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DO LIGHT PIONS NECESSARILY OBEY PCAC?*

Roger Dashen†
Institute for Advanced Study, Princeton, New Jersey 08540
and
National Accelerator Laboratory, Batavia, Illinois 60510

and

David J. Gross†
Joseph Henry Laboratories, Princeton University,
Princeton, New Jersey 08540
and
National Accelerator Laboratory, Batavia, Illinois 60510

Abstract

We show that contrary to recent arguments unitarity does not force massless pions to obey the Adler condition. A specific multiperipheral model serves as a counterexample.

†Alfred P. Sloan Foundation Research Fellow.

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I. INTRODUCTION

In a recent paper Rosenzweig and Veneziano[1] (hereafter referred to as RV) have argued that the Adler condition on pion amplitudes can be derived from the general principles of S-matrix theory. More specifically they argue that a certain inclusive sum rule will be violated as $m_\pi \rightarrow 0$ unless the Adler condition is satisfied in π-π scattering. If we accept their argument, then we are in the remarkable position that most of the soft pion theorems of PCAC and current algebra or, equivalently, chiral symmetry have been shown to follow from the general principles of S-matrix theory. Some time ago, Mandelstam[2] showed that the Adler condition alone is sufficient to obtain most results of current algebra.

From a field theoretic point of view it would be very surprising to have a "first principles", model independent derivation of the Adler condition. The pion in the σ-model formally satisfies the Adler condition. However, a theory based only on pions interacting through a $\lambda\pi^4$ coupling does not.

In addition to assuming the general principles of S-matrix theory, RV make a specific dynamical assumption of pole dominance. Although at first sight their assumption appears harmless, we will argue that it is not in general justified. In particular we will show that there is a simple multiperipheral model which serves as a counter-example to RV's assertions. Thus, it appears that we still do not have a model independent derivation of the Adler condition. However, it is still of considerable interest that an S-matrix assumption of pole dominance seems to be equivalent to PCAC.

The paper is organized as follows. In the next section we discuss some general properties of massless particles and infrared divergences. Then we discuss RV's calculation and our objection to their pole dominance assertion. The multiperipheral model mentioned above is developed in the last section and an appendix.

II. MASSLESS PARTICLES AND INFRARED PHENOMENA

The phenomenon discovered by RV is not unrelated to the infrared divergences which are well known to appear in theories with massless particles. Before discussing the specific calculation of RV it is therefore useful to look at some general features of the infrared problem.

Consider a theory which contains a boson of mass μ and let $A(k_1 \ldots k_n)$ be an on-shell scattering amplitude calculated with this theory. If we let μ tend to zero, one of two situations will occur: either

(i) As $\mu \to 0$ A will diverge for all values of the external momenta $k_1 \ldots k_n$

or

(ii) For general momenta $k_1 \ldots k_n$ A will remain finite. It may however be singular in μ for certain exceptional momenta.

Ordinary quantum electrodynamics is an example of case (i). There, as is well known, all the scattering amplitudes blow up as the photon mass goes to zero. We might call this situation a true infrared divergence. In this case the S-matrix ceases to exist in the limit $\mu \to 0$.

If we ignore wave function renormalization[3] (which in any case does not appear in the S-matrix), a massless pion will give rise to only the milder divergence of type (ii). Consider a theory of massive nucleons interacting with massless pions through the coupling $g\bar{N}\gamma_5 N\pi$. By looking at a few Feynman diagrams one can easily convince himself that for general values of the external momenta there is no infrared divergence in the S-matrix. The reason is that the nucleons can emit the pseudoscalar pions only in P-waves (as opposed to S-waves for photons). Therefore, the amplitude for a nucleon to emit a soft pion vanishes kinematically and there are, in general, no infrared infinities. We can further allow the pions to interact among themselves through a $\lambda\pi^4$ coupling. There is still no true infrared divergence because a pion cannot emit a single soft pion but rather must emit a pair. Two particle phase space keeps the pair emission from being infrared divergent. This is not to say, however, that this theory has no infrared infinities. For exceptional values of the external four momenta, the S-matrix is infrared divergent. As an example consider the π-π scattering diagram in Fig. (1). It is (infrared) finite everywhere except in the forward direction. The forward amplitude does, however, blow up for massless pions. By unitarity an infinite forward amplitude should imply an infinite π-π cross section. That the π-π cross section is, in fact, divergent is easily seen by squaring the diagram in Fig. (2) and integrating over phase space. The physics behind this infinite cross section is the same as that in Coulomb scattering. In the presence of a long range force cross sections computed with incoming particles in plane wave states (rather than wave packets) are infinite.[4]

The above remarks, although based on observations about Feynman diagrams in a particular model, are probably general. If there were a massless pseudoscalar hadron, the major qualitative effect would be that the S-matrix could become infinite in certain kinematic configurations.[3] These infinities would be the reflection, through unitarity, of infinite (although physically sensible) cross sections.

Massless pions which obey the Adler condition (as the real ones seem to) would not lead to <u>any</u> infrared singularities. In particular all cross sections would be finite. This is an essentially trivial consequence of the observation that the Adler condition makes soft pions decouple from everything. In fact, the Adler condition is probably equivalent to demanding that all hadron cross sections remain finite as $m_\pi \to 0$. But, since there is not anything unphysical about some cross-sections becoming infinite, one can hardly use this remark to "derive" the Adler condition.

Finally, what has been said in this section is based on knowledge derived from finite orders in perturbation theory. The argument of RV uses full non-linear unitarity, which of course is not perturbative. Thus, at this point we might still worry that some peculiar, distinctly non-perturbative, infrared phenomenon exists. Later we will argue, by means of a specific model, that there is no evidence for such an occurrance.

III. THE UNITARITY ARGUMENT OF ROSENZWEIG AND VENEZIANO

In this section we discuss the unitarity argument of RV. First we give a simplified, although somewhat imprecise, derivation of their principle result. We then point out where they make a physically questionable assumption about pole dominance. Finally, we briefly describe a multiperipheral model which disagrees with their result but is otherwise quite reasonable. The details of the model are presented in the following section and in the appendix.

The argument of RV can be paraphrased as follows. Consider that part of the π-π total cross section which comes from one of the pions fragmenting into a pion pair with mass s', while the other pion turns into a massive "nova" of mass M. The kinematics are shown in Fig. (3). We will consider only that region of phase space in which s' and t are small ($\lesssim 1$ GeV) while s and M are large ($>> 1$ GeV). Because t is small in this region, it may be reasonable to assume that the process is dominated by one pion exchange as shown in Fig. (4). In fact, RV argue that as $m_\pi \to 0$ pole dominance becomes an increasingly good approximation. Let us tentatively assume pole dominance and see what happens.

Clearly the total π-π cross section is greater than this partial cross-section integrated over our limited region of phase space. Some straightforward but tedious kinematics then leads to

$$\sigma_{\pi\pi}(s) > C\iiint_R \left(\frac{\Delta(t, m_\pi^2, M^2)}{\Delta^2(m_\pi^2, m_\pi^2, s)}\right)^{\frac{1}{2}} \left(1 - \frac{4m_\pi^2}{s'}\right)^{\frac{1}{2}} \sigma_{\pi\pi}(M^2) H(s') \frac{1}{(t-m_\pi^2)^2} dt\, ds'\, dM^2 \qquad (1)$$

where

Δ = usual triangle function

$\sigma_{\pi\pi}$ = π-π total cross section

C = numerical constant

$H(s') = \int |A(s', \cos\theta)|^2 d\cos\theta$

$A(s', \cos\theta)$ = Elastic π-π scattering amplitude

R = region of phase space defined by $-t_0 < t < t_{min}$, t_0 = fixed number $\lesssim 1(\mathrm{GeV})^2$,

$4m_\pi^2 < s' < s'_0$, s'_0 = fixed number $\lesssim 1(\mathrm{GeV})^2$

$x_0 s < M^2 < x_1 s$, $0 < x_0 < x_1 < 1$ are fixed numbers .

Here the term "fixed numbers" means something which remains fixed and finite in a limit, soon to be taken, where $s \to \infty$ and $m_\pi \to 0$.

Note that the derivation of Eq. (1) does not really depend on unitarity in the sense that the equation $\mathrm{Im}T = TT^\dagger$ was not used. Actually all that went into Eq. (1) was the fact that the total cross section is a sum of positive partial cross sections, plus the assumption of pole dominance. Some readers may be worried about double counting. That is, there may be some further low energy dipions in the "nova" which could confuse matters. In RV's work this difficulty is avoided through the use of momentum conservation sum rules for inclusive distributions. The difference between RV's more precise approach and ours is not significant for the problem at hand.

To proceed further, it is convenient to assume that

$$\sigma_{\pi\pi}(s) \sim s^\alpha (\log s)^\beta \tag{2}$$

as $s \to \infty$. Then dividing through by $\sigma_{\pi\pi}(s)$, setting $M^2 = xs$ and taking the limit $s \to \infty$ yields

$$1 > C \int_{x_0}^{x_1} \int_{4m_\pi^2}^{s_0'} \int_{-t_0}^{t_{min}} x^{\alpha+1} \left(1 - \frac{4m_\pi^2}{s'}\right)^{\frac{1}{2}} H(s') \frac{1}{(t-m_\pi^2)^2} \, dx ds' dt \tag{3}$$

where in this limit t_{min} has become

$$t_{min} = -x\left(\frac{s'}{1-x} - m_\pi^2\right) \quad . \tag{4}$$

The t integral is easily done, but we have to remember that the actual upper limit on s' is the smaller of s_0' and the value of s' for which $t_{min} = -t_0$. Doing this and introducing a new integration variable $y = s'/m_\pi^2$ yields

$$1 < C \int_{x_0}^{x_1} \int_4^{y_0} x^{\alpha+1} \left(1 - \frac{4}{y}\right)^{\frac{1}{2}} H(m_\pi^2 y) \left[\frac{1}{-(1-x) + \frac{xy}{1-x}} - \frac{1}{\frac{t_0}{m_\pi^2} + 1} \right] dx dy \tag{5}$$

$$y_0 = \min \left\{ s_0'/m_\pi^2, \frac{t_0(1-x)}{m_\pi^2 x} - (1-x) \right\} .$$

Let us now consider the limit $m_\pi \to 0$ with s_0' and t_0 fixed. In this limit y_0 tends to infinity and the result is

$$1 > (\text{const}) \; H(0) \; \log(\infty) \tag{6}$$

which is clearly a contradiction unless $H(0)$ vanishes. One can easily convince himself that for scattering of massless pions, the Adler condition is precisely the statement that $H(0) = 0$. This is RV's main result.

It is perhaps not surprising that, because of infrared problems, something funny happens when m_π is actually zero. The peculiar thing about Eq. (6) is that log (∞) is really log (m_π^{-2}) which means that the inequality will be violated for some small but finite pion mass unless the Adler condition is satisfied. Hence if we accept the assumptions which lead to Eq. (1) then the Adler condition for π-π scattering has been derived from "first principles".

Recalling that the basic ingredients of Eq. (1) were (i) the fact that the total cross section is a sum of positive partial cross sections and (ii) pole dominance in the region of phase space under consideration, it is evident that (i) is sacred but (ii) could be questioned. In this regard, it is important to keep in mind that we have assumed the pion pole dominates all the way from t_{min} out to $-t_0$ where t_0 is independent of m_π. A more conservative assumption would be that only those t-singularities with masses less than t_0 should count. As $m_\pi \to 0$ this would include not only the pion pole, but three pion effects, five pion effects and so on. There is no a-priori guarantee that the effect of this infinite number of multi-pion singularities is not at least as strong as that of the pole.

We will now proceed to argue that the pole does not in general dominate. Let us agree to call the square of the blob in the lower half of Fig. (4) the "off-shell cross section" $\sigma_{\pi\pi}(t, m^2)$ for a virtual pion of mass t incident on a real pion. In this language, the pole dominance assumption is equivalent to approximating $\sigma_{\pi\pi}(t, m^2)$ by the on shell cross section $\sigma_{\pi\pi}(m_\pi^2, M^2) \equiv \sigma_{\pi\pi}(M^2)$ as was done to obtain Eq. (1). In Fig. (5) we show some particular contributions to $\sigma_{\pi\pi}(t, M^2)$. When the pion mass is small so that the emitted pion pairs can carry off almost zero momenta, the singularities from the repeated pion exchange poles can pile up near $t = 0$, providing just the sort of multi pion effects alluded to above. Technically, these multiple pion poles can give $\sigma(t, M^2)$ a strong t-dependence. In fact, as we shall see, any damping of $\sigma(t, M^2)$ for $\frac{t}{M^2} \gg 1$ is sufficient to eliminate the logarithmic divergence in Eq. (5).

Since the diagrams shown in Fig. (6) are summed by the multiperipheral equation their properties can be studied in detail. Their presence does, in fact, invalidate the pole dominance assumption. Before going into this however, we should justify keeping the multiperipheral diagrams while ignoring other multi-pion effects. This is not hard. It is quite clear that RV's log m_π divergence is symptomatic of an infrared problem arising from the long range force associated with a massless pion. Now as $m_\pi \to 0$ the longest range contribution to the π-π cross section comes from those diagrams which have the largest number of single pion exchanges. These are, of course, just the multiperipheral diagrams. Also, the pairs emitted along the chain must carry off very small momenta if the force is to remain long ranged. Therefore, we can approximate the π-π elastic amplitude by constants as far as the leading long range terms are concerned. Thus a simple multiperipheral model based on pions interacting through a $\lambda\pi^4$ coupling should be an adequate guide to the leading behavior of pion amplitudes as $m_\pi \to 0$.

We have therefore studied a model in which $\sigma_{\pi\pi}$ is given by the multiperipheral chain shown in Fig. (6). The elastic $\pi\pi$ amplitudes are taken to be constants. Clearly the Adler condition is violated unless our constant π-π amplitude vanishes identically. The model does, however, satisfy the limited "unitarity" requirement that the total cross section is the sum of positive partial cross sections. Furthermore, one can obviously separate out a pion exchange pole as in Fig. (4). Thus the model satisfies the "sacred" assumptions of RV. However, for precisely the reasons alluded to above the pole does not dominate. There is nothing the matter with the model for any finite value of m_π. In particular, the total cross section as calculated in the model satisfies the Froissart bound[5] if the coupling is not too strong. When $m_\pi = 0$, the cross section blows up, but this is to be expected according to our general remarks about long range interactions.

In the next section, we study the multiperipheral equation in detail. To close this section, we remark that in a world where the Adler condition is satisfied, the assumption of pole dominance probably would be justified. If the Adler condition is satisfied the multiple emission of soft pions which leads to the t-dependence of $\sigma(t, M^2)$ would be surpressed. Thus, in the real world pole dominance may be a good approximation. However, to derive the Adler condition from pole dominance seems to be a circular argument. But, as mentioned before, it is quite interesting that an S-matrix "axiom" of pion pole dominance appears to be equivalent to a field theoretic "axiom" of PCAC.

IV. A MULTIPERIPHERAL MODEL

In the previous section we argued that as the pion mass vanishes the most singular contributions to $\sigma_{\pi\pi}$ are given by a simple multiperipheral model based on pions interacting through a $\lambda\pi^4$ coupling. In this section we shall study this model, in which $\sigma_{\pi\pi}$ is given by the multiperipheral chain shown in Fig. (6).

Since we are ultimately interested in letting $m_\pi \to 0$ we first set the mass of the produced pions equal to zero, while keeping the mass of the exchanged pions non zero. This majorizes the ~~c~~ cross section with finite produced pion mass, since the kernal of the integral equation that generates the multiperipheral graphs is proportional to the 2-body phase space of the produced pions which is increased as their mass vanishes. Of course setting the produced pion masses equal to zero does not cause $\sigma_{\pi\pi}$ to diverge since the range of the interaction remains finite.

One might argue that it is unrealistic to assume a pointlike $\lambda\pi^4$ coupling for an infinite range of $\pi\pi$ subenergies. However as the produced pion mass vanishes this objection becomes irrelevent. This is because the most singular part of $\sigma_{\pi\pi}$ will be determined by the π^4 amplitude near zero energy. As long as this amplitude is approximately constant from threshold to some (m_π-independent) energy Λ_0; one might as well take t_0 to be infinite, since the singular part of $\sigma_{\pi\pi}$ will only depend on the ratio of t_0 to the mass of the produced π's.

These approximations which we believe justified as $m_\pi \to 0$, are certainly sufficient to provide a counter-example to the "derivation" of the Adler condition. In addition they allow us to derive an explicit functional form for the total cross section with one pion off shell. This model was extensively studied about a decade ago as an example of a singular Bethe-Salpeter equation in the ladder approximation. The fundamental paper is that of Ref. 6 which dealt mainly with the bound state problem. We found the paper by Swift and Lee[7] (hereafter referred to as SL), which applied these methods to the scattering problem most useful. In the appendix we outline the solution of SL for the ladder graphs of the $\lambda\pi^4$ theory with massless bubbles--whose absorptive part is our multiperipheral model.

In this model the total $\pi\pi$ cross section with one pion off shell (mass t) for large energy s is given by:

$$\sigma_{\pi\pi}(t,s)_{\frac{s}{m_\pi^2}>>1} \approx \text{const.}\left(\frac{s}{m_\pi^2}\right)^{\nu_c-2} \ell n^{-3/2}\left(\frac{s}{m_\pi^2}\right) G\left(\frac{t}{m_\pi^2}\right) \tag{7}$$

$$G(z) \; \frac{\lambda}{16\pi^2\nu_c(\nu_c+1)} B\left(\frac{\nu_c-1}{2}, \frac{\nu_c-1}{2}\right) {}_2F_1\left(\frac{\nu_c-1}{2}, \frac{\nu_c-1}{2}; \nu_c+1, z\right) \tag{8}$$

$$\nu_c = \left(1+\frac{|\lambda|}{4\pi^2}\right)^{\frac{1}{2}} > 1 \quad . \tag{9}$$

The asymptotic behavior of this model is controlled by a fixed cut in angular momentum at $J = \nu_c - 1$. This leading singularity always lies above zero and for small enough coupling lies below one. Thus we can easily satisfy the Froissart bound in this model. The value of the cross section depends strongly on the off shell pion mass, t, for values of $|t|$ large compared with m_π^2. This dependence is contained in $G(z)$, which is normalized so that $G(1) = 1$; and arises due to the multi-pion thresholds in t, which have all accumulated at $t = m_\pi^2$ (since we have set the produced pion mass equal to zero). These singularities combine to damp $\sigma_{\pi\pi}(t, s)$ when $\frac{-t}{m\pi^2} >> 1$, since

$$\frac{\sigma_{\pi\pi}(t, s)}{\sigma_{\pi\pi}(m_\pi^2, s)} = G\left(\frac{t}{m_\pi^2}\right) \underset{\frac{-t}{m_\pi^2}>>1}{\approx} \left(\frac{m_\pi^2}{-t}\right)^{\frac{\nu_c - 1}{2}} \left(\frac{\nu_c - 1}{\nu_c + 1}\right) . \tag{10}$$

This factor is sufficient to remove the logarithmic divergence found by RV who made the approx $G\left(\frac{t}{m_\pi^2}\right) \approx G(1) = 1$. In fact if we insert $G\left(\frac{t}{m_\pi^2}\right)$ into Eq. (3), change variables to $y = \frac{s'}{m\pi^2}$, $z = \frac{t}{m\pi^2}$ and take the limit $m_\pi \to 0$ we have:

$$1 > C\ H(0) \int_{x_0}^{x_1} dx \int_y^\infty dy \int_{-\infty}^{-x\left(\frac{y}{1-x} - 1\right)} dz\ x^{\nu_c - 1} \sqrt{1 - \frac{4}{y}}\ \frac{G(z)}{(z-1)^2} . \tag{11}$$

The right hand side of (11) is certainly convergent with $G(z)$ given by (8), and one no longer concludes that $H(0) = 0$.

In conclusion this simple model illustrates that the one pion exchange pole is a poor approximation for values of the momentum transfer large compared to the pion mass, and that the combination of the multipion cuts can provide a convergence factor for large $\left|\frac{t}{m\pi^2}\right|$ that eliminates the logarithmic divergence of RV.

Acknowledgement

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References

1. C. Rosenzweig and G. Veneziano, Center of Theoretical Physics (MIT) publication 283.

2. S. Mandelstam, Phys. Rev. 165, 1884 (1968).

3. For massless pions in perturbation theory, the only type (i) infrared divergences come from self energy parts inserted on external lines. These are removed by multiplying by Z_2 when the on-shell S-matrix is constructed. The on-shell S-matrix is therefore formally finite except at exceptional values of the external momenta. However, there is an infrared difficulty in the pion propagator. It might be difficult to interpret a field theory of massless pions with a non-vanishing zero-energy π-π amplitude. We have not studied this question in any detail.

4. One usually thinks of very long range forces as being of particular importance in elastic scattering but not in inelastic processes such as that in Fig. (2). The physical interpretation of the infinite cross section associated with Fig. (2) is as follows. The phase space for a massless scalar particle to decay into two co-linear massless scalars is finite. Therefore, a massless scalar can decay into two others in free space. A pseudoscalar pion cannot, however, because of parity. Two pions, on the other hand, can exchange a pion and then both decay into co-linear pairs. The only role of the long range force (massless

pion exchange) is to carry off the odd parity.

5. We point this out because in their original derivation RV use the Froissart bound rather than the more restrictive assumption of Eq. (2). Thus, in addition to positivity, our multiperipheral model satisfies the high energy assumptions of RV.

6. A. Bastai, L. Bertocchi, S. Fubini, G. Furlan and M. Tonin Nuovo Cimento 30, 1512; 1532 (1963).

7. A. R. Swift and B. Lee, Journal of Math. Physics 5, 908 (1964).

Figure Captions

Fig. 1 A diagram for π-π scattering which has an infrared singularity in the forward direction, but only there.

Fig. 2 A diagram for $2\pi \to 4\pi$ which leads to an infinite total cross section for $\pi\pi$ scattering.

Fig. 3 Kinematics for $\pi\pi \to 2\pi$ + anything.

Fig. 4 The one pion exchange approximation for the amplitude in Fig. (4).

Fig. 5 Multiple soft pion emission processes leading to a strong t-dependence in $\sigma_{\pi\pi}(t, M^2)$, the square of the lower half of the diagram in Fig. (5).

Fig. 6 A multiperipheral model for $\sigma_{\pi\pi}$.

APPENDIX

In this appendix we shall outline the solution to the multiperipheral model discussed in Section IV. This solution is contained in Ref. 6, however we present it here for completeness and because SL do not actually work out the explicit form for the off shell π-π scattering amplitude.

Consider the forward scattering of an on shell pion of momenta p of an off shell pion of momentum q. The scattering amplitude, in our model, is given by the equation:

$$A(p,q) = A_0(p,q) + \int \frac{d^4k}{(2\pi)^4} \frac{A(p,k)\,V(k-q)}{(k^2-m_\pi^2)^2} \tag{12}$$

where $A_0(p,q) = \frac{1}{(2\pi)^4} V(p+q)$ and $V(p)$ is simply the basic massless bubble diagram. It can be taken to be the Fourier transform of a potential:

$$V(p) = \int d^4x\, e^{ip\cdot x} V(x) = \frac{(i\lambda)^2}{(2\pi)^4} \int \frac{d^4k}{(k^2)(k-p)^2} \quad . \tag{13}$$

It is useful to rewrite (12) as a differential equation by defining:

$$V(x)\ \Phi(x) = \int d^4q\ e^{iqx} A(p,q) \ . \tag{14}$$

One then derives for $\phi(x)$ the differential equation:

$$(\Box^2+m^2)^2\,[\,\Phi(x) - e^{ip\cdot x}\,] = V(x)\ \Phi(x) \tag{15}$$

which, if solved, gives $A(p,q)$ using:

$$A(p,q) = \frac{1}{(2\pi)^4} \int d^4x\ e^{-iq\cdot x} V(x)\Phi(x) \ . \tag{16}$$

One now performs a Wick rotation to the Euclidean metric, and expands $\Phi(x)$ in four dimensional spherical harmonics ($r = |x|$) $\Phi(x) = \sum_{n\ell m} \Phi_{n\ell m}(r)\, y^{m}_{\ell n}(\Omega)$. The potential is simply:

$$V(x) = \frac{\lambda^2}{(2\pi)^4} \frac{1}{r^4} \equiv \frac{a}{r^4} . \tag{17}$$

It then turns out[6, 7] that $\phi_{n\ell m} = \phi_\nu$ ($\nu = n+1$) satisfies the equation:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - m^2 - \frac{\nu^2}{r^2}\right] r\Phi_\nu(r) \equiv (O_\nu - m^2)^2\, r\phi_\nu = V(r) r\Phi_\nu . \tag{18}$$

Using the identity:

$$\begin{aligned}
(O_\nu - m^2)^2 - \frac{a}{r^4} &= \frac{1}{r}(O_{\nu_1} - m^2) r^2 (O_{\nu_2} - m^2)\frac{1}{r} \\
\nu_1^2 + \nu_2^2 &= 2(\nu^2+1) \\
\nu_1^2 \nu_2^2 &= (\nu^2-1)^2 - a
\end{aligned} \tag{19}$$

$$\begin{matrix}\nu_1 \\ \nu_2\end{matrix} = \bar{\nu} \mp w = \sqrt{\frac{1+\nu^2+\sqrt{(1-\nu^2)^2-a}}{2}} \mp \sqrt{1+\nu^2-\sqrt{(1-\nu^2)^2-a}} .$$

It follows that the solutions of (18) are $I_{\nu_i}(mr)$ or $K_{\nu_i}(mr)$. Imposing the boundary conditions one derives that:

$$\phi_\nu(r) = \frac{(2\pi)^2 y^{m}_{\ell n}(\Omega_p)^*}{2w\bar{\nu}} [\, I_{\nu_1}(mr) - I_{\nu_2}(mr)] . \tag{20}$$

We are now in a position to calculate $A(p, q)$. We simply insert (20) into (16) and carry out the integral. The result is that

$$A(p,q) = \frac{1}{16\pi^2} \sum_{\nu=1}^{\infty} C^{1}_{\nu-1}\left(\frac{p\cdot q}{m\sqrt{q^2}}\right) \frac{\nu a}{\bar{\nu} w} \left(\frac{q^2}{m^2}\right)^{\frac{\nu-1}{2}} \times$$

$$\left\{ e^{-\frac{i\pi}{2}(\nu+\nu_1)} \frac{\Gamma(\frac{\nu+\nu_1-1}{2})}{\Gamma(\frac{-\nu+\nu_1+3}{2})\Gamma(\nu+1)} {}_2F_1\left(\frac{\nu+\nu_1-1}{2}, \frac{\nu-\nu_1-1}{2}; \nu+1, \frac{q^2}{m^2}\right) - (\nu_1 \leftrightarrow \nu_2) \right\} \tag{21}$$

We are interested in the absorbtive part of $A(p, q)$ for large energies, i.e. $\frac{p \cdot q}{m^2} >> 1$. This can be calculated by performing a Watson-Sommerfeld transformation on (21) and keeping the leading singularity in the ν-plane. It is easily seen that this is located at $\nu = \nu_c = \sqrt{1 + \sqrt{a}}$; at which point ν_2 is analytic and $\nu_1 \approx \sqrt{\frac{8\nu_c(\nu_c^2-1)}{\nu_c^2 + 1}(\nu-\nu_c)}$. The result is Eq. (7).

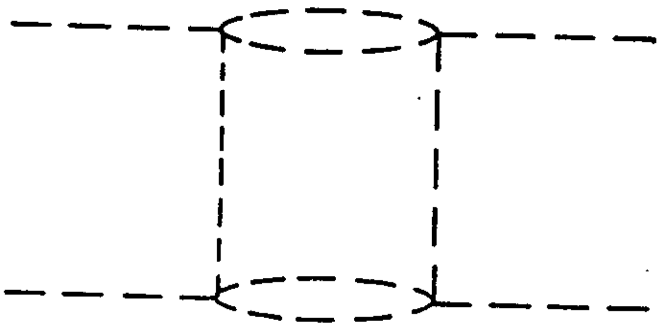


Fig. 1

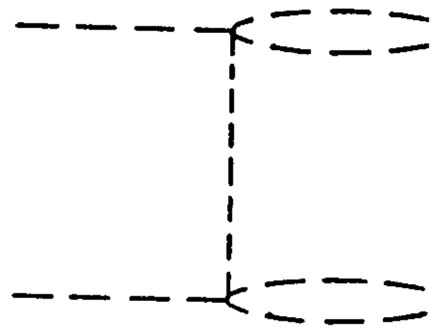


Fig. 2

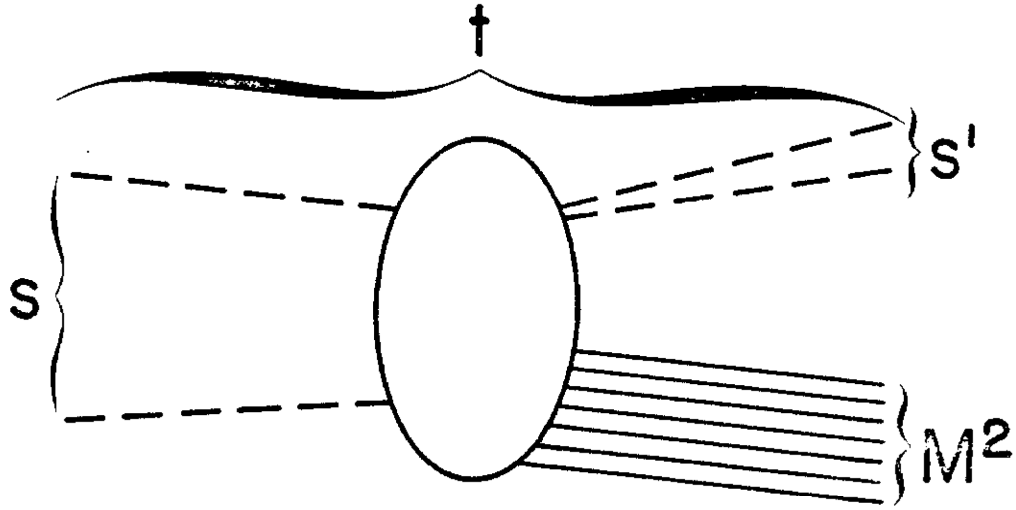


Fig. 3

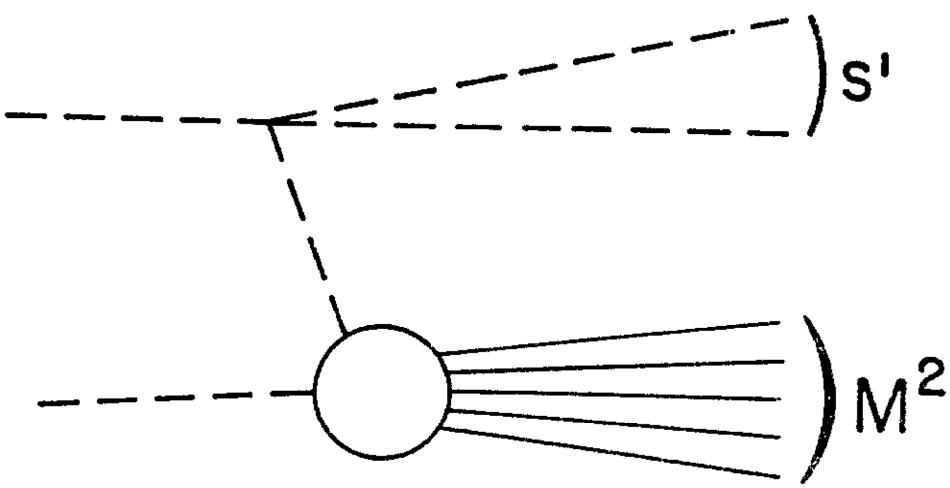


Fig. 4

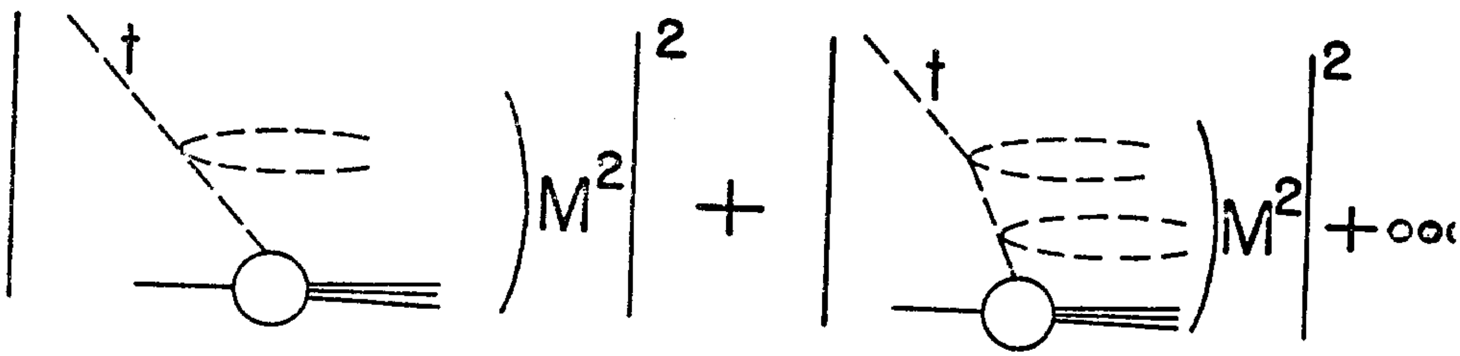


Fig. 5

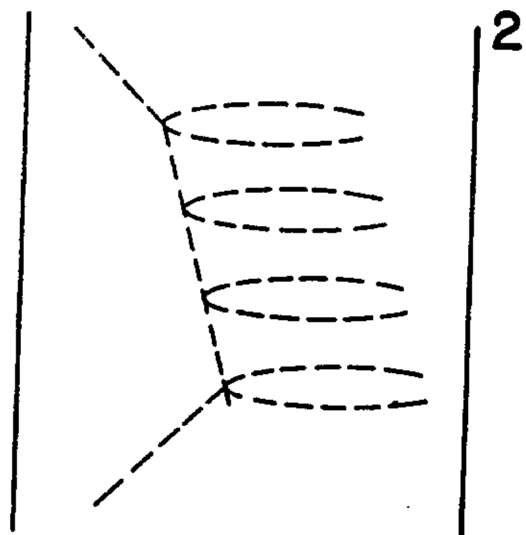
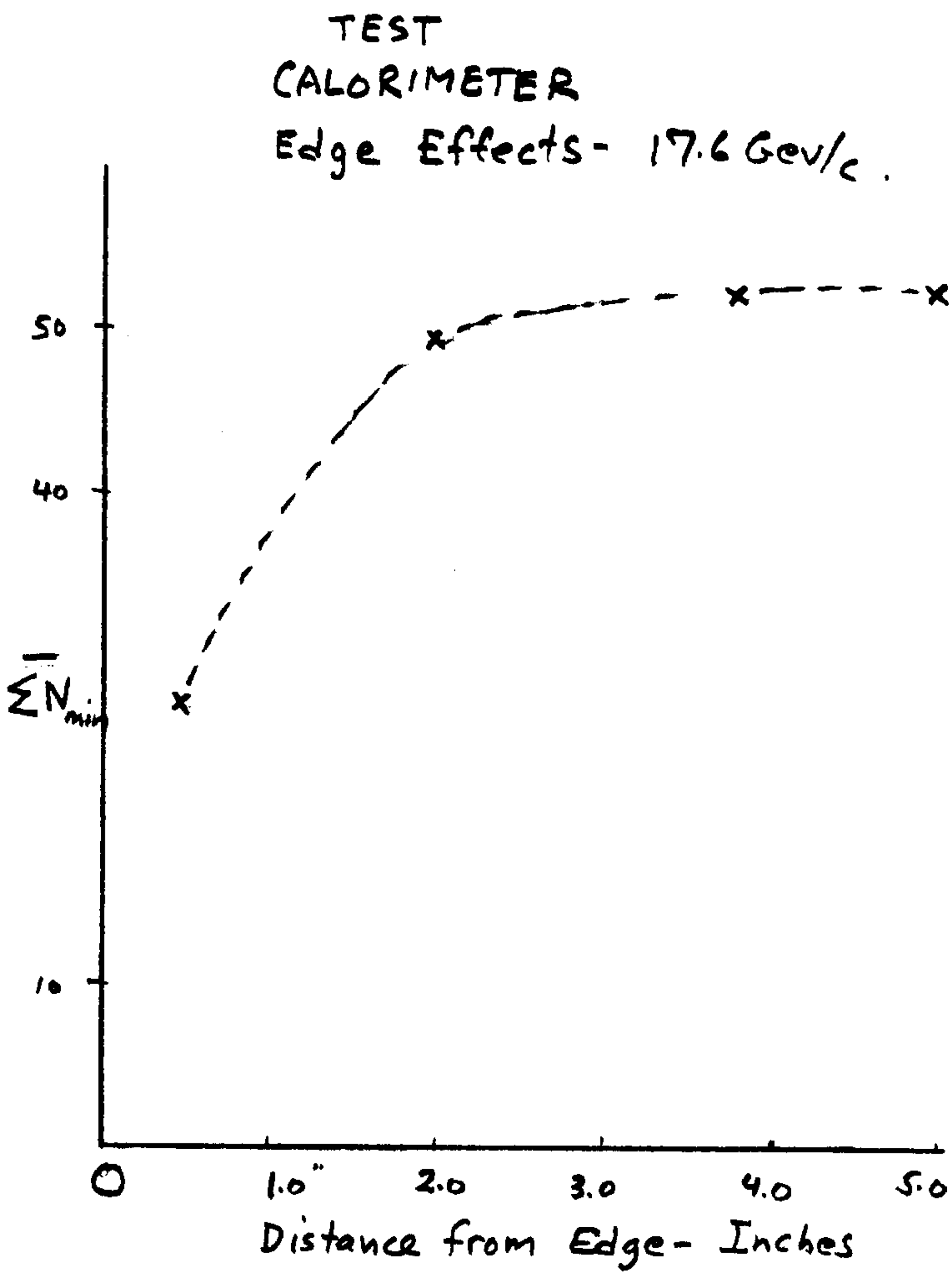


Fig. 6

Figure 13-8



TEST
CALORIMETER
Edge Effects - 17.6 Gev/c.
50
40
10
$\overline{\Sigma N_{min}}$
0
1.0"
2.0
3.0
4.0
5.0
Distance from Edge - Inches