise model which we use, and hopefully, may reduce some of the remaining areas of disagreement between the Sternheim-Silbar calculation and experiment.

(3) Our calculations suggest that the ratio R' $({}_{13}Al^{27})$ is larger than about .18 when the Weinberg parameter is in the currently interesting[2] range $\sin^2\theta_W \lesssim .35$. We do not attach great significance to the fact that this theoretical estimate of R' exceeds the upper bound of .14 reported by W. Lee,[1] since the discrepancy is easily of the order of uncertainties in the predictions of our production and charge-exchange models. We believe that a reasonably conservative statement is that if the hadronic neutral weak current has (up to isoscalar additions) the form of Eq. (43), and if $\sin^2\theta_W \lesssim .35$, then R' on an aluminum target is in the neighborhood of a fifteen percent effect. Thus, an experiment capable of measuring R' to a level of a few percent will provide a decisive test of Eq. (43), and if Eq. (43) is correct, should permit a crude determination of $\sin^2\theta_W$.

ACKNOWLEDGEMENTS

We wish to thank C. Baltay, M. A. B. Bég, K. M. Case, L. Hand, A. Kerman, B. W. Lee, W. Lee, H. J. Lipkin, S. B. Treiman and J. D. Walecka for helpful conversations. One of us (S. L. A.) wishes to acknowledge the hospitality of the Aspen Center for Physics, where initial parts of this work were done.

APPENDIX A. ONE-DIMENSIONAL SCATTERING PROBLEM

In this Appendix we solve the one-dimensional multiple scattering problem on which our approximate solution for pion three-dimensional multiple scattering is based.[22] We briefly recapitulate the formulation of the problem given in the text. We consider a uniform one dimensional medium extending from $x = 0$ to $x = L$, in which pions are uniformly initially produced moving (say) to the right. The pions move in the medium with inverse interaction length κ, and at each interaction the pions are forward-scattered with probability μ_+ and back-scattered with probability μ_-, with a concomitant multiplication of the pion number by a factor of λ. The probabilities μ_+ and μ_- satisfy the constraint

$$\mu_+ + \mu_- \leq 1 \; ; \tag{A.1}$$

when Eq. (A.1) holds with the inequality, pion absorption is present. The problem is to find the expected numbers $f_\pm$ of pions eventually emerging from the medium either moving to the right (f_+: no overall direction reversal) or to the left (f_-: overall direction reversal), normalized to unit integrated initial pion density.

We begin by remarking that since $f_+(f_-)$ is even (odd) in the direction-reversal-probability μ_-, it suffices to calculate

$$f = f_+ + f_- \; , \tag{A.2}$$

the expected amplitude for pions to emerge in either direction. We then recover $f_\pm$ by splitting f into parts even and odd in μ_-. To formulate the multiple scattering problem, we let $P(xj\,|\,yi)dx$ be the probability that a pion which, after collision $n-1$ was at coordinate y moving in direction

i ($i = l, r$ = left, right) is, after collision n, in an interval **dx** at **x** moving in direction j. From the definitions of κ and $\mu_\pm$ given above, one easily finds that P, which does not depend on n, is given by

$$\begin{aligned} P(xr|yr) &= \mu_+ \kappa \; e^{-\kappa(x-y)} \theta(x-y), \\ P(xl|yr) &= \mu_- \kappa \; e^{-\kappa(x-y)} \theta(x-y), \\ P(xl|yl) &= \mu_+ \kappa \; e^{-\kappa(y-x)} \theta(y-x), \\ P(xr|yl) &= \mu_- \kappa \; e^{-\kappa(y-x)} \theta(y-x), \end{aligned} \tag{A.3}$$

with θ the usual step function. Since the composition laws for <u>conditional probabilities</u> are the same as the quantum mechanical composition laws for probability amplitudes, it is convenient to introduce a Dirac state notation by writing

$$\begin{aligned} \langle xj|P|yi\rangle &= P(xj|yi) \;; \\ \langle xj|P^2|yi\rangle &= \int_0^L dz \sum_k \langle xj|P|zk\rangle \langle zk|P|yi\rangle \;, \\ \langle xj|P^n|yi\rangle &= \int_0^L dz \sum_k \langle xj|P|zk\rangle \langle zk|P^{n-1}|yi\rangle \;. \end{aligned} \tag{A.4}$$

Letting $\rho^{(0)}(yi)$ be the initial density of produced pions moving in direction i, we then find that the density $\rho^{(n)}(xj)$ of pions which have undergone exactly n collisions and are moving in direction j, is

$$\rho^{(n)}(xj) = \int_0^L dy \sum_i \langle xj|P^n|yi\rangle \; \rho^{(0)}(yi) \;. \tag{A.5}$$

The number of pions $N^{(n)}$ <u>emerging</u> from the medium after exactly n interactions is equal to the total number of pions present after n interactions, less the number of such pions which interact once more in the medium,

$$N^{(n)} = \int_0^L dx[\,\rho^{(n)}(xl) + \rho^{(n)}(xr)] \tag{A. 6}$$

$$- \int_0^L dx[\int_0^x dz\, \kappa\, e^{-\kappa\,(x-z)} \rho^{(n)}(xl) + \int_x^L dz\, \kappa\, e^{-\kappa\,(z-x)} \rho^{(n)}(xr)] \;.$$

Since each interaction multiplies the pion number by one factor of λ, the number $N^{(n)}$ must be weighted by λ^n in forming the expected number of pions leaving the medium. Taking $\rho^{(0)}(yi)$ to have the unit normalized value

$$\rho^{(0)}(yi) = \frac{1}{L}\,\delta_{i,r}\;, \tag{A. 7}$$

we get finally

$$f = \sum_{n=0}^{\infty} \lambda^n N^{(n)} \;. \tag{A. 8}$$

Eqs. (A. 3) - (A. 8) constitute the statement of our scattering problem. To write these equations more compactly, we introduce the additional notations

$$\begin{aligned} \langle zi|1|xj\rangle &= \delta\,(z-x)\;\delta_{ij}, \\ \langle z|P_{tot}|xr\rangle &= \kappa\; e^{-\kappa\,(z-x)}\theta\,(z-x)\;, \\ \langle z|P_{tot}|xl\rangle &= \kappa\, e^{-\kappa\,(x-z)}\theta\,(x-z)\;, \end{aligned} \tag{A. 9}$$

in terms of which Eqs. (A. 5) - (A. 8) take the form

$$\begin{aligned} f &= \iint_0^L dz\; dx\, \{[\,\delta\,(z-x) - \langle z|P_{tot}|xl\rangle\,] \sum_{n=0}^{\infty} \lambda^n \rho^{(n)}(xl) \\ &\qquad + [\,\delta(z-x) - \langle z|P_{tot}|xr\rangle\,] \sum_{n=0}^{\infty} \lambda^n \rho^{(n)}(xr)\}\,, \\ \sum_{n=0}^{\infty} \lambda^n \rho^{(n)}(xj) &= \frac{1}{L}\int_0^L dy\, \langle xj|\sum_{n=0}^{\infty} \lambda^n P^n|yr\rangle \\ &= \frac{1}{L}\int_0^L dy\, \langle xj|\,(1-\lambda\, P)^{-1}|yr\rangle \;. \end{aligned} \tag{A. 10}$$

Eq. (A. 10) can be further simplified by noting that

$$\delta(z-x) - \langle z|P_{tot}|xj\rangle = \left(1 - \frac{1}{\sigma_+ + \sigma_-}\right) \sum_i \langle zi|1|xj\rangle$$
$$+ \frac{1}{\sigma_+ + \sigma_-} \sum_i \langle zi|1-\lambda P|xj\rangle \ , \qquad \text{(A. 11)}$$

with

$$\sigma_{\pm} = \lambda \mu_{\pm} \qquad \text{(A. 12)}$$

Substituting Eq. (A. 11) into Eq. (A. 10) we obtain, finally,

$$f = \left(1 - \frac{1}{\sigma_+ + \sigma_-}\right) \langle (1-\lambda P)^{-1}\rangle_{AV} + \frac{1}{\sigma_+ + \sigma_-} \ , \qquad \text{(A. 13)}$$

with

$$\langle (1-\lambda P)^{-1}\rangle_{AV} = \frac{1}{L} \iint_0^L dz\, dy \sum_i \langle zi|(1-\lambda P)^{-1}|yr\rangle \ . \qquad \text{(A. 14)}$$

Eqs. (A. 12) - (A. 14) give a formal expression for f; to evaluate this expression explicitly we must determine the inverse operator appearing in Eq. (A. 14). Writing

$$\langle zi|(1-\lambda P)^{-1}|yj\rangle = \delta(z-y)\ \delta_{ij} + F(zi|yj) \qquad \text{(A. 15)}$$

and defining

$$f(yj) = \int_0^L dz \sum_i F(zi|yj) \ , \qquad \text{(A. 16)}$$

we find that Eq. (A. 14) can be expressed in terms of $f(yj)$ as

$$\langle (1-\lambda P)^{-1}\rangle_{AV} = 1 + \frac{1}{L}\int_0^L dy\ f(yr) = 1 + \frac{1}{L}\int_0^L dy\ f(yl) \ , \qquad \text{(A. 17)}$$

while the relation $(1-\lambda P)\ (1-\lambda P)^{-1} = 1$ implies that $f(yj)$ satisfies the integral equation

$$f(yj) = g(yj) + \int_0^L dz \sum_i f(zi) \langle zi|\lambda P|yj\rangle \ ,$$
$$g(yj) = \int_0^L dz \sum_i \langle zi|\lambda P|yj\rangle \ . \qquad \text{(A. 18)}$$

Referring back to Eq. (A. 3) for P, we easily see that $f(yj)$ and $g(yj)$ have the reflection symmetry

$$f(yl) = f(L-yr), \qquad g(yl) = g(L-y\,r) \quad . \tag{A.19}$$

Substituting Eq. (A.3) into Eq. (A.18) and using this symmetry, we find that Eq. (A.18) reduces to the single integral equation

$$f(yl) = (\sigma_+ + \sigma_-)(1-e^{-\kappa y}) + \int_0^y dz\,[\,\kappa\sigma_+ f(zl) + \kappa\sigma_- f(L-zl)\,]\; e^{-\kappa(y-z)} . \tag{A.20}$$

Multiplying Eq. (A.20) by $e^{\kappa y}$ and differentiating, we find the equivalent differential equation and boundary condition

$$\kappa f(yl) + f'(yl) = (\sigma_+ + \sigma_-)\,\kappa + \kappa\sigma_+ f(yl) + \kappa\sigma_- f(L-yl) \; , \tag{A.21}$$

$$f(0l) = 0 \; .$$

The solution to Eq. (A.21) has the form

$$f(yl) = \frac{\sigma_+ + \sigma_-}{1-(\sigma_+ + \sigma_-)}\,[\,1 - \frac{h(y)}{h(0)}\,] \quad , \tag{A.22}$$

with h a solution of the homogeneous equation

$$\kappa h(y) + h'(y) = \kappa\sigma_+ h(y) + \kappa\sigma_- h(L-y) \quad . \tag{A.23}$$

To solve Eq. (A.23), we try an exponential Ansatz of the form

$$h(y) = e^{\kappa\sigma y} + \mu e^{-\kappa\sigma y} \quad , \tag{A.24}$$

which we find gives a solution when σ and μ are related to κ and $\sigma_{\pm}$ by

$$\sigma = [\,(1-\sigma_+)^2 - \sigma_-^2\,]^{\frac{1}{2}} \quad ,$$

$$\mu = \frac{\sigma + 1 - \sigma_+}{\sigma_-}\; e^{\kappa\sigma L} \; . \tag{A.25}$$

It is now a matter of simple algebra to combine Eqs. (A.13), (A.17), (A.22), (A.24) and (A.25) to give our final result for f, f_+ and f_- ,

$$f = \frac{e^{\kappa\sigma L}-1}{\kappa\sigma L}\;\frac{1+\mu e^{-\kappa\sigma L}}{1+\mu} \tag{A.26}$$

$$= f_+ + f_- \quad ,$$

$$f_+ = \frac{e^{\kappa\sigma L}-1}{\kappa\sigma L}\;\frac{\mu^2 e^{-\kappa\sigma L}-1}{\mu^2-1},$$
$$f_- = \frac{e^{\kappa\sigma L}-1}{\kappa\sigma L}\;\frac{\mu(1-e^{-\kappa\sigma L})}{\mu^2-1}. \tag{A. 27}$$

As a check on Eq. (A. 27), we consider the special case in which there is no backward scattering, i.e., $\mu_- = 0$. We find

$$f_- = 0,$$
$$f_+ = \frac{1-e^{-\kappa L(1-\lambda\mu_+)}}{\kappa L(1-\lambda\mu_+)} = \frac{1}{L}\int_0^L dy\; e^{-\kappa y(1-\lambda\mu_+)}, \tag{A. 28}$$

which is just the elementary exponential decay law appropriate to the case of forward propagation with effective absorption constant $\kappa(1-\lambda\mu_+)$, averaged over the length of the one-dimensional medium.

Appendix B. Comparison of Approximate and Exact Scattering Solutions for a Uniform Spherical Geometry

In this Appendix we calibrate the accuracy of the approximate scattering solution used in the text by comparing the approximate solution with the exact scattering solution in the case of a simple geometry. We consider a uniform sphere of radius R composed of material which scatters isotropically. Particles ("pions") are produced uniformly throughout the sphere and propagate with inverse interaction length κ. At each interaction the particles scatter isotropically, with the particle number being simultaneously multiplied by a factor λ. We wish to find the expected number f of particles eventually emerging from the sphere, normalized to unit integrated initial particle density. We discuss successively the exact solution, two approximate solutions, and the numerical comparison.

1. Exact Solution

The formulation of the solution to the spherical problem is closely analagous to the formulation of the one-dimensional problem in Appendix A, and we omit all details. Corresponding to Eqs. (A.13), (A.17) and (A.18) we find[23]

$$f = (1 - \frac{1}{\lambda}) < (1 - \lambda P)^{-1} >_{AV} + \frac{1}{\lambda}, \qquad \text{(B.1)}$$

$$< (1-\lambda P)^{-1} >_{AV} = 1 + \frac{1}{\frac{4}{3}\pi R^3} \int_{|\underset{\sim}{y}| \leq R} d^3y\, f(\underset{\sim}{y}), \qquad \text{(B.2)}$$

$$f(\underset{\sim}{y}) = g(\underset{\sim}{y}) + \int_{|\underset{\sim}{z}| \leq R} d^3z\, f(\underset{\sim}{z})\, \lambda \frac{\kappa}{4\pi} \frac{e^{-\kappa |\underset{\sim}{z} - \underset{\sim}{y}|}}{|\underset{\sim}{z} - \underset{\sim}{y}|^2} \quad , \qquad \text{(B.3)}$$

$$g(\underset{\sim}{y}) = \int_{|\underset{\sim}{z}|\leq R} d^3z\, \lambda\, \frac{\kappa}{4\pi}\, \frac{e^{-\kappa|\underset{\sim}{z}-\underset{\sim}{y}|}}{|\underset{\sim}{z}-\underset{\sim}{y}|^2} \quad .$$

After spherical-averaging the scattering kernel, scaling out the sphere radius R and expressing the solution of the integral equation in iterative form, we find

$$f = 1 + 3\,(1 - \frac{1}{\lambda}) \int_0^1 u^2\, du \sum_{n=1}^{\infty} \lambda^n g^{(n)}(\rho, u)\ ;$$

$$g^{(0)} = 1, \qquad \text{(B. 4)}$$

$$g^{(n)}(\rho, u) = \rho \int_0^1 \frac{1}{2}\, v dv\; g^{(n-1)}(\rho, v)\, \frac{1}{u}\, [\,E_1(\rho\,|v-u|) - E_1(\rho\,(v+u))\,]\ ,$$

$$E_1(x) = \int_x^{\infty} dt\, \frac{e^{-t}}{t};$$

$$\rho = \kappa R.$$

Since we are only interested in values of λ which are smaller than 1, the series in Eq. (B. 4) is convergent and f is readily calculated by repeated numerical integration.

2. Approximate Solutions

We recall that the approximate scattering solution used in the text is obtained by projecting all forward- and backward- hemisphere scattering respectively onto the forward- and backward- directions, solving the resulting one dimensional scattering problem as a function of optical thickness, and then integrating over the distribution of optical thickness actually present. For the spherical problem considered here substitution of Eqs. (A. 25) and (A. 26) into this recipe gives the following approximate formula for f ,

$$f^{(1)} = \int_0^2 \frac{3}{8} u^2 du\, \frac{e^{\rho\sigma u} - 1}{\rho\sigma u}\quad \frac{1 + \mu e^{-\rho\sigma u}}{1 + \mu}\ ;$$

$$\sigma = (1 - \lambda)^{\frac{1}{2}}\ , \qquad \text{(B. 5)}$$

$$\mu = \frac{\sigma + 1 - \frac{1}{2}\lambda}{\frac{1}{2}\lambda}\quad e^{\rho\sigma u},$$

which is readily evaluated by a single numerical integration. We also include in our comparison the scattering approximation used by Sternheim and Silbar, in which all scattering is projected onto the forward direction. In this case the relevant one-dimensional solution becomes the pure-forward-scattering solution of Eq. (A.28) and we find a second approximate formula for f,

$$f^{(2)} = \int_0^2 \frac{3}{8} u^2 du \, \frac{e^{\rho(\lambda-1)u} - 1}{\rho(\lambda-1)u} \quad . \tag{B.6}$$

3. Numerical Comparison

Numerical results for f, $f^{(1)}$ and $f^{(2)}$ are given in Table IX for a wide range of values of λ and ρ. Agreement between the exact result f and the approximation $f^{(1)}$ used in the text is excellent over the entire range of parameters. The approximation $f^{(2)}$ used by Sternheim and Silbar is qualitatively correct, but develops significant deviations from the exact answer for large values of ρ. To interpret the parameter ρ in terms of nuclear size, we note that for a uniform spherical nucleus of radius $R \sim 1.3\, A^{1/3}$ fermi, and an interaction cross section characteristic of the peak of the (3, 3) resonance ($\sigma_{max} \sim 210$ mb = 21 fermi^2), we have

$$\rho \sim \frac{A}{\frac{4}{3}\pi R^3} \, \frac{2}{3} \sigma_{max} R \sim 2\, A^{1/3} \sim \begin{matrix} 6 \text{ Aluminum} \\ 12 \text{ Lead} \end{matrix} \quad . \tag{B.7}$$

Hence for aluminum our simple forward-back approximation solves the multiple scattering problem to an accuracy of better than 1%; even for the heaviest nuclei the approximation (with appropriate modifications to take neutron excess into account) should be good to better than 3%.

Appendix C. Miscellaneous Formulas

We collect here the formulas for cross sections and Pauli factors used in the text.

1. Cross Sections

For $\sigma_{\pi^+p}(W)$ we use the simple form

$$\sigma_{\pi^+p}(W) = \sigma_{(3,3)}(W) + 20\text{ mb}, \tag{C.1}$$

with the first term the resonant cross section and the second term a constant approximation to the nonresonant background. (This formula slightly overestimates the cross section at and below the resonant peak, and underestimates it above resonance.) For $\sigma_{(3,3)}(W)$ we use the Roper parameterization,[24]

$$\sigma_{(3,3)}(W) = \sigma_{MAX}\left(\frac{q_r}{q}\right)^2 \frac{(\frac{1}{2}\Gamma)^2}{(q_0-q_{0r})^2 + (\frac{1}{2}\Gamma)^2}, \tag{C.2}$$

with

$$q_0 = \frac{W^2 - M_N^2 + M_\pi^2}{2W}, \quad q \equiv q(W) = (q_0^2 - M_\pi^2)^{\frac{1}{2}},$$

$$q_{0r} = 1.921\, M_\pi, \qquad q_r = 1.640\, M_\pi, \tag{C.3}$$

$$\Gamma = \frac{1.262\, q^3/M_\pi}{(q_0+q_{0r})(1 + .504\, q^2/M_\pi^2)},$$

$$\sigma_{MAX} = \frac{8\pi}{q_r^2} \approx 185\text{ mb}.$$

2. Pauli Factors

We calculate the Pauli factors in the approximation of treating the nucleus as a collection of independent protons and neutrons with equal Fermi-sea radii $R_n = R_p = R \approx 1.6\, M_\pi$. Then the fraction of nucleons which can contribute, for given momentum transfer $\underset{\sim}{\Delta}$ to the nucleons, is the fraction of the volume of a sphere of radius R centered at $\underset{\sim}{0}$ which lies outside a second sphere of radius R centered at $\underset{\sim}{\Delta}$. That is,[8]

$$h(W,\phi) = \begin{cases} \frac{3}{4}\eta - \frac{1}{16}\eta^3 & \eta \le 2 \\ 1 & \eta \ge 2\,, \end{cases} \tag{C.4}$$

with[25]

$$\eta = \frac{|\underset{\sim}{\Delta}|}{R} \approx \frac{2q}{R}\ \sin(\tfrac{1}{2}\phi). \tag{C.5}$$

Performing the integrations over ϕ in Eq. (41), we find

$$\left.\begin{aligned} h_+ &= \alpha\frac{1}{\sqrt{2}}\frac{59}{70} - \alpha^3\frac{1}{\sqrt{2}}\frac{29}{420} \\ h_- &= \alpha\,\frac{136-59/\sqrt{2}}{70} - \alpha^3\,\frac{176-29/\sqrt{2}}{420} \end{aligned}\right\}\ \alpha \le 1;$$

$$\left.\begin{aligned} h_+ &= \alpha\frac{1}{\sqrt{2}}\frac{59}{70} - \alpha^3\frac{1}{\sqrt{2}}\frac{29}{420} \\ h_- &= 2 - \frac{4}{5}\alpha^{-2} + \frac{18}{35}\alpha^{-4} - \frac{4}{21}\alpha^{-6} - \alpha\frac{1}{\sqrt{2}}\frac{59}{70} + \alpha^3\frac{1}{\sqrt{2}}\frac{29}{420} \end{aligned}\right\}\ 1 \le \alpha \le \sqrt{2};$$

$$\left.\begin{aligned} h_+ &= 1 - \frac{4}{5}\alpha^{-2} + \frac{18}{35}\alpha^{-4} - \frac{4}{21}\alpha^{-6} \\ h_- &= 1 \end{aligned}\right\}\ \sqrt{2} \le \alpha; \tag{C.6}$$

with

$$\alpha = q/R\,. \tag{C.7}$$

For the production Pauli factor $g(W,k^2)$ we use the expression[26]

$$k_0 = \frac{W^2 - M_N^2 - |k^2|}{2W},\quad k = [\,k_0^2 + |k^2|\,]^{1/2}\,,$$

$$g(W,k^2) = \frac{1}{2k}\left\{\frac{3k^2+q^2}{2R} - \frac{5k^4 + q^4 + 10\,k^2q^2}{40\,R^3}\right\},\quad k+q \le 2R; \tag{C.8}$$

$$g(W,k^2) = \frac{1}{4qk}\left\{(q+k)^2 - \frac{4}{5}R^2 - \frac{(k-q)^3}{2R} + \frac{(k-q)^5}{40\,R^3}\right\},\quad k-q \le 2R \le k+q;$$

$$g(W,k^2) = 1\,,\qquad 2R \le k-q.$$

Table I. Nuclear density parameters[a, b].

Nucleus $_Z T^A$	c	c_1	R fermi[c]	$R\rho(0)$ fermi^{-2}
$_5B^{10}$	1	0	2.45	.251
$_6C^{12}$	1.333	0	2.41	.268
$_7N^{14}$	1.667	0	2.46	.263
$_8O^{16}$	1.600	0	2.75	.247
$_{13}Al^{27}$	2.000	.667	1.76	.241

a. The data are taken from H. R. Collard, L. R. B. Elton and R. Hofstadter, in Landolt-Börnstein: Numerical Data and Functional Relationships; Nuclear Radii, edited by K.-H. Hellwege (Springer, Berlin, 1967), New Series, Group I, Vol. 2.

b. The density $\rho(r)$ is normalized so that $\int d^3r\, \rho(r) = A$.

c. For the first four nuclei, the r.m.s. charge radius is equal to R. For aluminum, the r.m.s. charge radius corresponding to the listed parameters is 2.91 fermi.

Table II. Calculations of R' $({}_{13}Al^{27})$ based on Eq. (15).

$\sin^2\theta_W$	$R(N_T)$ [a]	R' $({}_{13}Al^{27})$ No variations	Pauli factors $h \to 1$	σ_{ABS} $\times .5$	σ_{ABS} $\times 2$	Isoscalar multipoles omitted	Simulated Albright et. al. lower bound on R' $({}_{13}Al^{27})$
0	.697	.422	.396	.411	.435	.422	.408
.1	.573	.346	.325	.337	.356	.346	.321
.2	.465	.280	.264	.273	.289	.280	.245
.3	.374	.225	.212	.220	.232	.225	.179
.4	.300	.180	.170	.176	.186	.180	.123
.5	.242	.146	.138	.143	.150	.145	.078
.6	.200	.122	.115	.119	.125	.120	.043
.7	.175	.108	.102	.106	.110	.106	.019
.8	.166	.104	.099	.102	.107	.102	.004
.9	.174	.111	.106	.109	.113	.108	.000
1.0	.198	.128	.123	.126	.131	.125	.006

a. The numbers in this column are slightly smaller than those plotted in curve a of S. Adler, Phys. Rev. D (to be published) because we have reduced the axial-vector mass parameter [See Eq. (44)] from 1.0 to 0.9 GeV/c in the present calculation, and have also weighted the production cross sections according to the actual neutron/proton ratio in aluminum.

Table III. Charged pion to neutral pion ratio $\bar{\bar{r}}(\nu_\mu \mu^- T)$.

$\bar{\bar{r}}(\nu_\mu \mu^- \; n+p)$ pure (3, 3) approximation	$\bar{\bar{r}}(\nu_\mu \mu^- N_T)$ with $I=\frac{1}{2}$ corrections	$\bar{\bar{r}}'(\nu_\mu \mu^- \; {}_{13}Al^{27})$ from Eq. (15) and charged pion analog	$\bar{\bar{r}}'(\nu_\mu \mu^- \; {}_{13}Al^{27})$ from Eq. (48)
5	3.67	2.63	2.68

Table IV. Simulated $\bar{d}({}_{13}Al^{27};k^2)$ and $\bar{A}({}_{13}Al^{27};k^2)$ obtained from electroproduction and charge exchange correction models.

	$k_{10}^{L} = 2$ GeV/c		$k_{10}^{L} = 6$ GeV/c	
k^2 (GeV/c)^2	$\bar{d}({}_{13}Al^{27};k^2)$	$\bar{A}({}_{13}Al^{27};k^2)$	$\bar{d}({}_{13}Al^{27};k^2)$	$\bar{A}({}_{13}Al^{27};k^2)$
0	.191	.606	.188	.608
.1	.181	.688	.179	.682
.2	.169	.702	.167	.694
.3	.164	.702	.162	.694
.4	.160	.698	.158	.690
.6	.157	.694	.154	.684
.8	.156	.690	.152	.680
1.0	.155	.690	.150	.676
1.4	.155	.688	.147	.672
1.8	.156	.686	.145	.666

Table V. Test of first averaging approximation for $R'({}_{13}Al^{27})$.

$\sin^2\theta_W$	$R'({}_{13}Al^{27})$ From Eq. (23)	$R'({}_{13}Al^{27})$ From Eq. (15)
0	.433	.422
.1	.355	.346
.2	.288	.280
.3	.232	.225
.4	.186	.180
.5	.151	.146
.6	.127	.122
.7	.113	.108
.8	.109	.104
.9	.116	.111
1.0	.134	.128

Table VI. Test of second averaging approximation for $R'(_{13}Al^{27})$.

			$R'(_{13}Al^{27})$	
$\sin^2\theta_W$	$\bar{\bar{r}}(\nu_\mu \nu_\mu N_T)$	$\bar{\bar{r}}(\nu_\mu \mu^- N_T)$	From Eq. (24)	From Eq. (15)
0	.692	3.67	.432	.422
.1	.697	3.67	.356	.346
.2	.707	3.67	.289	.280
.3	.727	3.67	.234	.225
.4	.763	3.67	.189	.180
.5	.820	3.67	.154	.146
.6	.903	3.67	.129	.122
.7	1.01	3.67	.116	.108
.8	1.12	3.67	.112	.104
.9	1.19	3.67	.119	.111
1.0	1.22	3.67	.136	.128

Table VII. Averaged charge exchange parameters for various nuclei.

Nucleus	$\bar{\bar{a}}$	$\bar{\bar{c}}$	$\bar{\bar{d}}$
${}_5B^{10}$	.846	.0363	.125
${}_6C^{12}$	.811	.0450	.138
${}_7N^{14}$	.790	.0498	.144
${}_8O^{16}$	.807	.0460	.139
${}_{13}Al^{27}$	.724	.0642	.162

Table VIII. Effect of modifications of the model on $\bar{\bar{d}}({}_{13}Al^{27})$.

	No variations	Nucleon motion included	Pauli factors $h \rightarrow 1$	σ_{ABS} $\times .5$	σ_{ABS} $\times 2$
$\bar{\bar{d}}({}_{13}Al^{27})$	.162	.142	.187	.175	.145

Table IX. Comparison of exact and approximate multiple scattering solutions.

λ	ρ	f[Eq. (B.4)]	$f^{(1)}$[Eq. (B.5)]	$f^{(2)}$[Eq. (B.6)]
.5	.5	.827	.827	.835
	1	.687	.686	.707
	2	.489	.488	.527
	4	.290	.289	.332
	8	.154	.153	.182
	16	.0790	.0773	.0930
.6667	.5	.878	.877	.885
	1	.766	.764	.789
	2	.584	.583	.638
	4	.370	.368	.445
	8	.203	.200	.262
	16	.105	.102	.138
.8333	.5	.935	.934	.940
	1	.867	.865	.885
	2	.733	.731	.789
	4	.524	.522	.638
	8	.310	.307	.445
	16	.166	.161	.262
.9167	.5	.966	.966	.969
	1	.928	.927	.940
	2	.845	.842	.885
	4	.678	.676	.789
	8	.446	.443	.638
	16	.251	.244	.445

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6. We adhere to the notations of Ref. 5 wherever possible.
7. A detailed numerical calculation of the effect of Fermi motion on the production cross sections indicates a substantial broadening of the resonance and a simultaneous shift of the resonance center to lower excitation energies. Both effects increase with increasing k^2. The effective upper edge of the resonance, however, is not shifted and so an integration over experimental data from the effective threshold (which differs greatly from the threshold for pion production on nucleons at rest) to a fixed upper cutoff of

$$(k_0^L)_{MAX} = \frac{(1.47\ \mathrm{GeV})^2 - M_N^2 + |k^2|}{2M_N}$$

includes virtually the entire resonance. The area under the resonance obtained this way is essentially the same as the area obtained when Fermi motion is neglected. Hence we expect production Fermi motion effects to be relatively unimportant once the excitation energy has been integrated out, provided, of course, that one is not too close to a kinematic threshold for (3, 3)-resonance production.

8. S. M. Berman, in Proceedings of the International Conference on Theoretical Physics of Very High Energy Phenomena, CERN (1961).

9. This restriction is of course not necessary in principle. The extension of our multiple scattering model to take a neutron excess into account will be given elsewhere (S. L. Adler, to be published).

10. Since Eq. (15) involves an integration over excitation energy k_0^L, we expect the Fermi motion smearing of the production cross section to be relatively unimportant, and neglect it in the applications of Eq. (15) in Section 3.

11. The dominant resonant vector and axial-vector multipoles lead to different angular dependences of the production cross section.

12. Some acceptance dependence in $\bar{A}$ could be tolerated, since it would tend to cancel between the numerator and denominator of R'.

13. We are indebted to A. Kerman for a discussion about this point.

14. When Pauli effects are included, the forward peak is washed out, but the backward peak remains.

15. S. Weinberg, Phys. Rev. Letters 19, 1264 (1967); Phys. Rev. Letters 27, 1688 (1971); A. Salam, Elementary Particle Theory, edited by N. Svartholm (Almquist Forlay, Stockholm, 1968), p. 367; G. 't Hooft, Nuclear Physics B35, 167 (1971).

16. This is strictly true only when $N_T=Z(p+n)$, whereas in the production calculation we have used $N_T= 13p + 14n$. The numerical effect of this change is small.

17. C. H. Albright, B. W. Lee, E. A. Paschos and L. Wolfenstein, Phys. Rev. D7, 2220 (1973). Here G, θ_C and α denote, respectively, the Fermi constant, Cabibbo angle and fine structure constant.

18. The ratio 3.67 also includes the (small) effect of taking account of the actual n/p ratio in aluminum.

19. The corresponding prediction for incident antineutrinos is 2.32.

20. S. L. Adler, Ann. Phys. 50, 189 (1968), Eq. (4E.7).

21. Qualitatively, both nucleon Fermi motion and the deviations of the scattering angular distribution from pure "forward-back" would be expected to produce an angular smearing of the result of Eq. (51).

22. For a nice pedagogical discussion of one-dimensional multiple scattering, see G.M. Wing, An Introduction to Transport Theory (Wiley, New York, 1962).

23. The methods leading to these equations are discussed in K. M. Case and P. F. Zweifel, Linear Transport Theory (Addison-Wesley, Reading, Mass., 1967). See especially Section 3.6.

24. L. D. Roper, Phys. Rev. Letters 12, 340 (1960).

25. We have approximated $\underset{\sim}{\Delta}$ by the isobar frame momentum transfer. The approximation is bad only when η is so large that $h = 1$.

26. Eq. (C.8) is obtained from Eq. (6C.6) of Ref. 18.