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Low Energy Theorem For $\gamma+\gamma \to \pi+\pi+\pi$

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ABSTRACT

We use the partially-conserved axial-vector current (PCAC) hypothesis to show that the matrix elements for $\gamma+\gamma \to \pi^0+\pi^0+\pi^0$ and $\gamma+\gamma \to \pi^0+\pi^++\pi^-$ vanish in the soft-π^0 limit. This, combined with photon gauge-invariance, implies low energy theorems relating these matrix elements to the matrix elements for $\gamma+\gamma \to \pi^0$ and $\gamma \to \pi^0+\pi^++\pi^-$. Since the magnitude of the former is determined by the π^0 lifetime, while the ratio of the latter to the former is determined in a model-independent way by isospin and low energy theorem arguments, a model-independent prediction for the $\gamma+\gamma \to \pi+\pi+\pi$ amplitude can be given. Our results agree with those of Aviv, Hari Dass and Sawyer in the neutral case, but not in the charged case. We give a diagrammatic and effective Lagrangian interpretation of our formulas which explains the discrepancy.

The reaction $\gamma+\gamma \to \pi+\pi+\pi$ is of interest, both because it may be observable in electron-positron colliding beam experiments[1], and because it is relevent to theoretical unitarity calculations[2] of a lower bound on the decay rate of $K_L^0 \to \mu^+\mu^-$. In recent papers, Aviv, Hari Dass and Sawyer[3] and Yao[4] have applied effective Lagrangian methods to calculate the matrix elements for the neutral and charged cases of $\gamma+\gamma \to \pi+\pi+\pi$. The fact that Refs. (3) and (4) are in disagreement has prompted us to repeat the calculation by standard current-algebra -PCAC methods.[5] Our results agree with Ref. (3) [but not with Ref. (4)] in the neutral case $\gamma+\gamma \to \pi^0+\pi^0+\pi^0$, and disagree with both Refs. 3 and 4 in the more interesting charged case $\gamma+\gamma \to \pi^0+\pi^++\pi^-$. After briefly discussing our method and results, we explain the reasons for our disagreement with the earlier calculations.

We begin with the simple, but powerful observation that the matrix elements $\mathcal{M}^{0+-} \equiv \mathcal{M}\left[\gamma(k_1) + \gamma(k_2) \to \pi^0(q_0) + \pi^+(q_+) + \pi^-(q_-)\right]$ and $\mathcal{M}^{000} \equiv \mathcal{M}\left[\gamma(k_1) + \gamma(k_2) \to \pi^0(q_0) + \pi^0(q_0') + \pi^0(q_0'')\right]$ vanish in the single soft π^0 limit $q_0 \to 0$, with the remaining two pions held on mass shell. To see this, we follow the standard PCAC procedure[6] of writing the reduction formula describing $\mathcal{M}^{0+-}$ or $\mathcal{M}^{000}$ with the pi-zero off shell, and then replacing the pi-zero field π^0 by the divergence of the axial-vector current $(M_\pi^2 f)^{-1}\, \partial_\lambda \mathfrak{F}_3^{5\lambda}$. (The normalization constant f is given by $f \approx f_\pi/(\sqrt{2}\, M_\pi^2) \approx 0.68\, M_\pi$, with f_π the charged-pion decay amplitude.) Because the corresponding axial charge F_3^5 commutes with the electromagnetic current, no equal-time commutator terms are picked up when the derivative ∂_λ is brought outside the T - product in the reduction formula. Integration by parts then makes the derivative act on the π^0 wave function, producing a factor $q_{0\lambda}$. Thus both $\mathcal{M}^{0+-}$ and $\mathcal{M}^{000}$ are proportional to q_0, and since they contain no pole terms which become singular as $q_0 \rightarrow 0$, they vanish in this limit. Note that this argument cannot be affected by the presence of divergence anomalies[7] in $\partial_\lambda \mathfrak{F}_3^{5\lambda}$, since all divergence anomalies vanish when the four-momentum q_0 associated with $\partial_\lambda \mathfrak{F}_3^{5\lambda}$ vanishes.[8,9]

In addition to the soft π^0 limit which we have just derived, we know that $\mathcal{M}^{0+-}$ and $\mathcal{M}^{000}$ must be gauge-invariant. That is, they are bilinear forms in ϵ_1 and ϵ_2 (the polarization vectors of the two photons) and vanish when either ϵ_1 is replaced by k_1 or ϵ_2 is replaced by k_2. We can now invoke the standard lore of current algebra low energy theorems,[5] which tells us that since we know three independent pieces of information about the low energy behavior of $\mathcal{M}^{0+-}$ and $\mathcal{M}^{000}$ (the $q_0 \rightarrow 0$ limit, gauge invariance for photon 1, and gauge invariance for photon 2), we can determine $\mathcal{M}^{0+-}$ and $\mathcal{M}^{000}$ from their pion pole diagrams up to an error of order $O(q_0 k_1 k_2)$ at least.[10] In particular the terms in $\mathcal{M}^{0+-}$ and $\mathcal{M}^{000}$ <u>quadratic</u> in the momenta k_1, k_2, q_0, $q_+(q_0')$, $q_-(q_0'')$ are completely determined. The relevant pion pole diagrams are illustrated in Fig. 1. The pion-pion scattering amplitudes which appear are evaluated from the current algebra expression[11,12]

$$\mathcal{M}\,[\,\pi^a \to \pi^b(q_b) + \pi^c(q_c) + \pi^d(q_d)]$$

$$= i f^{-2}\left\{\delta_{bc}\,\delta_{ad}[\,(q_b+q_c)^2 - M_\pi^2] + \delta_{bd}\,\delta_{ac}[\,(q_b+q_d)^2 - M_\pi^2] + \delta_{cd}\,\delta_{ab}[\,(q_c+q_d)^2 - M_\pi^2]\right\}, \tag{1}$$

while the $\gamma+\gamma \to \pi^0$ and $\gamma \to \pi^0+\pi^++\pi^-$ amplitudes are expressed in terms of coupling constants F^π and $F^{3\pi}$ defined by

$$\mathcal{M}\,[\,\gamma(k_1)+\gamma(k_2) \to \pi^0] = i\, k_1^\alpha k_2^\beta\, \epsilon_1^\gamma\, \epsilon_2^\delta\, \epsilon_{\alpha\beta\gamma\delta}\ F^\pi,$$
$$\mathcal{M}\,[\,\gamma(k_1) \to \pi^0 + \pi^+(q_+) + \pi^-(q_-)] = i\, k_1^\alpha\, \epsilon_1^\beta\, q_+^\gamma\, q_-^\delta\, \epsilon_{\alpha\beta\gamma\delta}\ F^{3\pi}. \tag{2}$$

The coupling constant F^π is related to the π^0 lifetime by [13]

$$\tau_{\pi^0}^{-1} = (M_\pi^3 / 64\pi)\,(F^\pi)^2\ ; \tag{3}$$

comparison with experiment gives $|F_\pi| = (\alpha/\pi)(0.66 \pm 0.08\ M_\pi)^{-1}$, with α the fine structure constant. While the coupling constant $F^{3\pi}$ has not been measured, both the theory of PCAC anomalies[14] and model-independent isospin and low-energy theorem arguments (see below) predict

$$eF^{3\pi} = f^{-2}\, F^{\pi}, \quad e = (4\pi\alpha)^{1/2}. \tag{4}$$

Combining Eqs. (1) and (2) with the appropriate propagators to form the pion pole diagrams, and adding the unique second degree polynomial which guarantees gauge invariance and vanishing of the matrix elements as $q_0 \to 0$, we get the following predictions for $\mathcal{M}^{0+-}$ and $\mathcal{M}^{000}$:

$$\mathcal{M}^{000} = i f^{-2} F^{\pi} k_1^{\alpha} k_2^{\beta} \epsilon_1^{\gamma} \epsilon_2^{\delta} \epsilon_{\alpha\beta\gamma\delta} \left[1 - \frac{(q_0+q_0')^2 + (q_0+q_0'')^2 + (q_0'+q_0'')^2 - 3M_\pi^2}{(q_0+q_0'+q_0'')^2 - M_\pi^2} \right]$$

$$= i f^{-2} F^{\pi} k_1^{\alpha} k_2^{\beta} \epsilon_1^{\gamma} \epsilon_2^{\delta} \epsilon_{\alpha\beta\gamma\delta} \left[\frac{-M_\pi^2}{(k_1+k_2)^2 - M_\pi^2} \right],$$

(when 3 final pions are on shell)

(5a)

$$\mathcal{M}^{0+-} = i f^{-2} F^{\pi} k_1^{\alpha} k_2^{\beta} \epsilon_1^{\gamma} \epsilon_2^{\delta} \epsilon_{\alpha\beta\gamma\delta} \left[1 - \frac{(q_+ + q_-)^2 - M_\pi^2}{(q_0+q_++q_-)^2 - M_\pi^2} \right]$$

$$-ieF^{3\pi} \epsilon_1^{\gamma} \epsilon_2^{\delta} \left\{ \left[\frac{(2q_+-k_2)_\delta}{k_2^2 - 2q_+\cdot k_2}\, k_1^{\alpha} (q_+-k_2)^{\sigma} q_-^{\tau} - \frac{(2q_--k_2)_\delta}{k_2^2 - 2q_-\cdot k_2}\, k_1^{\alpha} q_+^{\sigma} (q_--k_2)^{\tau} \right] \epsilon_{\alpha\gamma\sigma\tau} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \gamma \leftrightarrow \delta \end{pmatrix} + (k_1-k_2)^{\alpha} q_0^{\tau} \epsilon_{\alpha\gamma\delta\tau} \right\}. \tag{5b}$$

These equations are our basic results.[15]

According to Eq. (5), $\mathcal{M}^{000}$ is suppressed relative to $\mathcal{M}^{0+-}$, in agreement with the conclusion of Aviv et al. We disagree with the result for $\mathcal{M}^{000}$ quoted by Yao, who has (through an apparent algebraic error) replaced $-M_\pi^2$ in Eq. (5a) by $-4M_\pi^2$. In the case of strictly massless pions, our result for $\mathcal{M}^{000}$ becomes the simple statement that the terms in the matrix element quadratic in the external momenta vanish. This result can be immediately generalized to the reaction $\gamma+\gamma \to n\pi^0$, as follows: The PCAC argument given above tells us that in the limit when any one π^0 has zero four-momentum, with the other $n-1$ π^0's on mass shell, the matrix element $\mathcal{M}(\gamma+\gamma\to n\pi^0)$ must vanish. In addition, gauge invariance implies that $\mathcal{M}$ must vanish when either of the photon four-momenta k_1, k_2 vanishes. Taking four-momentum conservation into account, this gives us $n+2-1 = n+1$ independent conditions on the low-energy behavior of $\mathcal{M}$. Since for massless, neutral pions the pion pole diagrams (tree diagrams) sum to a constant, independent of pion four-momenta, the $n+1$ conditions can be satisfied only if $\mathcal{M}(\gamma+\gamma \to n\pi^0)$ vanishes[16] up to terms which are at least of order $(\text{momentum})^{n+1}$.

Our result for $\mathcal{M}^{0+-}$ in Eq. (5b) disagrees with the formulas quoted by Aviv et al. and by Yao, both of which overlook the class of pole diagrams proportional to $F^{3\pi}$. The formula of Aviv et al. also has the 1 in the square bracket multiplying F^π replaced by 1/3. In order to better understand this latter discrepancy, it is helpful to have a diagrammatic interpretation of the various terms in Eq. (5b). This is given in Fig. 2, which illustrates the lowest order perturbation theory contributions to $\mathcal{M}^{000}$ and $\mathcal{M}^{0+-}$ in the Gell-Mann - Levy σ- model.[17] The first and fourth rows give just the lowest order contributions to the pole diagrams of Fig. 1. The σ- pole diagrams in the second row can clearly be represented as matrix elements of the effective Lagrangian

$$i\,\mathcal{L}^{\sigma}_{\text{eff}} = \frac{i}{16}\, f^{-2} F^{\pi} F^{\alpha\beta} F^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta}\, \pi^{0} \underset{\sim}{\pi}\cdot\underset{\sim}{\pi}\,, \tag{6}$$

with $F^{\alpha\beta}$ the electromagnetic field-strength tensor. As a check, we note that $\pi^0 \underset{\sim}{\pi}\cdot\underset{\sim}{\pi} = (\pi^0)^3 + 2\pi^0\pi^+\pi^-$, and since the matrix element of $(\pi^0)^3$ has a Bose symmetry factor of 6, the contributions of Eq. (6) to $\mathcal{M}^{000}$ and to $\mathcal{M}^{0+-}$ are in the correct ratio of 3:1. Let us turn next to the five-point functions in the third row. Aviv et al. assume that these are represented by the same effective Lagrangian structure as in Eq. (6). If this were so, a five-point contribution of $-2\, f^{-2}\mathcal{M}^{\pi}$ to $\mathcal{M}^{000}$ would imply a corresponding contribution of $-(2/3)f^{-2}\mathcal{M}^{\pi}$ to $\mathcal{M}^{0+-}$, which would then combine with the σ- pole diagram to give a total non-pole contribution of $(1/3)\, f^{-2}\mathcal{M}^{\pi}$. This is the origin of the 1/3 in the formula of Aviv et al. In actual fact, however, we find that the five-point diagrams are not described by Eq. (6), but rather by the effective Lagrangian

$$i\,\mathcal{L}_{\mathrm{eff}}^{5\text{-pt}} = \frac{-i}{2}\, e\, F^{3\pi} (\partial^{\alpha} A^{\gamma})\, A^{\delta}\, \epsilon_{\alpha\gamma\delta\tau} (\partial^{\tau} \pi^{0})\, \underset{\sim}{\pi}\cdot\underset{\sim}{\pi}\,. \tag{7}$$

Eq. (7) still couples the three final pions through a pure I=1 state, as required by G-parity. In the charged pion case, Eq. (7) obviously leads to the five-point contribution listed in the third row of Fig. 2(b). Although not gauge-invariant by itself, this contribution combines with the pole terms in the fourth row of Fig. 2(b) (which are also not by themselves gauge-invariant) to give a gauge-invariant sum. In the neutral case, using the fact that the matrix element of $\partial^{\delta}\pi^{0}\,(\pi^{0})^{2}$ is $2\,i(q_0 + q_0' + q_0'') = 2i(k_1+k_2)$ and using Eq. (4) to eliminate $F^{3\pi}$ in terms of F^{π}, we find that Eq. (7) just gives the gauge-invariant contribution $-2f^{-2}\,\mathcal{M}^{\pi}$, as required.[18] Finally, we note that while Yao obtains the correct value of 1 for the constant term in the square bracket multiplying F^{π}, he gets this by using an incorrect effective Lagrangian, which does not respect the Δ I=1 rule, to generalize from the neutral to the charged case. The moral is that effective Lagrangians must be handled with caution. When ambiguities arise as to the form of the effective Lagrangian, they must be resolved by reference back to the basic current algebra relations, which the effective Lagrangian is supposed to represent.

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[4] Tsu Yao, Physics Letters 35B, 225 (1971).

[5] S. L. Adler and R. F. Dashen, Current Algebras and Applications to Particle Physics, (Benjamin, New York, 1968), Chapters 2 and 3.

[6] The reasoning is identical to that used to obtain the soft-π^0 theorem for η-decay; see D. G. Sutherland, Physics Letters 23, 384 (1966) and S. L. Adler, Physical Review Letters 18, 519 (1967) for details.

[7] S. L. Adler, Physical Review 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969); W. A. Bardeen, Physical Review 184, 1848 (1969).

[8] For example, the $\gamma+\gamma \rightarrow \pi^0$ matrix element given in Eq. (2) below may be rewritten, by using four-momentum conservation, in the form

$\mathcal{M}[\gamma+\gamma \rightarrow \pi^0] = i q_0^\alpha k_2^\beta \epsilon_1^\gamma \epsilon_2^\delta \epsilon_{\alpha\beta\gamma\delta} F^\pi$, and so vanishes when $q_0 = 0$. The effect of the PCAC anomaly on this reaction is to make soft pion calculations give $F^\pi \neq 0$.

[9]In fact, the diagrammatic analysis given below in Fig. 2 shows that the soft-π^0 limit of $\mathcal{M}[\gamma+\gamma \rightarrow \pi+\pi+\pi]$ involves only axial-vector Ward identities for ring diagrams which have pseudoscalar (and in some cases scalar) vertices in addition to vector vertices and the axial-vector vertex. These Ward identities are known not to have anomalies; see W.A. Bardeen, Ref. 7 and R.W. Brown, C.-C. Shih and B.L. Young, Physical Review 186, 1491 (1969).

[10]Since the matrix elements in question are even functions of the external four-momenta, the error will actually be of order $(\text{momentum})^4$.

[11]S. Weinberg, Physical Review Letters 17, 616 (1966). In addition to current algebra, the derivation of Eq. (1) assumes an isoscalar "σ-term".

[12]We use the notation and metric conventions conventions of J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields (McGraw-Hill Book Co., New York, 1965), pp. 377-390.

[13]For a discussion of experimental evidence on the sign of F^π, see F.J. Gilman, Physical Review 184, 1964 (1969) and S.L. Adler, π^0 Decay, in Proceedings of the Third International Conference on High Energy Physics and Nuclear Structure, S. Devons, ed. (Plenum Press, New York, 1970), p. 647.

[14]In a renormalizable fermion triplet model which satisfies PCAC, the anomaly predictions for F^{π} and $F^{3\pi}$ individually are $F^{\pi} \approx -(\alpha/\pi)\, f^{-1}\, 2\overline{Q}$ and $F^{3\pi} \approx -(e/4\pi^2)\, f^{-3}\, 2\overline{Q}$. The quantity $\overline{Q}$, which is the average charge of the non-strange triplet particles, drops out in the ratio. See S.L. Adler, Ref. 7; S.L. Adler and W.A. Bardeen, Physical Review 182, 1517 (1969); and R. Aviv and A. Zee (to be published).

[15]In the curly-bracketed terms in Eq. (5b), we have specialized to the case in which the charged pions are on mass shell: $q_+^2 = q_-^2 = M_\pi^2$.

[16]This generalizes the result of E.S. Abers and S. Fels, Physical Review Letters 26, 1512 (1971).

[17]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[18]We emphasize that this consistency check means that Eq. (4) is a model-independent result, since it is required by Eq. (5), together with the fact that the only two-derivative 2γ-3π couplings consistent with the $\Delta I = 1$ rule are given by Eqs. (6) and (7). An alternative derivation of Eq. (4) can be obtained by applying standard current-algebra-PCAC methods directly to $\mathcal{M}[\gamma \rightarrow \pi^0 + \pi^+ + \pi^-]$; anomalies do not enter because they make a contribution of higher order in momentum. See R. Aviv and A. Zee, Ref. 14, for further details.

FIGURE CAPTIONS

Figure 1: Pion pole diagrams for (a) the neutral and (b) the charged cases.

Figure 2: Lowest order diagrams contributing to (a) $\mathcal{M}^{000}$ and (b) $\mathcal{M}^{0+-}$ in the Gell-Mann-Lévy σ- model. The single solid line propagating around each loop denotes the nucleon. In this order of perturbation theory, $f^{-1} = g_r/M_N$, with g_r the pion-nucleon coupling constant and with M_N the nucleon mass. [The large black dot at the four-pion vertices denotes the pion-pion scattering amplitude of Eq. (1). To lowest order in perturbation theory, this arises as the sum of a direct four-pion interaction (coming from the term $(\underset{\sim}{\pi}\cdot\underset{\sim}{\pi})^2$ in the σ- model Lagrangian) and of pole terms involving σ- mesons exchanged between pairs of pions.]

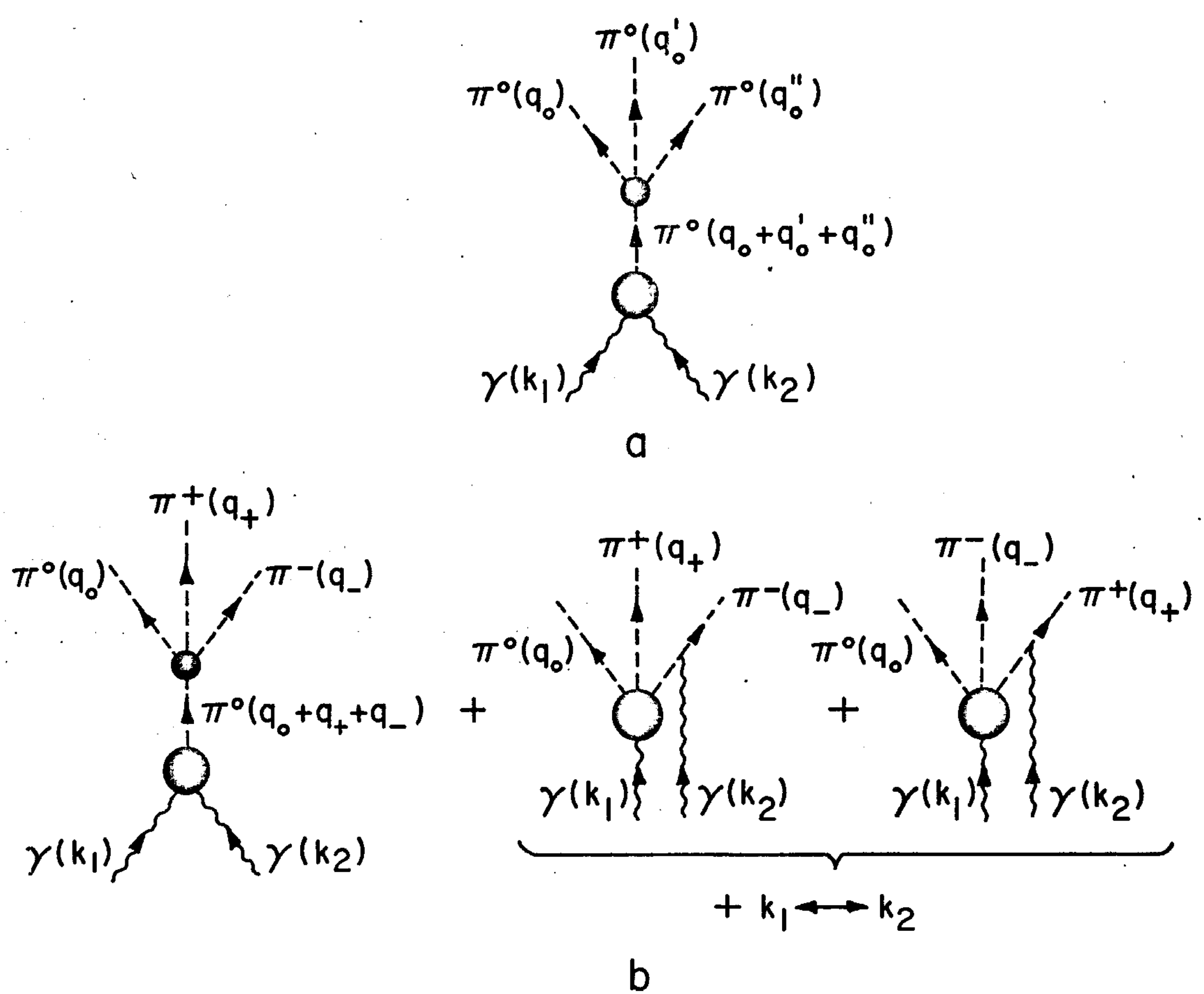
π°(q₀')
π°(q₀)
π°(q₀'')
π°(q₀+q₀'+q₀'')
γ(k₁)
γ(k₂)
a
π+(q+)
π°(q₀)
π−(q−)
π°(q₀+q++q−)
γ(k₁)
γ(k₂)
+
π+(q+)
π−(q−)
π°(q₀)
γ(k₁)
γ(k₂)
+
π−(q−)
π+(q+)
π°(q₀)
γ(k₁)
γ(k₂)
+ k₁ ⟷ k₂
b

Fig. 1

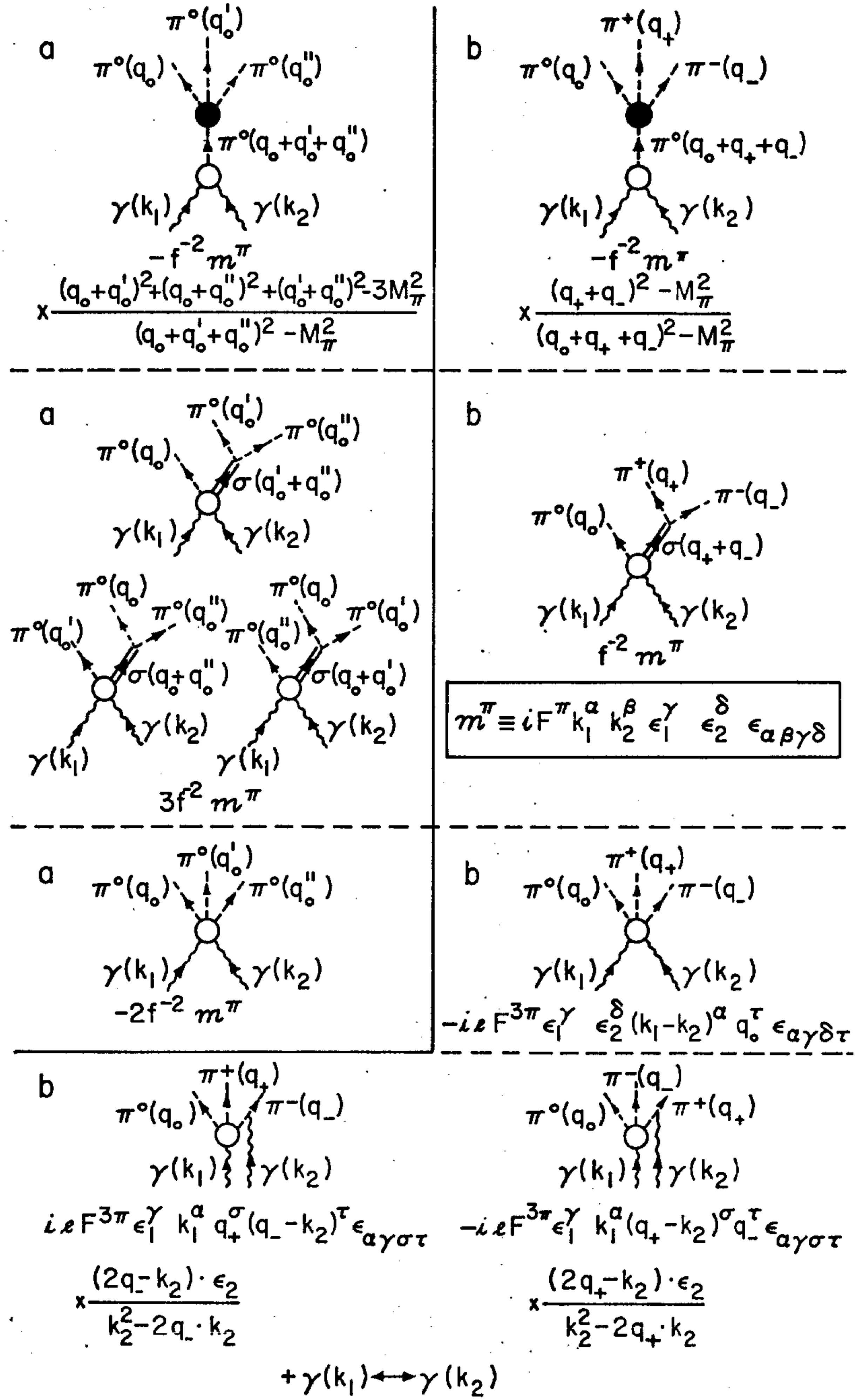
a
π°(q₀') π°(q₀) π°(q₀'') π°(q₀+q₀'+q₀'') γ(k₁) γ(k₂)
−f⁻² m^π × [(q₀+q₀')²+(q₀+q₀'')²+(q₀'+q₀'')²−3M_π²] / [(q₀+q₀'+q₀'')² −M_π²]
b
π⁺(q₊) π°(q₀) π⁻(q₋) π°(q₀+q₊+q₋) γ(k₁) γ(k₂)
−f⁻² m^π × [(q₊+q₋)² −M_π²] / [(q₀+q₊+q₋)²−M_π²]
a
σ(q₀'+q₀'') σ(q₀+q₀'') σ(q₀+q₀')
3f² m^π
b
σ(q₊+q₋)
f⁻² m^π
m^π ≡ iF^π k₁^α k₂^β ε₁^γ ε₂^δ ε_αβγδ
a
−2f⁻² m^π
b
−ieF^{3π} ε₁^γ ε₂^δ (k₁−k₂)^α q₀^τ ε_αγδτ
b
ieF^{3π} ε₁^γ k₁^α q₊^σ (q₋−k₂)^τ ε_αγστ × (2q₋−k₂)·ε₂ / (k₂²−2q₋·k₂)
−ieF^{3π} ε₁^γ k₁^α (q₊−k₂)^σ q₋^τ ε_αγστ × (2q₊−k₂)·ε₂ / (k₂²−2q₊·k₂)
+γ(k₁)↔γ(k₂)

Fig. 2