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SPONTANEOUSLY BROKEN GAUGE SYMMETRIES

PART III - EQUIVALENCE

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ABSTRACT

We discuss the equivalence of the S-matrix in the R- and U-gauge formulations of spontaneously broken gauge theories.

We give definitions of the U-gauge Green's functions in terms of the R-gauge ones, for both abelian and nonabelian cases. Based on the equivalence theorem, we give a renormalization prescription of the U-gauge formulation.

I. INTRODUCTION

In this paper, we wish to demonstrate the equivalence of the S-matrix in the R- and U-gauge formulations of spontaneously broken gauge theories. We have discussed the advantages and disadvantages of the two formulations in a previous paper (Part II).

We shall carry out this demonstration by expressing Green's functions in the U-gauge in terms of those in the R-gauge. What we shall show in this paper is a concrete realization of the remarks made previously by Weinberg,⁽¹⁾ and Salam and Strathdee⁽²⁾ about the equivalence of the two formulations. But more importantly, the present work gives definitions of the U-gauge Green's functions in terms of the well-defined R-gauge ones.

This paper is organized as follows. In the next section we consider the equivalence of the two formulations for the Abelian model considered previously. In Sec. III, we give some illustrations of the equivalence and formulate the renormalization prescription in the U-gauge. In Sec. IV, we deal with the generalization to nonabelian cases.

It is empirically known that the T-matrix for the Abelian case computed in the U-gauge is finite.^(1,3) This is a corroboration of our general arguments in this paper.

II. ABELIAN CASE

We recall the model discussed in ref. (4). It consists of the gauge boson A_μ , coupled to a complex scalar field ϕ :

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + (\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi \\ & - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2 - \delta\mu^2(\phi^*\phi), \end{aligned} \quad (2.1)$$

with $\mu^2 < 0$. The symmetric vacuum is unstable, and an asymmetric vacuum becomes stable. Let v be the vacuum expectation of ϕ . We can adjust the phase of ϕ so that v is real.

The R-gauge formulation is suggested by the parametrization

$$\phi = \frac{1}{\sqrt{2}}(v + \psi + i\chi), \quad (2.2)$$

where ψ and χ are real fields with null vacuum expectation value, and requires the subsidiary condition

$$\partial^\mu A_\mu(x) = 0. \quad (2.3)$$

The U-gauge formulation, on the other hand, is based on the choice of fields

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2}}(u + \rho) e^{i\zeta}, \\ B_\mu &= A_\mu - \frac{1}{e} \partial_\mu \zeta, \end{aligned} \quad (2.4)$$

where ρ and ζ are real. Under the gauge transformation of the second kind, we have

$$\begin{aligned} B_\mu &\rightarrow B_\mu \\ \rho &\rightarrow \rho \\ \zeta &\rightarrow \zeta^\theta = \zeta + \theta \end{aligned} \tag{2.5}$$

The constant u is to be so adjusted that $\langle \rho \rangle_0 = \langle \zeta \rangle_0 = 0$. (Only in the tree approximation, does one have $u = v$). The gauge condition is chosen to be

$$\zeta = c \tag{2.6}$$

where c is a constant, which we shall choose to be zero.

Since

$$\int [d\theta] \prod_x \delta[\zeta^\theta(x)] = 1$$

we may write the generating functional of the U-gauge Green's functions as

$$\begin{aligned} \exp i Z_U[J_\mu, K] &= \int [dB_\mu][d\rho][d\zeta] \prod_x \delta[\zeta(x)] J[\rho] \\ &\times \exp i \left\{ S_U[B_\mu, \rho] + \int d^4x [J_\mu B^\mu + K\rho] \right\} \end{aligned} \tag{2.7}$$

where S_U is the action expressed in terms of the U-gauge variables, and $J[\rho]$ is the Jacobian of the functional transformation

$(\psi, \chi) \rightarrow (\rho, \zeta)$:

$$J[\rho] \sim \exp\left\{\delta'(0) \int d^4x \ln\left[1 + \frac{1}{u} \rho(x)\right]\right\} \quad (2.8)$$

We may restrict the source J_μ to be transverse in Eq. (2.7):

$$\partial^\mu J_\mu = 0. \quad (2.9)$$

Since

$$\int [d\theta] \prod_x \delta[\partial^\mu A_\mu^\theta(x)] = \text{constant}, \quad (2.10)$$

independent of A_μ , we may insert the factor (2.10) on the right hand side of Eq. (2.7), and revert to the R-gauge variables. We obtain thereby

$$\begin{aligned} \exp i Z_\sigma [J_\mu, K] &= \int [dA_\mu][d\psi][d\chi] \prod_x \delta(\partial^\mu A_\mu(x)) \\ &\times \exp i \left\{ S_R [A_\mu, \psi, \chi] + \int d^4x \left[-J_\mu A^\mu + K \left(\sqrt{(v + \psi)^2 + \chi^2} - u \right) \right] \right\} \end{aligned} \quad (2.11)$$

Equation (2.11) allows us to evaluate the U-gauge Green's functions by the Feynman rules of the R-gauge.

It follows immediately from Eq. (2.11) that the transverse parts of the vector meson propagators are the same in both gauges. In particular, if we write

$$\lim_{k^2 \rightarrow 0} \Delta_{\mu\nu}(k; U) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) \frac{X_3}{k^2 - m^2} + k_\mu k_\nu \text{ term},$$

$$\lim_{k^2 \rightarrow 0} \Delta_{\mu\nu}(k; R) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) \frac{Z_3}{k^2 - m^2} + k_\mu k_\nu \text{ term},$$

we have

$$X_3 = Z_3 \quad (2.12)$$

Let us examine the scalar source term in Eq. (2.11). It is

$$\begin{aligned} & K \left(\left((\nu + \psi)^2 + \chi^2 \right)^{1/2} - u \right) \\ &= K \left[(\nu - u) + \psi + \frac{1}{2\nu} \chi^2 - \frac{1}{2\nu^2} \psi \chi^2 + \dots \right] \end{aligned} \quad (2.13)$$

The scalar propagators behave near $k^2 = \mu^2$ like

$$\begin{aligned} \lim_{k^2 \rightarrow \mu^2} \Delta(k^2; U) &= \frac{X_2}{k^2 - \mu^2}, \\ \lim_{k^2 \rightarrow \mu^2} \Delta(k^2; R) &= \frac{Z_2}{k^2 - \mu^2}, \end{aligned} \quad (2.14)$$

The ratio $(X_2/Z_2)^{1/2}$ is not equal to one, due to the possibility of exciting the physical scalar meson by the nonlinear terms in Eq. (2.14).

Since the series on the right hand side of Eq. (2.13) is infinite, X_2 contains high order divergences. The mass shifts of the scalar boson in both gauges are the same, however.

From Eq. (2.11) we can compute a Green's function in the U-gauge.

After the amputation of external lines, and after letting external momenta on the mass shell, we obtain

$$\chi_2^{\frac{1}{2}E_s} T(U) = Z_2^{\frac{1}{2}E_s} T(R) \quad (2.15)$$

where E_s is the number of external scalar lines and $T(U)$ is the T-matrix in the U-gauge. Note that Eq. (2.15) holds between the T-matrices. There are no such simple relations between proper vertices in the two formulations.

III. RENORMALIZATION

The Lagrangian in the U-gauge is

$$\begin{aligned}
 \mathcal{L} = & -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{(eu)^2}{2} B_\mu^2 \\
 & + \frac{1}{2}(\partial_\mu \rho)^2 - \frac{1}{2}(2\lambda u^2)\rho^2 - \frac{1}{4}\rho^4 - \lambda u\rho^3 \\
 & + \frac{1}{2}e^2 B_\mu^2(2u\rho + \rho^2) - \frac{1}{2}\delta\mu^2 - u\delta\mu^2 \rho
 \end{aligned} \tag{3.1}$$

The discussion in the last section implies that the T-matrix becomes finite if we choose $\delta\mu^2$ so that the vacuum expectation value of ρ is zero, and renormalize fields and constants according to

$$\begin{aligned}
 B_\mu &= B_\mu^r X_3^{1/2}, \\
 \rho &= \rho^r X_2^{1/2}, \quad u = v^r (X_2')^{1/2}, \\
 e &= e_r X_1 X_2^{-1} X_3^{-1/2}, \\
 \lambda &= \lambda_r X_4 X_2^{-2}.
 \end{aligned} \tag{3.2}$$

We will now discuss how X_1 , X_4 and X_2' may be chosen.

Let $e^2 u \Gamma_{1,2}(U)$ be the on-shell T-matrix element with one scalar and two vector particles. We will define X_1 by

$$\Gamma_{1,2} = \frac{X_2}{X_1} \left(\frac{X_2}{X_2'} \right)^{1/2} \tag{3.3}$$

Then the renormalized T-matrix:

$$e^2 u \Gamma_{1,2}(U) X_3 X_2^{1/2} = e_N^2 v_N \quad (3.4)$$

is finite. Similarly, we write $\lambda u \Gamma_{3,0}(U)$ for the on-shell T-matrix for three scalar particles, and define X_4 by

$$\Gamma_{3,0}(U) = \frac{1}{X_4} \left(\frac{X_2}{X_2'} \right)^{1/2} \quad (3.5)$$

We may choose X_2' so that the renormalized vector propagator has the low energy limit:

$$\lim_{k_\mu \rightarrow 0} \frac{1}{X_3} \Delta_{\mu\nu}(k) = -g_{\mu\nu} \frac{1}{(e_N v_N)^2} \quad (3.6)$$

The physical masses m^2 , and μ^2 are finite functions of e_r , λ_r and v_r .

The ratios $(X_2/Z_2)^{1/2}$ and $(X_2'/Z_2')^{1/2}$ may be computed perturbatively from the structure of Eq. (2.13), where Z_2 is defined by Eq. (2.13) and Z_2' by the relation $v = v_r (Z_2')^{1/2}$ [The ratio (Z_2/Z_2') is finite]. We have

$$\left(\frac{X_2}{Z_2} \right)^{1/2} = 1 + i\lambda \left[\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(p-k)^2} \right]_{p^2=\mu^2} - i \frac{1}{2v^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2}, \quad (3.7)$$

in the one loop approximation. The requirement that $\langle \rho \rangle_0 = 0$

translates into

$$\delta Z_\sigma / \delta K = 0 \quad (3.8)$$

Combining Eqs. (3.8), (2.7) and (2.13), we see that

$$u = v + \frac{i}{2v} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2},$$

or

$$\left(X_2' / Z_2' \right)^{1/2} = 1 + \frac{i}{2v^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2}, \quad (3.9)$$

in the one loop approximation.

There is an interesting check on Eq. (3.9). In the one loop approximation, the transverse self energy of the vector boson in the R-gauge is given by, with $m^2 = (ev)^2$, $\mu^2 = \lambda v^2$,

$$\begin{aligned} \Sigma_{\mu\nu}(p; R) &= i4e^2 (ev)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p-k)^2 - \mu^2} \frac{1}{k^2 - m^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \\ &\quad - ie^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p-k)^2 - \mu^2} \frac{1}{k^2} 4k_\mu k_\nu \\ &\quad + ie^2 g_{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \left[\frac{1}{k^2 - \mu^2} + \frac{1}{k^2} \right] = \Sigma_{\mu\nu}(p; U) + ie^2 g_{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2}. \end{aligned}$$

This difference between the self energies in the two gauges must be accountable by the difference in the first order expressions. From Eq. (3.9) we see that

$$e^2 u^2 = e^2 v^2 + ie \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2}$$

IV. NON-ABELIAN CASE

In this section we will derive the generalization of Eq. (2.11) to the nonabelian case. We shall use the concepts and notations developed in Sec. III of Part II. Corresponding to the decomposition of generators (3.11), we can write a finite transformation as

$$g = e^{i\alpha \cdot L} = e^{i\beta \cdot l} e^{i\gamma \cdot \underline{t}} \quad , \quad g \in G \tag{4.1}$$

where $\{\alpha\}$ and $\{\beta, \gamma\}$ are parametrizations of the group manifold.

Under this transformation, ξ and ρ defined by Eq. (II 3.11):

$\phi = D[\exp i \xi \cdot \underline{t}] (\nu + \rho)$
transform nonlinearly⁽⁵⁾ as

$$\xi \rightarrow \xi'(\xi, g) \tag{4.2}$$

where $\xi'(\xi, g)$ is defined by

$$g e^{i\xi \cdot \underline{t}} = e^{i\underline{t} \cdot \xi'(\xi, g)} e^{i\underline{u}(\xi, g) \cdot \underline{l}} \quad , \tag{4.3}$$

and

$$\rho \rightarrow \rho'(g) = D[e^{i\underline{l} \cdot \underline{u}(\xi, g)}] \rho \tag{4.4}$$

The vector fields $B_{\underline{m}\mu}$ and $C_{\underline{m}\mu}$ defined by

$$\{ B_{\underline{m}\mu} \cdot \underline{t} + C_{\underline{m}\mu} \cdot \underline{l} \} = e^{-i\xi \cdot \underline{t}} A_{\underline{m}\mu} \cdot \underline{l} e^{i\xi \cdot \underline{t}} + \frac{i}{g} e^{-i\xi \cdot \underline{t}} \partial_{\mu} e^{i\xi \cdot \underline{t}} \tag{4.5}$$

transform like

$$\begin{aligned}
 \underline{B}_\mu \cdot \underline{t} + \underline{C}_\mu \cdot \underline{l} &\rightarrow \underline{B}'_\mu(g) \cdot \underline{t} + \underline{C}'_\mu(g) \cdot \underline{l} \\
 &= e^{i\underline{u} \cdot \underline{l}} [\underline{B}_\mu \cdot \underline{t} + \underline{C}_\mu \cdot \underline{l}] e^{-i\underline{u} \cdot \underline{l}} \\
 &\quad - \frac{i}{g} (\partial_\mu e^{i\underline{u} \cdot \underline{l}}) e^{-i\underline{u} \cdot \underline{l}} \quad , \quad \underline{u} = \underline{u}(\underline{\xi}, g)
 \end{aligned}
 \tag{4.6}$$

We define the U-gauge Jacobian Δ_U by

$$\Delta_U[\underline{C}_\mu, \underline{\xi}] \int [dg] \prod_x [\delta(\partial^\mu \underline{C}'_\mu(x, g)) \delta(\underline{\xi}'(x, g))] = 1 \tag{4.7}$$

where dg is the invariant Hurwitz measure over the group manifold.

The generating functional of the U-gauge Green's functions is

$$\begin{aligned}
 \exp i Z_U[\underline{I}_\mu, \underline{J}_\mu, K] &= \int [d\underline{C}_\mu] [d\underline{B}_\mu] [d\underline{\xi}] [d\underline{\rho}] \Delta_U[\underline{C}_\mu, \underline{\xi}] \\
 &\times J_\phi[\underline{\rho}, \underline{\xi}] \prod_x \delta(\partial^\mu \underline{C}_\mu(x)) \delta(\underline{\xi}(x)) \\
 &\times \exp i \left\{ S_U[\underline{B}_\mu, \underline{C}_\mu, \underline{\rho}] + \int d^4x [K \cdot \underline{\rho} - \underline{I}_\mu \cdot \underline{C}^\mu - \underline{J}_\mu \cdot \underline{B}^\mu] \right\}
 \end{aligned}
 \tag{4.8}$$

where $J_\phi[\underline{\rho}, \underline{\xi}]$ is the Jacobian of the functional transformation

$\phi \rightarrow (\underline{\rho}, \underline{\xi})$. Because of the delta functions in the integrand of Eq. (4.8)

we may replace Δ_U by 1, and J_ϕ by $J_\phi[\underline{\rho}, 0]$.

We may insert

$$\Delta_L[\underline{A}_\mu] \int [dg] \prod_x \delta[\partial^\mu \underline{A}'_\mu(x; g)] = 1 \tag{4.9}$$

in the integrand of Eq. (4.8) and revert to the original variable A_μ and ϕ (which we shall assume to be real). We obtain, making use of the invariance of Δ_U , Δ_L and S under the transformation,

$$\begin{aligned} \exp i Z_U &= \int [dA_\mu] [d\phi] \Delta_L [A_\mu] \prod_x \delta(\partial^\mu A_\mu(x)) e^{i S_R [A_\mu, \phi]} \\ &\times \Delta_U [C_\mu, \xi] \int [dq] \prod_x [\delta(\partial^\mu C'_\mu(q)) \delta(\xi'(q))] \\ &\times \exp i \int d^4x [K \cdot \rho'(q) - \underline{I}^\mu \cdot C'_\mu(q) - \underline{J}^\mu \cdot B'_\mu(q)] \end{aligned} \quad (4.10)$$

where ρ , ξ , B_μ and C_μ on the right hand side are to be regarded as nonlinear functionals of A_μ and ϕ .

Let $g_0 = g(\beta_0, \gamma_0)$ be such that

$$\xi'_\mu(x; \xi, g_0) = 0, \quad (4.11)$$

and

$$\partial^\mu C'_\mu(x; g_0) = 0. \quad (4.12)$$

Then Eq. (4.10) becomes

$$\begin{aligned} \exp i Z_U &= \int [dA_\mu] [d\phi] \Delta_L [A_\mu] \prod_x \delta(\partial^\mu A_\mu(x)) \\ &\exp i \left\{ S_R [A_\mu, \phi] + \int d^4x [K \cdot \rho'(q) - \underline{I}^\mu \cdot C'_\mu(q) - \underline{J}^\mu \cdot B'_\mu(q)] \right\} \end{aligned} \quad (4.13)$$

where $\rho(g_0)$, $C'_\mu(g_0)$ and $B'_\mu(g_0)$ need still be expressed in terms of A_μ and ϕ . Equation (4.11) is satisfied if we choose

$$\gamma_0 = - \xi \quad (4.14)$$

in which case, we have also $u(\xi, g_0) = \beta_0$. The parameters β_0 are then determined by the requirement that C'_μ :

$$l \cdot C'_\mu(g_0) = e^{i\beta_0 \cdot l} l_\mu C_\mu e^{-i\beta_0 \cdot l} - \frac{i}{g} (\partial_\mu e^{i\beta_0 \cdot l}) e^{-i\beta_0 \cdot l} \quad (4.15)$$

be divergenceless.

As an illustration, we shall derive the expressions for $\rho(g_0)$, $C'_\mu(g_0)$ and $B'_\mu(g_0)$ for the model discussed in Sec. IV of Part II; SU(2) gauge bosons interacting with an isotriplet of scalar mesons ϕ , with $\langle \phi_3 \rangle_0 = v$. First, we have

$$\begin{aligned} (\xi_1, \xi_2) &= (\phi_2, -\phi_1) \frac{1}{\sqrt{\phi_1^2 + \phi_2^2}} \text{Arcsin} \frac{\sqrt{\phi_1^2 + \phi_2^2}}{u + \rho} \\ &= -(\gamma_1, \gamma_2)_0 \end{aligned} \quad (4.16)$$

and

$$\rho = \sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2} - u. \quad (4.17)$$

The parameter β_0 is determined from

$$\partial^\mu C'_\mu(g_0) = \partial^\mu (C_\mu + \frac{1}{g} \partial_\mu \beta_0) = 0$$

so that

$$\beta_0 = -g \frac{1}{\partial^2} \partial^\mu C_\mu. \quad (4.18)$$

The fields C_μ and $B_{\mu\mu}$ may be computed from Eq. (4.5):

$$\begin{aligned}
 C_\mu &= \frac{1}{2} T_N \tau^3 \left[e^{-i \underline{\xi} \cdot \underline{\tau} / 2} A_{\underline{\mu} \cdot \underline{\tau}} e^{i \underline{\xi} \cdot \underline{\tau} / 2} \right. \\
 &\quad \left. + \frac{2i}{g} \partial_\mu (e^{-i \underline{\xi} \cdot \underline{\tau} / 2}) e^{i \underline{\xi} \cdot \underline{\tau} / 2} \right] \quad (4.19) \\
 &= A_\mu^3 + (\underline{\xi} \times A_\mu)^3 - \frac{1}{2g} (\underline{\xi} \times \partial_\mu \underline{\xi})^3 + \dots
 \end{aligned}$$

and

$$\begin{aligned}
 B_\mu^\perp &= \frac{1}{2} T_N \tau^\perp \left[e^{-i \underline{\xi} \cdot \underline{\tau} / 2} A_{\underline{\mu} \cdot \underline{\tau}} e^{i \underline{\xi} \cdot \underline{\tau} / 2} \right. \\
 &\quad \left. + \frac{2i}{g} (\partial_\mu e^{-i \underline{\xi} \cdot \underline{\tau} / 2}) e^{i \underline{\xi} \cdot \underline{\tau} / 2} \right] \quad (4.20) \\
 &= A_\mu^\perp - \frac{1}{2} \partial_\mu \xi^\perp + (\underline{\xi} \times A_\mu)^\perp + \dots
 \end{aligned}$$

So, finally, we have

$$\begin{aligned}
 \rho'(q_0) &= \rho = \sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2} - u \\
 C'_\mu(q_0) &= (g_{\mu\nu} - \partial_\mu \partial_\nu / \partial^2) C^\nu \quad (4.21)
 \end{aligned}$$

and

$$[B'_\mu(q_0)]^\perp = \frac{1}{2} T_N \tau^\perp \left[e^{i \beta_0 \tau_3 / 2} B_{\underline{\mu} \cdot \underline{\tau}} e^{-i \beta_0 \tau_3 / 2} \right]$$

where β_0 , C_μ and B_μ are to be expressed in terms of A_μ and ϕ by the use of Eqs. (4.16) - (4.20).

Incidentally,

$$\beta_0 = -g \frac{1}{\partial^2} \left[(\partial^\mu \underline{\xi} \times A_\mu)^3 - \frac{1}{2g} (\underline{\xi} \times \partial^2 \underline{\xi})^3 \right] + \dots$$

In the nonabelian case $X_3 \neq Z_3$ in general, and the equivalence of the T-matrix in two gauges is expressed as

$$\chi_2^{\frac{1}{2}E_s} \chi_3^{\frac{1}{2}E_v} T(U) = Z_2^{\frac{1}{2}E_s} Z_3^{\frac{1}{2}E_v} T(R) \quad (4.22)$$

where E_v is the number of external vector lines. Equation (4.22) shows that the T-matrix in the U-gauge is finite after renormalization, and the T-matrix in the R-gauge is unitary and devoid of infrared divergences.

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