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PART I. CONFORMAL INVARIANCE AND SHORT-DISTANCE BEHAVIOR OF FIELD THEORIES

# INTRODUCTION TO CONFORMAL INVARIANCE

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## THE MECHANICS OF CONFORMAL INVARIANCE

Every canonical field theory has associated with it an energy momentum tensor having the following properties:

$$\theta_{\mu\nu}(x) = \theta_{\mu\nu}(x), \frac{\partial}{\partial x_\nu}\theta_{\mu\nu} = D \qquad (1)$$

This object is always a local function of the underlying fields of the theory and 1 allows one to construct additional conserved tensors:

$$\frac{\partial}{\partial x_\lambda} M_{\lambda\mu\nu}(x) = 0, M_{\lambda\mu\nu} = x_\mu\theta_{\lambda\nu} - x_\nu\theta_{\lambda\mu} \qquad (2)$$

The two conservation laws then lead to 10 conserved integrals of the motion

$$P_\mu = \int d\bar{x}\, \theta_{0\mu}(t, \bar{x}), M_{\mu\nu} = \int d\bar{x}\, M_{0\mu\nu}(t, \bar{x})$$

which correspond precisely to the 10 generators of the Poincaré group. In general there are no further conserved quantities associated with space time transformations.

Let us, however, specialize to theories for which $\theta_{\mu\nu}$ is traceless. Such theories are perforce massless, for, if we consider single particle matrix elements of $\theta_{\mu\nu}$, we have

$$\langle p \mid \theta_{\mu\nu} \mid p\rangle = p_\mu p_\nu$$

or

$$\langle p \mid \theta \mid p\rangle = p^2 = m^2$$

so that $\theta = 0$ implies $m^2 = 0$. More importantly, the tracelessness of $\theta_{\mu\nu}$ plus its symmetry and conservation allow us to construct five new conserved currents

$$S_\mu = x^\lambda\theta_{\lambda\mu}, \qquad \partial^\mu S_\mu = \theta = 0$$

$$K_{\mu\nu} = (2x_\mu x_\lambda - x^2 g_{\mu\lambda})\theta_\nu{}^\lambda, \qquad \partial^\nu K_{\mu\nu} = x_\mu\theta = 0$$

Associated with these currents are five new constants of the motion

$$D = \int d\bar{x}\, x^\lambda\theta_{\lambda 0}(t, \bar{x})$$

$$K_\mu \quad \int dx\, (x_\mu x_\lambda \;\cdots\; x^2 g_{\mu\lambda})\theta_0{}^\lambda(x, t) \qquad (3)$$

Therefore, massless theories at most can have 15 conserved quantities associated with space-time transformations. This suggests that we look for a 15 generator space-time symmetry group, including the 10 generator Poincaré group as a subgroup in the expectation that this larger group will apply to massless theories.

It is in fact rather easy to find such a group. We simply adjoin to the known translations and Lorentz transformations the following five-parameter set of transformations

$$\text{Dilatations: } x_\mu \to x_\mu$$

$$\text{Special Conformal: } x_\mu \to \frac{x_\mu - c_\mu x^2}{1 - 2c\cdot x + c^2x^2} \tag{4}$$

The new transformations are nonlinear and do not leave the Minkowski interval unchanged:

$$\text{Dilatations: } (x - y)^2 \to \lambda^2(x - y)^2$$

$$\text{Special Conformal: } (x - y)^2 \to (x - y)^2/\sigma(x)\sigma(y) \tag{5}$$

$$\sigma(x) = 1 - 2c\cdot x + c^2x^2$$

It is apparent that the conformal transformations may change spacelike intervals into timelike intervals and leave only the light cone invariant. There do exist, however, scalar invariants of these transformations, but they involve at least four space-time points. A typical invariant is the cross ratio

$$R(x_1x_2x_3x_4) = \frac{(x_1 - x_2)^2(x_3 - x_4)^2}{(x_1 - x_4)^2(x_2 - x_3)^2} \tag{6}$$

From the transformation laws of Equation 4, plus the familiar laws for translation and Lorentz transformation, one can read off the infinitesimal generators of the 15 transformations we are considering:

$$\begin{aligned} &P_\mu \to i\partial_\mu \qquad && M_{\mu\nu} \to i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ &D \to ix\cdot\partial \qquad && K_\mu \to i(2x_\mu x\cdot\partial - x^2\partial_\mu) \end{aligned} \tag{7}$$

Using this representation we can compute the group algebra, and, indeed from the fact that the algebra closes, demonstrate that we are really dealing with a group. The full algebra, apart from the familiar commutators of $M_{\mu\nu}$ and $P_\mu$ among themselves, is

$$\begin{aligned} [P_\mu, D] &= iP_\mu \qquad [M_{\mu\nu}, D] = 0 \\ [K_\mu, D] &= -iK_\mu \qquad [K_\mu, K_\nu] = 0 \\ [M_{\mu\nu}, K_\lambda] &= i(g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu) \\ [K_\mu, P_\nu] &= -2i(g_{\mu\nu}D + M_{\mu\nu}) \end{aligned} \tag{8}$$

Dirac noticed a long time ago that this algebra, known as the algebra of the conformal group, is isomorphic to $SO(4,2)$. The identification which realizes this correspondence is

$$g_{AB} = \begin{array}{cccccc} (+ & - & - & -, & & +) \\ 0 & 1 & 2 & 3 & 5 & 6 \end{array}$$

$$[J_{AB}, J_{CD}] = i(g_{BC}J_{AD} - g_{AC}J_{BD} + g_{BD}J_{CA} - g_{AD}J_{CB})$$
$$J_{\mu\nu} = M_{\mu\nu},\ J_{65} = D \tag{9}$$
$$J_{5\mu} = \tfrac{1}{2}(P_\mu - K_\mu), \qquad J_{6\mu} = \tfrac{1}{2}(P_\mu + K_\mu)$$

One can directly see how this isomorphy arises by considering six vectors $\eta_A$ whose metric tensor is the $g_{AB}$ given above. Under $SO(4,2)$ transformations on $\eta_A$, $\eta^2 = \eta_A g^{AB} \eta_B$ is of course preserved. In particular, light-cone *rays* are transformed into one another. Now light-cone rays are specified by four coordinates and we can choose the unique standard form

$$\eta_\mu = K(x_\mu, \tfrac{1}{2}(1 + x^2), \tfrac{1}{2}(1 - x^2)) \tag{10}$$

to establish a correspondence between six-dimensional light-cone rays and four-dimensional vectors. The $SO(4,2)$ transformations take light-cone rays into other light-cone rays and thereby generate transformations on $x_\mu$. It is not difficult to show that these are just the transformations we have been discussing.

The $SO(4,2)$ correspondence simplifies many calculations. For example, the $SO(4,2)$ scalar invariant is just $\eta_1 \cdot \eta_2$ by virtue of Equation 9:

$$\eta_1 \cdot \eta_2 \equiv -\tfrac{1}{2}K_1K_2(x_1 - x_2)^2$$

so that although $(x_1 - x_2)^2$ is not invariant

$$\frac{(\eta_1 \cdot \eta_2)(\eta_3 \cdot \eta_4)}{(\eta_1 \cdot \eta_4)(\eta_2 \cdot \eta_3)} = \frac{(x_1 - x_2)^2(x_3 - x_4)^2}{(x_1 - x_4)^2(x_2 - x_4)^2}$$

is.

In order to study how field theories behave under conformal transformations, it is necessary to classify the possible transformation laws of fields. We make use of the remark that if we exclude the translations from consideration, the point $x = 0$ is carried into itself and we expect the fields at $x = 0$ to be carried into linear combinations of themselves. Consequently, under the infinitesimal generators of dilatations and conformal transformations, respectively, we have

$$\delta_D\varphi(0) = \widehat{D}\varphi(0)$$
$$\delta_{K_\mu}\varphi(0) = \widehat{K}_\mu\varphi(0)$$

when $\widehat{D}$ and $\widehat{K}_\mu$ are numerical matrices. The group algebra matrices are as follows:

$$[\widehat{D}, \widehat{K}_\nu] = \widehat{K}_\nu \qquad [\widehat{K}_\mu, \widehat{K}_\nu] = 0$$
$$[\widehat{D}, \textstyle\sum_{\mu\nu}] = 0 \qquad [\textstyle\sum_{\mu\nu}, \widehat{K}_\lambda] = g_\mu \widehat{K}_{\lambda\nu} - g_{\nu\lambda}K_\mu$$

where $\sum_{\mu\nu}$ is the anlog of $\widehat{K}_\mu$ and $\widehat{D}$ for the special Lorentz generators $M_{\mu\nu}$.

The commutator of $\widehat{D}$ and $\widehat{K}_\mu$ implies that

$$e^{\widehat{D}a}\widehat{K}_\mu e^{-\widehat{D}a} = e^a\widehat{K}_\mu \tag{11}$$

for any $a$. Since $\widehat{K}_\mu$ is a finite dimensional matrix, it satisfies a polynomial characteristic equation $P(\widehat{K}_\mu) = 0$. But then Equation 11 implies that $P(e^a\widehat{K}_\mu) = 0$ as well, for any $a$. This is possible only if the characteristic polynomial is actually a monomial, $\widehat{K}_\mu{}^n = 0$. Therefore $\widehat{K}_\mu$ is nilpotent. A particularly convenient choice of nilpotent matrix is $\widehat{K}_\mu = 0$. This, it turns out, is forced on you if the fields $\varphi$ belong to a single irreducible representation of the Lorentz group. In that case, which is appropriate to the study of a canonical field, it turns out that $\widehat{D}$ is just a numerical multiple of the identity, $\widehat{D} = dI$. The number $d$, known as the dimension of the field, is otherwise undetermined.

In order to reconstruct the infinitesimal transformation law of the field for arbitrary $x$ from its transformation law at $x = 0$ we need only use the group algebra of Equation 8 to show that

$$e^{-iPx}De^{iP\cdot x} = D + x\cdot P$$

$$e^{-iP\cdot x}K_\mu e^{iP\cdot x} = K_\mu + 2x_\mu D + x^\nu M_{\mu\nu} + 2x_\mu x\cdot P - x^2 P_\mu$$

This allows us to show that the general infinitesimal transformation law is

$$\begin{aligned} \delta_D\varphi(x) &= (d + x\cdot\partial)\varphi(x) \\ \delta_{K_\mu}\varphi(x) &= (2x_\mu x\cdot\partial - x^2\partial_\mu + 2x_\mu d + 2x^\nu \textstyle\sum_{\mu\nu})\varphi(x) \end{aligned} \tag{12}$$

We reemphasize that we are taking $\varphi(x)$ to be a field of definite spin so that we may take $\widehat{K}_\mu = 0$ and $\widehat{D} = dI$. Therefore, the only new parameter needed to characterize the behavior of the field under conformal transformations is $d$, the dimension.

## Canonical Treatment

Now that we know how fields transform under canonical transformations, we must ask how to construct a Lagrangian such that the resulting field theory is invariant under the conformal group. Under a general variation $\delta\varphi$ of the fields, the change in the Lagrangian is

$$\begin{aligned} \delta\mathfrak{L} &\equiv \frac{\delta\mathfrak{L}}{\delta\varphi}\,\delta\varphi + \frac{\delta\mathfrak{L}}{\delta\partial_\mu\varphi}\,\partial_\mu\delta\varphi \\ &= \partial_\mu\left[\frac{\delta\mathfrak{L}}{\delta\partial_\mu\varphi}\,\delta\varphi\right] \end{aligned}$$

if we use the Euler-Lagrange equations of motion. On the other hand, if, as an *identity*,

$$\delta\mathfrak{L} = \partial^\mu\Lambda_\mu$$

then the current

$$J_\mu = \Pi_\mu\delta\varphi - \Lambda_\mu \tag{13}$$

is conserved. The canonical commutation relations of the fields also guarantee that $J_\mu$ is the generating current of the transformation $\varphi \to \varphi + \delta\varphi$. Similarly, if a symmetry-breaking term is added to the Lagrangian so that the best one can say is

$$\delta\mathfrak{L} = \partial^\mu\Lambda_\mu + \Delta$$

then the same current $J_\mu$ still generates the transformation $\varphi \to \varphi + \delta\varphi$, but the current is of course not conserved, but satisfies

$$\partial^\mu J_\mu = \Delta$$

Let us apply these remarks to scale transformations, recalling that the fundamental field transforms as

$$\delta\varphi = (d + x\cdot\partial)\varphi$$

Evidently $\partial_\mu\phi$ will transform with dimension $d + 1$, and monomials in $\varphi$ and its derivatives will transform with a dimension equal to the sum of the dimensions of its factors. Thus

$$\begin{aligned}\delta M &= (d_M + x\cdot\partial)M \\ &= (4 + x\cdot\partial)M + (d_M - 4)M \\ &= \partial^\mu(x_\mu M) + (d_M - 4)M\end{aligned}$$

If the total Lagrangian is a sum of monomials $\mathfrak{L}_i$, then the current $J_\mu$ of Equation 12, with the identification $\Lambda_\mu = x_\mu\mathfrak{L}$, satisfies

$$\partial^\mu J_\mu = \sum_i (d_i - 4)\mathfrak{L}_i$$

so that scale transformations are a symmetry of the theory if $d_i = 4$ for all the monomials in $\mathfrak{L}$. If we make the standard assignment of $d = 1$ for Bose fields and $d = 3/2$ for Fermi fields, this means that scale invariance is a symmetry provided that all coupling constants are dimensionless.

In most interesting cases we can make trivial additions to the scale transformation generating current (hereafter called $S_\mu$) such that $S_\mu = \theta_{\mu\nu}x^\nu$, where $\theta_{\mu\nu}$ is a satisfactory energy-momentum tensor for the theory having the property that $\theta = \theta_\mu{}^\mu$ involves only those terms in $\mathfrak{L}$ with dimensional coupling constants.

For conformal transformations, the story is similar but the algebra is more elaborate. One is ultimately able to rearrange the generating current $(K_{\mu\nu})$ of conformal transformations so that it takes the form

$$K_{\mu\nu} = (2x_\mu x_\lambda - g_{\mu\lambda}x^2)\theta^\lambda{}_\nu$$

with $\theta_{\mu\nu}$ being the same energy-momentum tensor as before.

We are of course interested in finding the constraints on the Green's functions of the theory which are implied by the existence of these generating currents. A convenient method of doing this is to consider the identity

$$\frac{\partial}{\partial x_\mu}\langle T(S_\mu(x)\phi(y_1)\cdots\phi(y_n))\rangle = \langle T(\theta(x)\phi(y_1)\cdots\phi(y_n))\rangle$$
$$+\ \delta(x_0 - y_{10})\langle T([S_0(x),\ \phi(y_1)]\phi(y_2)\cdots\phi(y_n))\rangle + \text{permutations}$$

If scale invariance were an exact symmetry, the term involving $\theta$ would vanish, of course, but we want to consider what happens in theories where the particle masses are not zero. The left-hand side of this equation may be eliminated by integrating $d^4x$. The equal time commutators are evaluated by observing that $S_\mu$ is the generating current for scale transformations so that

$$\delta(x_0 - y_0)[S_0(x),\ \phi(y)] = \delta(x - y)(d + x\cdot\partial)\phi(x)$$

This then yields an equation relating an amplitude for $n$ fields to an amplitude for $n$ fields plus an insertion of the operator $\theta$.

This relation is most easily appreciated in momentum space, where it reads

$$\left[4 - nd - \sum^n p\cdot\frac{\partial}{\partial p_i}\right]\Gamma^{(n)}(p_1\cdots p_n) = -i\Gamma_\theta{}^{(n)}(0;p_1\cdots p_n) \qquad \textbf{(14)}$$

when $\Gamma^{(n)}$ is the one-particle-irreducible Green's function for $n$ particles and $\Gamma_\theta^{(n)}$ is the one-particle-irreducible Green's function for $n$ particles plus an insertion of $\theta$ (carrying zero four momentum) and $d$ is still the dimension of the field $\phi$.

Carrying out a similar treatment of the conformal generating current yields the more formidable equation

$$\sum_i \left[ -2d \frac{\partial}{\partial p_i{}^\mu} - 2p_i \cdot \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_i{}^\mu} + p_{i\mu} \frac{\partial^2}{\partial p_i{}^2} \right] \Gamma^{(n)}(p_1 \cdots p_n)$$
$$= -2i \frac{\partial}{\partial k^\mu} \Gamma_\theta{}^{(n)}(k; p_1 \cdots p_n)|_{k=0} \tag{15}$$

To simplify the equation, we have taken scalar fields so that the spin matrix $\sum_{\mu\nu}$ appearing in Equation 11 may be dropped.

If $\theta$ is zero, Equations 13 and 14 are differential equations for $\Gamma^{(n)}$ which can be solved to yield the general form of $\Gamma^{(n)}$ which is consistent with exact scale and conformal invariance. We shall shortly discuss what this general form looks like and what its phenomenological consequences might be. The first problem to surmount is that in interesting theories, the particle masses are not zero and $\theta$ is therefore not identically zero. On the other hand, $\theta$ is usually a "soft" operator—that is to say, of dimension less than four—and Weinberg's theorem guarantees that in the deep Euclidian region (all mementa large and spacelike) $\Gamma_\theta{}^{(n)}$ vanishes more rapidly than $\Gamma^{(n)}$. Therefore, in appropriate asymptotic regions one would expect to recover the predictions of a theory in which $\theta$ does vanish. This is the basis of all practical applications of scale and conformal invariance.

## Construction of Conformal Invariant Green's Functions

Having argued that in appropriate high energy limits, scale and/or conformal invariance is reestablished as an exact symmetry, it is appropriate to ask what restrictions this places on Green's functions. It would be possible to solve the relevant Ward identities directly, but it is more transparent to use the $SO(4,2)$ formalism established in the first section.

We want to consider fields, $\tilde{\phi}(\eta)$, depending on the six-vector, $\eta$, instead of the usual fields, $\phi(x)$, depending on the Minkowski four-fector, $x$. We have previously established that the correspondence between $\eta$ and $x$ is that all the points lying along a given light-cone ray ($\eta^2 = \eta'^2 = 0$, $\eta = c\eta'$) correspond to the same $x$ according to the relation

$$\eta = K\left(x, \frac{1+x^2}{2}, \frac{1-x^2}{2}\right) \tag{16}$$

Consequently, the fields $\tilde{\phi}(\eta)$ need only be defined on the light cone, $\eta^2 = 0$. Furthermore, since $\eta$ and $\lambda\eta$ correspond to the same Minkowski point, $x$, the dependence of $\tilde{\phi}$ on the overall scale of $\eta$ should not be a true dynamical degree of freedom. This can be achieved by requiring that

$$\tilde{\phi}(\lambda\eta) = \lambda^d \tilde{\phi}(\eta) \tag{17}$$

when $d$ is the dimension of the underlying field $\phi(x)$ and making the identification

$$\phi(x) = \frac{1}{K^d} \tilde{\phi}(\eta) \tag{18}$$

with $K$ as defined in Equation 16. With this identification it is easy to check that applying the linear transformations of $SO(4,2)$ to $\tilde{\phi}(\eta)$ induces the correct conformal transformations on $\phi(x)$.

With this background it is easy to construct conformally invariant Green's functions. Let us illustrate with the two-point function of two different scalar fields:

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{K_1^{d_1}K_2^{d_2}}\langle\tilde{\phi}_1(\eta_1)\tilde{\phi}_2(\eta_2)\rangle$$

By invariance under $SO(4,2)$, we have

$$\langle\tilde{\phi}_2(\eta_1)\tilde{\phi}_2(\eta_2)\rangle = F(\eta_1 \cdot \eta_2)$$

while the homogeneity equation (Equation 16) requires that

$$F(\lambda\eta_1 \cdot \eta_2) = \lambda^{d_1}F(\eta_1 \cdot \eta_2) = \lambda^{d_2}F(\eta_1 \cdot \eta_2)$$

This requires, firstly, that $d_1 = d_2 = d$ and, secondly, that

$$F(\eta_1 \cdot \eta_2) = C(\eta_1 \cdot \eta_2)^d$$

Recalling from the first section that $\eta_1 \cdot \eta_2 \propto K_1K_2(x_1 - x_2)^2$, we find that

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = C[(x_1 - x_2)^2]^d$$

Conformal invariance has supplied us with the "selection rule" $d_1 = d_2$ and completely determined the functional form. Actually, scale invariance alone would have given the same functional form had we *assumed* that $d_1 = d_2$.

If we carry out the same analysis for a three-point function we find the similar result

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = C(x_1 - x_2)^{(d_1+d_2-d_3)}\cdot(x_1 - x_3)^{(d_1+d_3-d_2)}\cdot(x_2 - x_3)^{(d_2+d_3-d_1)}$$

Once again, the functional form is determined up to a constant but this time conformal invariance is essential to the result. Finally at the level of four-point functions arbitrary form factors depending on cross ratios come in and the functional form, though severely restricted, is not completely determined.

We have so far discussed the fairly trivial case of scalar fields and had to construct objects which transformed as $SO(4,2)$ scalars. If we deal with nonzero fields we must construct $SO(4,2)$ tensors and worry about projecting out the physical components of the tensors. The algebra is more complicated, but the essential result that two- and three-point functions are determined up to multiplicative constants remains true in most cases.

## Anomalies

In the second section we gave a naive canonical treatment of the Ward identities for scale and conformal invariance. It is by now well known that it is in general illegal as a result of the underlying divergences of perturbation theory to make free use of canonical commutators and equations of motion in deriving Ward identities. The true Ward identities will differ from the canonical ones by anomalous terms which are calculable only by special arguments. We now turn to the question of whether the scale and conformal Ward identities are afflicted with such anomalies.

The simplest way to discuss this problem is to use the normal product method of Zimmermann. The essence of this method is that it allows you to use the equations of motion and equal-time commutators as long as you keep proper track of the subtractions needed to define products of operators at a point.

We will try to give the idea of Zimmermann's method as succintly as possible: Consider an operator such as $\phi\Box\phi$, where $\phi$ is a scalar field. It has dimension four, and one can easily verify by power-counting that its two- and four-point matrix elements require subtraction. On the other hand, we might use equations of motion to replace $\Box\phi$ by $\mu^2\phi$, in which case we would have $\phi\Box\phi = \mu^2\phi^2$. The operator $\phi^2$, however, has dimension two and only its two-particle matrix element needs subtraction, and using the equality $\phi\Box\phi = \mu^2\phi^2$ is bound to lead to trouble. We could, however, perform *unneeded* subtractions on $\phi^2$, treating it *as if* it had dimension four ( a notation four this is $[\phi^2]_d$ where $d$ is the dimensionality assigned to $\phi^2$ for purposes of determining how many subtractions it receives). Zimmermann's point is that the equality $[\phi\Box\phi]_4 = \mu^2[\phi^2]_4$ is consistent because although we have used the equations of motion, we have not incorrectly changed the subtraction procedure. Finally one can show that an object such as $[\phi^2]_4$ can be reexpressed in terms of operators with standard subtractions as follows:

$$[\phi^2]_4 = a[\phi^2]_2 + b[\phi^4]_4 + c[\partial_\mu\phi\, \partial^\mu\phi]_4 \tag{19}$$

The sum is over all operators (of the right spin, of course) of dimension less than or equal to four) and $a$, $b$, $c$ are functions of the renormalized coupling constants of the theory.

In the Ward identities of Equations 13 and 14, $\theta$ is formally of dimension less than four, but is cast up by using the equations of motion within operators of dimension four ($\theta_{\mu\nu}$) itself). The Zimmermann algorithm would then be that the Ward identities are correct so long as we understand by $\theta$ the object $[\theta]_4$. But, as we see from Equation 18, $[\theta]_4$ contains operators which are truly of dimension four and it is not possible to argue that the $\theta$ terms in the Ward identities vanish in some asymptotic limit.

Let us consider in a little more detail the scale invariance Ward identity, Equation 13, in a scalar field theory. In this case $\theta = \mu^2\phi^2$, an operator of dimension two, and we must deal with

$$[\theta]_4 = a(\lambda)[\theta]_2 + b(\lambda)[\phi^4]_4 + c(\lambda)[\partial_\mu\phi\partial^\mu\phi]_4$$

where $\lambda$ is the coupling constant of the theory. We encounter zero momentum matrix elements of $[\theta]_4$ on the right-hand side of the Ward identity, and we would like to show that the dimension four constituents of $[\theta]_4$ can be re-expressed in a useful manner. For example, the zero four momentum insertion of $[\phi^4]_4$ simply counts the number of $\phi^4$ interactions there are in a given graph. If $\lambda$ is the $\phi^4$ coupling constant, then we may say

$$\Gamma^{(n)}_{[\phi^4]_4}(0;\cdot P_1\cdots P_n) \equiv \lambda\frac{\partial}{\partial\lambda}\Gamma^{(n)}(P_1\cdots P_n)$$

The zero four momentum insertion of $[\partial_\mu\phi\partial^\mu\phi]_4$ can be seen, by virtue of a topological identity, just to count the number of external legs or

$$\Gamma^{(n)}_{[\partial_\alpha\phi\partial^\alpha\phi]_4}(0;\cdot P_1\cdots P_n) = n\Gamma^{(n)}(P_1\cdots P_n)$$

These two relations allow one to rewrite the Ward identity as

$$\left[4 - n(1+\gamma(\lambda)) - \sum P_i\cdot\frac{\partial}{\partial P_i} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right]\Gamma^{(n)}(P_1\cdots P_n) \tag{20}$$
$$= -i\alpha(\lambda)\,\Gamma^{(n)}_{[\theta]_2}(0;\cdot P_1\cdots P_n)$$

where the functions $\alpha$, $\beta$, $\gamma$ are unknown functions of the $\beta$ term arising from $[\phi^4]_4$ and the $\gamma$ term arising from $[\partial_\mu\phi\partial^\mu\phi]_4$. Since the operator $[\theta_2]$ *is* of dimension less than four, we *can* argue that it is asymptotically negligible and that $\Gamma^{(n)}$ does asymptotically satisfy a homogeneous Ward identity. The anomaly terms $\beta$ and $\gamma$ seriously modify the conclusions one may draw from this fact. Only if $\beta = 0$ because of a special choice of coupling constant can one directly get a simple result: In that case one recovers the naive scaling Ward identity but with an anomalous dimension $d = 1 + \gamma$, assigned to the underlying field.

One can also discuss in a similar fashion the Ward identities for conformal invariance. Once again, the Zimmermann algorithm suggests that the naive Ward identity is correct so long as we replace $\theta$ by $[\theta]_4$. Then the problem is to reexpress the dimension four parts of $[\theta]_4$ in a useful form. This is now more complicated because one is not simply taking the zero momentum matrix element of $\theta$. Nevertheless there is still a topological identity which simplies $[\partial_\mu\phi\partial^\mu\phi]_4$:

$$i\frac{\partial}{\partial k^\mu}\Gamma^{(n)}_{[\partial_\mu\phi\partial^\mu\phi]_4}(k;\cdot P_1\cdots P_n)_{k=0} \equiv \sum\frac{\partial}{\partial P_i{}^\mu}\Gamma^{(n)}(P_1\cdots P_n) \tag{21}$$

This identity just replaces the canonical dimension of the underlying field by the anomalous dimension, $1 + \gamma$. No similar reduction appears possible for $[\phi^4]_4$. Therefore the final form for the conformal Ward identity is

$$\sum_i\left[-2(1+\gamma)\frac{\partial}{\partial P_i{}^\mu} - 2P_i\cdot\frac{\partial}{\partial P_i}\frac{\partial}{\partial P_i{}^\mu} + P_\mu{}^i\frac{\partial^2}{\partial P_i{}^2}\right]\Gamma^{(n)}(P_1\cdots)$$

$$= -2i\frac{\partial}{\partial k^\mu}[\Gamma^{(n)}_{[\theta]_2+\beta[\phi^4]_4}(k;\cdot P_1\cdots)]|_{k=0}$$

Clearly the right-hand side of this equation becomes asymptotically negligible only if $\beta = 0$. Only in this special case is the effective $\theta$ insertion of dimension less than four and only then can one recover asymptotically the homogeneous conformal Ward identity. In that special case, just as for the scale invariance Ward identity, the only effect of anomalies appears in the occurrence of anomalous dimensions $d = 1 + \gamma$, for the underlying field.

## Fixed Points

The results of the last section suggest that the asymptotic behavior of Green's functions is scale and conformal invariant in the usual sense only if the coupling constant is adjusted such that $\beta(\lambda) = 0$. This would mean that for randomly chosen values of the coupling, one should never see any vestiges of these symmetries. It turns out that a closer inspection of the Ward identities reveals that this conclusion is too pessimistic.

Consider the "corrected" scaling Ward identity of Equation 19 in an asymptotic region where $\Gamma^{(n)}_{[\theta]_2}$ may be neglected relative to $\Gamma^{(n)}$ itself. Then we have the homogeneous equation

$$\left[4 - n - \sum P_i\cdot\frac{\partial}{\partial P_i} + \beta\frac{\partial}{\partial\lambda} + n\gamma\right]\Gamma^{(n)}(P_1\cdots) = 0$$

The operator $4 - n - \sum P_i\cdot(\partial/\partial P_i)$ is nothing more than the operation of uniformly scaling all the momenta in $\Gamma^{(n)}$ and an equivalent way of writing the Ward identity in

$$\left[\eta \frac{\partial}{\partial \eta} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] \Gamma^{(n)}(\eta P) = 0$$

This equation is trivial to solve since it is just a first order partial differential equation. The result is

$$\Gamma^{(n)}(\eta P_1 \cdots \eta P_n;\lambda) = \Gamma^{(n)}(P_1 \cdots P_n;\bar{\lambda}(\lambda, \eta)) \exp\left(- n \int_{\lambda}^{\bar{\lambda}} d\lambda' \frac{\gamma(\lambda')}{\beta(\lambda')}\right) \quad (22)$$

with

$$\bar{\lambda}(\lambda, 1) = \lambda$$

$$\left(\eta \frac{\partial}{\partial \eta} + \beta(\lambda) \frac{\partial}{\partial \lambda}\right) \bar{\lambda} = 0$$

This equation for $\bar{\lambda}$ can be recast in the more helpful form

$$\frac{d}{dt} \bar{\lambda} = -\beta(\bar{\lambda})$$

$$t = \ln \eta, \bar{\lambda}(0) = \lambda$$

from which one sees directly that as $\eta \to +\infty$, $\bar{\lambda}$ approaches a zero of $\beta$, say $\lambda_0$. Then from Equation 21, one sees that the behavior of $\Gamma^{(n)}(\eta P; \lambda)$ is identical to that of $\Gamma^{(n)}(P; \lambda_0)$ in the limit $\eta \to \infty$. But since $\beta(\lambda_0) = 0$, $\Gamma^{(n)}(P; \lambda_0)$ will, according to the scale and conformal invariance Ward identities, satisfy the naive requirements of scale and conformal invariance with anomalous dimensions for the fields. In other words, even though anomalies appear to destroy asymptotic scale and conformal invariance, except for those special values of coupling constant where $\beta = 0$, one can use the scaling Ward identity to establish that the asymptotic behavior of the theory for *any* coupling constant is ultimately the same as for those coupling constants when $\beta = 0$, thus reinstating scale and conformal invariance as asymptotic symmetries. An interesting challenge is to construct a theory where one asymptotically reinstates not just conformal invariance, but the wider class of symmetries implicit in the parton model, thus uncovering a field theoretic justification for the parton model.

## DISCUSSION

Dr. Alex Harvey (*Queens College, City University of New York*): In the application of the 15-parameter conformal group to classical field theory, such as the Maxwell field, there are two well-known problems. One is that the conformal group has transformations that convert timelike intervals into spacelike intervals so that there is trouble with causality. The other is that nonzero mass is excluded because mass is not a conformal invariant quantity. Do these two problems of the classical theory have relevance to the work being discussed here?

Dr. Callan: Yes. Physical mass leads to just plain explicit breaking of conformal and scale invariance. You must look for some appropriate limit of a realistic field theory in which conformal invariance is recovered. Causality is carried over from the realistic theory in the limit.

The problem is compounded by the fact that in an interesting quantum field theory there are infinite masses secretly buried in the bottom of the formulas that are associated with the renormalization of the theory. Both physical masses and regulator masses can break conformal invariance and each type of mass leads to its own funny problems. The latter kind lead to the anomalies that we had to talk around so much.