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INTERNAL SYMMETRIES AND MODEL-INDEPENDENT RELATIONS FOR INCLUSIVE PROCESSES*

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ABSTRACT

Predictions are given for the isospin and SU(3) dependence of single-particle and multiparticle inclusive cross sections in hadronic and photoproduction processes. Hadronic isospin relations can test consistency of experimental data. SU(3) relations measure SU(3) symmetry breaking. In photoproduction and electroproduction, isospin relations can be used to look for isotensor electromagnetic currents. For example, in the reaction $\gamma + d \rightarrow 2\pi + X$, inequalities relate the cross sections for the different 2π charge states. These must hold at all values of pion momenta if the photon has only isoscalar and isovector components.

We wish to call attention to a class of model-independent symmetry relations for inclusive reaction cross sections. Relations that depend only upon isospin invariance of strong interactions are expected to be satisfied experimentally and may be useful to test the internal consistency of experimental data. Relations for electromagnetic processes may test the assumption that the electromagnetic current contains only isoscalar and isovector components. Relations that follow from SU(3) symmetry may give information about SU(3) symmetry breaking.

The symmetry relations discussed here can be obtained by straight-forward application of a "maximum-complexity" theorem[1] to specific cases. The physical content of the maximum-complexity theorem is analogous to the statement that an initial state that contains only s and p waves cannot produce a final-state angular distribution more complicated than $A + B\cos\theta + C\cos^2\theta$. For isospin, consider the inclusive reaction

$$A + B \to C_{IM} + X, \tag{1}$$

where C_{IM} denotes the set of states within an isospin multiplet having isospin I, the subscript M is the eigenvalue of I_z, and X is everything else. The maximum-complexity theorem requires the isospin dependence of the cross section σ_{IM} for the inclusive reaction (1) to be given by a polynomial in M,

$$\sigma_{IM} = \sum_{n=0}^{2I_{AB}^{(max)}} a_{In} M^n , \tag{2}$$

of degree equal to twice the maximum isospin $I_{AB}^{(max)}$ in the initial state. Relations between the cross sections are then obtained if the number of free parameters is less than the number of independent experimental cross sections; i.e., if $I > I_{AB}^{(max)}$. For initial states involving available beams on nucleon targets, $I_{AB}^{(max)}$ is at least 1, and relations are obtained only for states having $I \geq \frac{3}{2}$. Such isospin multiplets are available only as resonances and not as stable particles. For deuteron targets, $I_{AB}^{(max)}$ can be as low as $\frac{1}{2}$, and relations are obtainable for inclusive single-pion production as well as for resonance production. Such relations can therefore be used as consistency tests on separation of resonances from background and on unscrambling of deuteron data.

The multiplet C_{IM} need not be a single particle or resonance. It could be a multiparticle system such as a nucleon-pion or multipion state. Then the cross sections σ_{IM} for the production of a given isospin eigenstate are not directly measurable, except for the cases of maximum and minimum charge. However, the sums of cross sections for a given value of M and all possible values of I are expressible in terms of observable cross sections as shown below. For these sums, inequalities can be obtained from Eq. (2). For example,

$$\sigma_M \equiv \sum_{I=M}^{I_{max}} \sigma_{IM} \geq \sigma_{I'M} = \sum_{n=0}^{2I_{AB}^{(max)}} a_{I'n} M^n, \tag{3}$$

where I_{max} is the maximum isospin obtainable for the multiparticle system and I' is any value of I between M and I_{max}.

Isospin equalities. We first consider equalities obtained by applying Eq. (2) to initial states with $I_{AB}^{(max)} = \frac{1}{2}$. The inclusive cross section σ_{IM} must then be a linear function of M. The relations obtained are

$$\sigma(K^{\pm}d \to \pi^{+}X) + \sigma(K^{\pm}d \to \pi^{-}X) = 2\sigma(K^{\pm}d \to \pi^{0}X), \tag{4a}$$

$$\sigma(pd \to \pi^{+}X) + \sigma(pd \to \pi^{-}X) = 2\sigma(pd \to \pi^{0}X), \tag{4b}$$

$$\sigma(\bar{p}d \to \pi^{+}X) + \sigma(\bar{p}d \to \pi^{-}X) = 2\sigma(\bar{p}d \to \pi^{0}X), \tag{4c}$$

$$\sigma(A_{1/2}d \to \Sigma^{+}X) + \sigma(A_{1/2}d \to \Sigma^{-}X) = 2\sigma(A_{1/2}d \to \Sigma^{0}X), \tag{4d}$$

$$\sigma(A_{1/2}d \to \Delta_M X) = a_0 + a_1 M, \tag{4e}$$

where Σ can also be any Y^* resonance, $A_{1/2}$ is any particle with isospin $\frac{1}{2}$, such as K^+, K^-, p, or $\bar{p}$, and Δ_M is the Δ state with $I_z = M$.

For $I_{AB}^{(max)} = 1$, the cross section is a quadratic function of M, and relations for inclusive Δ-production can be obtained.

$$\sigma(A_{1/2}N \to \Delta^{+}X) + \frac{1}{3}\sigma(A_{1/2}N \to \Delta^{-}X) = \sigma(A_{1/2}N \to \Delta^{0}X) + \frac{1}{3}\sigma(A_{1/2}N \to \Delta^{++}X) \tag{5a}$$

$$\sigma(\pi^\pm d \to \Delta^+ X) + \frac{1}{3}\sigma(\pi^\pm d \to \Delta^- X) = \sigma(\pi^\pm d \to \Delta^0 X) + \frac{1}{3}\sigma(\pi^\pm d \to \Delta^{++} X) \quad (5b)$$

$$\sigma(\gamma d \to \Delta^+ X) + \frac{1}{3}\sigma(\gamma d \to \Delta^- X) = \sigma(\gamma d \to \Delta^0 X) + \frac{1}{3}\sigma(\gamma d \to \Delta^{++} X). \quad (5c)$$

U-spin equalities. The same arguments can be applied to U spin or V spin if SU(3) symmetry is assumed. Because the photon is a U-spin scalar, the condition analogous to (2) for relations with U spin is most easily satisfied in photoproduction experiments on protons, for which $U_{AB}^{(max)} = \frac{1}{2}$. However, simple equalities involving octet particles in the final state are not available because the Λ, Σ^0, π^0, and η are not eigenstates of U spin or V spin. Thus the most useful relations are those for decuplet baryon production, namely

$$\sigma(\gamma p \to \Delta^0 X) + \sigma(\gamma p \to \Xi^{*0} X) = 2\sigma(\gamma p \to Y^{*0} X), \quad (6a)$$

$$\sigma(\gamma p \to \{\Delta^-, Y^{*-}, \Xi^{*-}\Omega^-\}X) = a_0 + a_1 U_z, \quad (6b)$$

$$\sigma(\gamma n \to Y^{*-} X) + \frac{1}{3}\sigma(\gamma n \to \Omega^- X) = \sigma(\gamma n \to \Xi^{*-} X) + \frac{1}{3}(\gamma n \to \Delta^- X), \quad (6c)$$

$$\sigma(K^\pm p \to Y^{*-} X) + \frac{1}{3}\sigma(K^\pm p \to \Omega^- X) = \sigma(K^\pm p \to \Xi^{*-} X) + \frac{1}{3}\sigma(K^\pm p \to \Delta^- X), \quad (6d)$$

$$\sigma(\pi^\pm p \to Y^{*-} X) + \frac{1}{3}\sigma(\pi^\pm p \to \Omega^- X) = \sigma(\pi^\pm p \to \Xi^{*-} X) + \frac{1}{3}\sigma(\pi^\pm p \to \Delta^- X). \quad (6e)$$

When the same approach is applied to multiparticle inclusive processes, equalities are not usually obtained, but useful inequalities can be derived. For example, relations (4e) and (5) hold when the Δ is replaced by any nucleon-pion system in the $I = \frac{3}{2}$ state.

However, only the $p\pi^+$ and $n\pi^-$ states are pure $I = \frac{3}{2}$ states; the $p\pi^0$, $p\pi^-$, $n\pi^0$ and $n\pi^+$ states are mixtures of isospin $\frac{1}{2}$ and $\frac{3}{2}$. Thus the equalities (4e) and (5) are not valid unless the $N\pi$ system is known to be in a resonance with $I = \frac{3}{2}$. To obtain inequalities, we first note that the sums of the cross sections over all states having the same charge are simply expressed in terms of the cross sections for the isospin eigenstates. The sum contains no interference terms between different isospins because I and M are simultaneously measurable. Then Eq. (3) gives

$$\sigma_{1/2} \equiv \sigma(\pi^+ nX) + \sigma(\pi^0 pX) = \sigma_{3/2,\,1/2} + \sigma_{1/2,\,1/2} \geq \sigma_{3/2,\,1/2}, \tag{7a}$$

$$\sigma_{-1/2} \equiv \sigma(\pi^0 nX) + \sigma(\pi^- pX) = \sigma_{3/2,\,-1/2} + \sigma_{1/2,\,-1/2} \geq \sigma_{3/2,\,-1/2}\,. \tag{7b}$$

The inequalities (7) can be combined with the relations (4e) and (5) for $\sigma_{3/2,M}$ to obtain inequalities relating multiparticle cross sections. In the simplest case, $I_{AB}^{(max)} = \frac{1}{2}$ and the cross sections $\sigma_{3/2,M}$ lie on a straight line when plotted against M, as shown in Fig. 1(a). The sums $\sigma_{\pm 1/2}$ must then lie <u>above</u> this straight line, which can be determined by the two points $\sigma_{3/2,\pm 3/2}$. Thus we obtain the inequalities

$$\sigma(K^\pm d \to \pi^+ nX) + \sigma(K^\pm d \to \pi^0 pX) \geq \frac{2}{3}\sigma(K^\pm d \to \pi^+ pX) + \frac{1}{3}\sigma(K^\pm d \to \pi^- nX), \tag{8a}$$

$$\sigma(K^\pm d \to \pi^0 nX) + \sigma(K^\pm d \to \pi^- pX) \geq \frac{2}{3}\sigma(K^\pm d \to \pi^- nX) + \frac{1}{3}\sigma(K^\pm d \to \pi^+ pX) \tag{8b}$$

When $I_{AB}^{(max)} = 1$, the curve of $\sigma_{3/2,\,M}$ vs M is a parabola, which is not completely determined by the two points $\sigma_{3/2}$ and $\sigma_{-3/2}$. However, inequalities are still obtainable by using the inequalities (7) and noting that all cross sections must be positive. This is easily seen in the extreme case in which $\sigma_{-3/2} = 0$, shown in Fig. 1(b). The lowest parabola that passes through the point $\sigma_{3/2,\,-3/2} = 0$ and keeps $\sigma_{3/2,\,-1/2} \geqslant 0$ passes through the point $\sigma_{3/2,\,-1/2} = 0$ and has the form

$$\sigma_{3/2,\,M} = \frac{1}{6}\,\sigma_{3/2,\,3/2}(M + \tfrac{3}{2})(M + \tfrac{1}{2}). \tag{9}$$

This gives the inequality

$$\sigma_{1/2} = \sigma_{3/2,\,1/2} + \sigma_{1/2,\,1/2} \geqslant \frac{1}{3}\,\sigma_{3/2}. \tag{10}$$

We now apply this approach to the general case for reactions where $I_{AB}^{(max)} = 1$, for example

$$K^{\pm} + p \to \pi + N + X, \qquad p + p \to \pi + N + X, \tag{11a, b}$$

$$\bar{p} + p \to \pi + N + X, \qquad \pi + d \to \pi + N + X, \tag{11c, d}$$

$$\gamma + d \to \pi + N + X. \tag{11e}$$

The equalities (5) apply to these reactions when the Δ is replaced by the πN system in the $I = \frac{3}{2}$ state. Thus the inequalities (7) give

$$\sigma(\pi^+ nX) + \sigma(\pi^0 pX) \geq \frac{1}{3}\left[\sigma(\pi^+ pX) - \sigma(\pi^- nX)\right], \tag{12a}$$

$$\sigma(\pi^0 nX) + \sigma(\pi^- pX) \geq \frac{1}{3}\left[\sigma(\pi^- nX) - \sigma(\pi^+ pX)\right]. \tag{12b}$$

The procedure used to obtain the inequalities (12) can be applied to any multiparticle inclusive process from initial states with $I_{AB}^{(max)} = 1$, such as multipion production. The experimental cross sections σ_M at two conjugate values $M = \pm\mathcal{M}$ can be used to determine two of the three parameters of the parabola. The third parameter cannot be determined completely, but bounds are obtained by requiring all cross sections to be positive. The equation for the parabola can be written

$$\sigma_M(I_{AB}^{(max)} = 1) \geq \tfrac{1}{2}(\sigma_{\mathcal{M}} + \sigma_{-\mathcal{M}})(M/\mathcal{M})^2 + \tfrac{1}{2}(\sigma_{\mathcal{M}} - \sigma_{-\mathcal{M}})(M/\mathcal{M}) + x(1 - M^2/\mathcal{M}^2), \tag{13}$$

where x is an undetermined parameter, $M < |\mathcal{M}|$, and $|\mathcal{M}|$ need not be the maximum value of $|M|$ for the system under consideration. For three-pion production, the cases $|\mathcal{M}| = 2$ and $|\mathcal{M}| = 3$ give nontrivial relations.

The inequality becomes an equality when the cross sections come from a pure isospin eigenstate, e.g., a resonance; however, these are not of great physical interest because of the experimental difficulties of separating resonances from background. By requiring the cross section (13) to be positive for some value of M, which we denote for convenience by -m, we find a lower bound on x.

$$\sigma_{-m}(I_{AB}^{(max)} = 1) > \tfrac{1}{2}(\sigma_{\mathcal{M}} + \sigma_{-\mathcal{M}})(m/\mathcal{M})^2 - \tfrac{1}{2}(\sigma_{\mathcal{M}} - \sigma_{-\mathcal{M}})(m/\mathcal{M}) + x\left[1 - (m^2/\mathcal{M}^2)\right] \geq 0. \tag{14}$$

Thus for any allowed value of $|m| < |\mathcal{M}|$, the condition on x is

$$x \geq \tfrac{1}{2}(1 - m^2/\mathcal{M}^2)^{-1}\left[(\sigma_{\mathcal{M}} - \sigma_{-\mathcal{M}})(m/\mathcal{M}) - (\sigma_{\mathcal{M}} + \sigma_{-\mathcal{M}})(m/\mathcal{M})^2\right]. \qquad (15)$$

Substituting Eq. (15) into Eq. (13) gives

$$\sigma_M(I_{AB}^{(max)} = 1) \geq \left(\frac{M + m}{2\mathcal{M}}\right)\left[\sigma_{\mathcal{M}}\left(\frac{\mathcal{M} + M}{\mathcal{M} + m}\right) - \sigma_{-\mathcal{M}}\left(\frac{\mathcal{M} - M}{\mathcal{M} - m}\right)\right], \qquad (16)$$

where m should be chosen to give the best inequality and, as before, $|M| < |\mathcal{M}|$ and $|m| < |\mathcal{M}|$.

Setting $\mathcal{M} = \pm\frac{3}{2}$ and $m = M = \pm\frac{1}{2}$ gives the inequalities (12). For $\mathcal{M} = 2$, the next case of interest, we find

$$\sigma_M(I_{AB}^{(max)} = 1) \geq \tfrac{1}{4}(M + m)\left[\sigma_2(2 + M)/(2 + m) - \sigma_{-2}(2 - M)/(2 - m)\right], \qquad (17)$$

where $|M| < 2$ and $|m| < 2$. This relation applies to multipion inclusive processes, such as the two-pion reactions

$$K^{\pm} + p \to \pi(k_1) + \pi(k_2) + X, \qquad p + p \to \pi(k_1) + \pi(k_2) + X, \qquad (18a, b)$$

$$\bar{p} + p \to \pi(k_1) + \pi(k_2) + X, \qquad \gamma + d \to \pi(k_1) + \pi(k_2) + X, \qquad (18c, d)$$

where the momenta k_1 and k_2 are specified to distinguish between the two pions. The choice $|\mathcal{M}| = 2$ is convenient experimentally, since the doubly charged states are most easily identified. Substituting M values into Eq. (17) and trying m = 0 and m = ±1 gives the inequalities

$$\sigma_1(I_{AB}^{(max)} = 1) \geq \tfrac{1}{2}[\sigma_2 - \sigma_{-2}], \qquad \sigma_{-1}(I_{AB}^{(max)} = 1) \geq \tfrac{1}{2}[\sigma_{-2} - \sigma_2], \qquad (19a, b)$$

$$\sigma_1(I_{AB}^{(max)} = 1) \geq \tfrac{1}{4}[\tfrac{3}{2}\sigma_2 - \tfrac{1}{2}\sigma_{-2}], \qquad \sigma_{-1}(I_{AB}^{(max)} = 1) \geq \tfrac{1}{4}[\tfrac{3}{2}\sigma_{-2} - \tfrac{1}{2}\sigma_2], \qquad (19c, d)$$

$$\sigma_0(I_{AB}^{(max)} = 1) \geq \tfrac{1}{4}[\tfrac{2}{3}\sigma_2 - 2\sigma_{-2}], \qquad \sigma_0(I_{AB}^{(max)} = 1) \geq \tfrac{1}{4}[\tfrac{2}{3}\sigma_{-2} - 2\sigma_2], \qquad (19e, f)$$

where, for the case of the two-pion reactions (18),

$$\sigma_{\pm 2} \equiv \sigma(AB \to \pi^{\pm}\pi^{\pm}X), \qquad \sigma_{\pm 1} \equiv \sigma(AB \to \pi^{\pm}\pi^{0}X) + \sigma(AB \to \pi^{0}\pi^{\pm}X), \qquad (20a, b)$$

$$\sigma_0 \equiv \sigma(AB \to \pi^{+}\pi^{-}X) + \sigma(AB \to \pi^{-}\pi^{+}X) + \sigma(AB \to \pi^{0}\pi^{0}X). \qquad (20c)$$

The inequalities (19) also hold for the multipion inclusive processes

$$A + B \to n\pi + X; \qquad I_{AB}^{(max)} = 1, \qquad (21)$$

provided that σ_M includes all the n-pion states of charge M. Additional inequalities are obtainable for the n-pion case by substituting $|m| = 3, 4, \cdots, n$ into Eq. (17) and trying different values for m.

All the relations derived here hold for any set of fixed values of the momenta of the outgoing particles. Thus in a given experiment they can be tested at each point in the energy spectrum and angular distribution. Note that the inequalities (11e) and (18d) could test for the presence of an isotensor component in the electromagnetic current. Such equalities as Eq. (5c), which involve resonance production, are useless for such tests because ambiguities in separating resonances from background are always greater than the effect tested. Pais[3] has recently suggested extensive tests of isospin properties of currents by inequalities in exclusive reactions.

These model-independent relations follow from isospin and U-spin invariance, respectively, and will hold in any model (e.g., in the Mueller-Regge model) if the model does not violate isospin or SU(3) symmetry. Additional model-dependent symmetry relations have been obtained from particular models.[4] Those usually follow from assumptions that limit the quantum numbers in a particular channel to those of allowed (non-exotic) Regge trajectories, or to be those of the Pomeron in the case of a diffractive process.

REFERENCES

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†On leave from The Weizmann Institute, Rehovoth, Israel.

[1]M. Peshkin, Phys. Rev. 121, 636 (1961).

[2]To prove (2), express σ_{IM} as a sum of irreducible tensor polynomials $P_L(I, M)$, analogous to the spherical harmonics Y_{LO}. The Wigner-Eckart theorem implies that the maximum L in the sum is not greater than $2I_{AB}^{(max)}$. Details are given in Ref. 1. [Eq. (2.16) should read $Q_2 = \frac{3}{2}J_z^2 - \frac{1}{2}J(J+1)$].

[3] A. Pais (to be published). We thank Dr. Pais for calling our attention to the interest in this question.

[4] R. N. Cahn and M. B. Einhorn (to be published).

FIGURE CAPTION

Fig. 1. Cross section vs. M. In (a), $I_{AB}^{(max)} = \frac{1}{2}$. In (b), $I_{AB}^{(max)} = 1$.

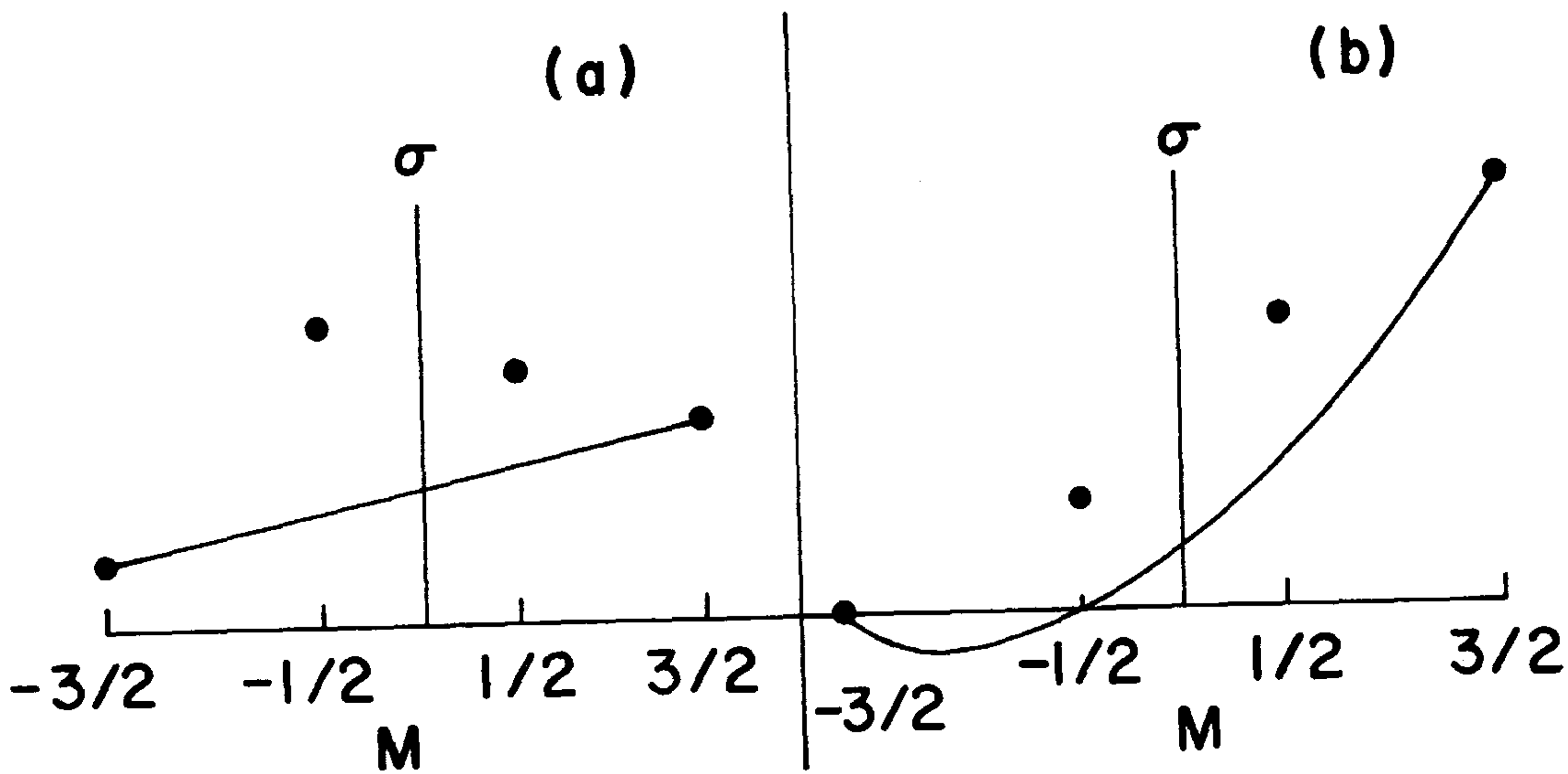

Fig. 1.