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GROUP THEORETICAL PROPERTIES OF DUAL RESONANCE MODELS

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GROUP THEORETICAL PROPERTIES OF DUAL RESONANCE MODELS*

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I. INTRODUCTION

One of the remarkable things about duality is that it leads to the formulation of very esthetic theoretical ideas although it has its roots in the structure of strong-interaction data. Surely this marriage of conceptual beauty with experimental observation is no accident. The first steps towards the construction of amplitudes that were "dual" have been excellently described in several review articles[1]; in these lectures we would rather like to show the emergence of a very fundamental group theoretical structure that seems to underlie all dual resonance models (DRM) built to date. Since no DRM duplicates the data closely enough, we would like to understand how to add the missing ingredients without affecting the properties we like about the more primitive models (like factorization, crossing, Regge behavior, etc.).

As we are only at the beginning of our understanding of duality, we can only talk at the moment about mesons and ask the more pragmatic reader to bear with us while we try to unravel this very mysterious concept.

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*Lectures presented at the 1971 Boulder Summer School.

The other purpose of these notes is to familiarize the reader with the mathematical techniques used in deriving DRM's. Hence the character of what follows will be rather technical as it must be at this stage of the art.

II. MATHEMATICAL PRELIMINARIES

The work of Koba and Nielsen[2] has shown the relevance of projective transformations in dual resonance models (DRM). These transformations are generated in the complex plane by real Möbius transformations which are locally isomorphic to the more familiar SU(1,1) group, the non-compact partner of SU(2). We concentrate from now on in the study of SU(1,1) so as to understand its role in DRMs in greater detail.[3]

If h is an element of SU(1,1) it is in one-to-one correspondence with the pseudounitary unimodular 2×2 matrix

$$h \rightarrow \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \qquad |\alpha|^2 - |\beta|^2 = 1, \tag{1}$$

where α and β are complex numbers and the star denotes complex conjugation. Its Lie algebra is generated by the operators L_0, L_+, and L_- which obey

$$\left[L_0, L_\pm\right] = \pm L_\pm \quad ; \qquad \left[L_+, L_-\right] = -L_0 \tag{2}$$

and has a Casimir operator

$$L^2 = L_+L_- + L_-L_+ - L_0^2. \tag{3}$$

In the complex z-plane, h corresponds to

$$z \rightarrow (hz) \equiv z' = \frac{\alpha z + \beta}{\alpha^* + \beta^* z}. \tag{4}$$

In particular it maps any point on the unit circle onto the unit circle. In order to construct the representations of the SU(1,1) algebra, we choose a certain representation for the generators

$$L_0 = z \frac{d}{dz} ; \qquad L_{\pm} = \frac{1}{\sqrt{2}} z^{\pm 1} \left(z \frac{d}{dz} \mp J \right) \tag{5}$$

for which

$$L^2 = -J(J+1) \tag{6}$$

is automatically a c-number. It can be shown that there exists basically two types of unitary representations of the algebra: those for which the spectrum of L_0 is unbounded and those for which it is bounded. For reasons that will become clear later we concentrate on the latter ones. There we again have two subdivisions since L_0 can be bounded either from above or below. These unitary irreducible representations (UIR) of the algebra are

1. $\underline{D_J^{(+)}}$, where J is a real negative number, as required by unitarity, and the spectrum of eigenvalues of L_0 is bounded below

$$L_0 = -J, \ -J+1, \ -J+2, \ \ldots ,$$

and it is spanned by the states

$$| J, m)_+ = \sqrt{\frac{(m-1-2J)!}{m!}} \, z^{m-J} . \tag{7}$$

Note that

$$L_- | J, 0)_+ = 0, \tag{8}$$

and the states are generated by successive application of L_+ on $|J,0)_+$.

2. $\underline{D_J^{(-)}}$, where again J is real and negative and the spectrum of L_0 is bounded above

$$L_0 = J, \ J-1, \ J-2, \ \ldots \ .$$

It is spanned by the basis

$$|J,m)_- = \sqrt{\frac{(m-1-2J)!}{m!}}\, z^{-m+J} \tag{9}$$

and

$$L_+ |J,0)_- = 0 \tag{10}$$

so that the states are generated by the successive application of L_- on $|J,0)_-$. The connection of these representations to the DRM's is achieved in the following way. Introduce the operator functions[4]

$$F_\rho(z) = \sum_{m=0}^{\infty} a_\rho^{(m)} |J,m)_+ \qquad \epsilon D_J^{(+)} \tag{11a}$$

$$\tilde{F}_\rho(z) = \sum_{m=0}^{\infty} a_\rho^{(m)\dagger} |J,m)_- \qquad \epsilon D_J^{(-)} \tag{11b}$$

where the coefficients of the basis vectors are harmonic oscillator operators[5] obeying

$$\left[a_\rho^{(m)}, a_\sigma^{(n)}\right] = \left[a_\rho^{(m)\dagger}, a_\sigma^{(n)\dagger}\right] = 0$$

$$\left[a_\rho^{(m)}, a_\sigma^{(n)\dagger}\right] = g_{\rho\sigma}\delta_{n,m} \qquad n,m = 0,1,\ldots \tag{12a}$$

we use the metric $g_{oo} = -g_{ii} = -1$ so that we immediately see that the $a_0^{(n)\dagger}$ will introduce negative norm states in the theory. (This disease plagues all relativistic theories.) The vacuum state $|0>$ is defined by

$$a_\mu^{(n)}|0> = 0 \qquad n = 0,1,\ldots \tag{12b}$$

Furthermore consider the case n = 0 to be describing a translational mode,[6] that is, let $J = -\epsilon/2$ where ϵ is a positive infinitesimal. Then, when written in terms of the canonical coordinates,

$$q_\rho = \frac{1}{\sqrt{\epsilon}} \left[a_\rho^{(0)\dagger} + a_\rho^{(0)} \right] \tag{13a}$$

$$p_\rho = \frac{i\sqrt{\epsilon}}{2} \left[a_\rho^{(0)\dagger} - a_\rho^{(0)} \right]. \tag{13b}$$

$F_\rho(z)$ and $\tilde{F}_\rho(z)$ are separately singular as $\epsilon \to 0$; however, this singularity is absorbed by taking their sum

$$Q_\rho(z) = F_\rho(z) + \tilde{F}_\rho(z) \tag{14}$$

$$= q_\rho + ip_\rho \ln z + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[a_\rho^{(n)\dagger} z^{-n} + a_\rho^{(n)} z^{n} \right], \tag{15}$$

which can be loosely interpreted as the dual generalization of a coordinate. Another quantity of interest is the "generalized momentum"

$$P_\rho(z) = -i\, z \frac{d}{dz} Q_\rho(z) \tag{16}$$

$$= p_\rho + i \sum_{n=1}^{\infty} \sqrt{n} \left[a_\rho^{(n)\dagger} z^{-n} - a_\rho^{(n)} z^{+n} \right]. \tag{17}$$

The relevant representation of the SU(1,1) operators is now obtained by taking the matrix elements of the operators Eq. (5) between the states Eq. (11a) or equivalently Eq. (11b):

$$L_0 = (F | L_0 | F) = \sum_{m=0}^{\infty} (m + \frac{\epsilon}{2})\, a^{(m)\dagger} \cdot a^{(m)} \tag{18a}$$

$$L_+ = (F|L_+|F) = \sum_{m=0}^{\infty} \sqrt{\frac{1}{2}(m+\epsilon)(m+1)}\, a^{(m+1)\dagger} \cdot a^{(m)} \tag{18b}$$

$$L_- = (F|L_-|F) = \sum \sqrt{\frac{1}{2}(m+\epsilon)(m+1)}\, a^{(m)\dagger} \cdot a^{(m+1)}. \tag{18c}$$

Another more elegant way of obtaining the representation of the SU(1,1) operators is to consider the Fourier coefficients of the square of the "generalized momentum"[7]

$$L_{-m} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{dz}{z}\, z^{-m} : P_\mu(z) P^\mu(z) : \tag{19}$$

where z is on the unit circle and the normal ordering applies to the periodic modes only. Specializing expression (19) to $m = 0, \pm 1$, we obtain the usual representation of the SU(1,1) generators, namely

$$L_0 = 2\left\{ \frac{1}{2} p^2 + \sum_{m=1}^{\infty} m\, a^{(m)\dagger} \cdot a^{(m)} \right\} \tag{20a}$$

$$L_{+1} = 2\left\{ i\, p \cdot a^{(1)\dagger} + \sum_{m=1}^{\infty} \sqrt{m(m+1)}\, a^{(m+1)\dagger} \cdot a^{(m)} \right\} \tag{20b}$$

$$L_{-1} = 2\left\{ -i\, p \cdot a^{(1)} + \sum_{m=1}^{\infty} \sqrt{m(m+1)}\, a^{(m)} \cdot a^{(m+1)\dagger} \right\}. \tag{20c}$$

Here, unlike the previous representation we have already taken ϵ to zero. However, for calculational purposes we prefer to use Eqs. (18) and let $\epsilon \to 0$ only at the end of all calculations.[8]

For general integer m, Eq. (19) yields

$$L_m = 2\left\{ i\sqrt{m}\, p \cdot a^{(m)\dagger} + \sum_{n=1}^{\infty} \sqrt{n(m+n)}\, a^{(n+m)\dagger} \cdot a^{(n)} - \frac{1}{2} \sum_{n=1}^{m-1} \sqrt{n(m-n)}\, a^{(m-n)\dagger} \cdot a^{(n)\dagger} \right\} \tag{21}$$

These operators were first found by Virasoro[9] in conjunction with the ghost compensation mechanism that occurs in the DRM's. They form among themselves the so-called Virasoro algebra[10]

$$\left[L_m, L_n\right] = 2(n-m)\,L_{n+m} + \frac{4}{3}\,n(n^2-1)\,\delta_{n,-m}\,. \tag{22}$$

The generators $1/2n\ L_{\pm n}$ and $1/n\ L_0 + (n^2-1)/3n$ form, for a given n, an SU(1,1) algebra and generate finite transformations of the form

$$z \underset{n}{\rightarrow} \qquad z' = \left[\frac{\alpha z^n + \beta}{\alpha^* + \beta^* z^n}\right]^{1/n} \qquad n = 1, 2, \ldots \tag{23}$$

At the present moment, however, the relevance of this algebra to duality has not been clarified although it is suspected to be very deep. All we can say is that it acts as a gauge group for dual models. More will be said on this in the course of these lectures.

The major part of the mathematical equipment needed in dual "modelry" has now been presented, and we turn our attention to the problem of the construction of dual factorizable tree amplitudes.

III. GROUP THEORETICAL RULES FOR THE CONSTRUCTION OF DUAL AMPLITUDES

We wish to emphasize that the rules we will enunciate in this section[11] are not the product of very deep insight but rather of a detailed analysis of the N-point generalization of the Veneziano amplitude. In addition, we believe them to be necessary but not sufficient.

1. Associate with the absorption of a particle of momentum k_μ, with various quantum numbers collectively labelled by $\{\lambda\}$, a vertex operator $V(k_\mu, \{\lambda\}; z)$, where $z = e^{-i\tau}$.

2. In order to preserve the correct selection rules at each vertex, we require that V transforms under the groups which generate $\{\lambda\}$ as the field of the absorbed particle.

3. At this stage, the dynamical assumption of duality is expressed in terms of an additional transformation requirement. Namely we demand that

$$[L_0, V(k, \{\lambda\}; z)] = - z \frac{d}{dz} V(k, \{\lambda\}; z) \qquad (24)$$

$$[L_\pm, V(k, \{\lambda\}; z)] = - \frac{z^{\pm 1}}{\sqrt{2}} \left(z \frac{d}{dz} \mp J_s \right) V(k, \{\lambda\}; z), \qquad (25)$$

where J_s in this case is a scalar function depending on the various quantum numbers of the particle

$$J_s = J_s(m^2, j, c^{\{\lambda\}}); \qquad (26)$$

here j is the spin and $c^{\{\lambda\}}$ represents the Casimir operators of the groups which generate $\{\lambda\}$. This means that the additional feature of dual vertices is that they are labelled by the Casimir operator of SU(1,1).

If T is a finite unitary transformation of SU(1,1), it follows that

$$T\, V(k,\{\lambda\};z)\, T^{\dagger} = |\alpha^{*} + \beta^{*} z|^{2J_s} V(k,\{\lambda\};z') \tag{27}$$

with

$$z' = \frac{\alpha z + \beta}{\alpha^{*} + \beta^{*} z} . \tag{28}$$

4. An arbitrary number of particles can interact in a dual manner only if their dual vertices have the same SU(1,1) spin, i.e.,

$$J_s(m_1^2, j_{(1)}, \ldots) = J_s(m_2^2, j_{(2)}, \ldots), \tag{29}$$

which implies, as we shall see later, relations between the various quantum numbers of particles. The origin of this requirement becomes clear where one tries to build amplitudes out of these dual vertices.

5. The factorizable dual amplitude for the scattering of an arbitrary number of particles in a given order is just given in the tree approximation by the vacuum expectation value of the product of their dual vertices taken so as to make an SU(1,1) invariant.[12] The amplitude corresponding to Fig. 1 is then given by

$$A_N(k_1, \ldots, k_N) = \int \!\! \ldots \!\! \int dz_1 \ldots dz_N\, K_N(z_1, \ldots, z_N)\, \delta^{(4)}\left(\sum_1^N k_i\right) \times$$
$$\langle 0 | V(k_1,\{\lambda\}_1;z_1)\, V(k_2,\{\lambda\}_2;z_2) \ldots V(k_N,\{\lambda\}_N;z_N) | 0 \rangle . \tag{30}$$

The requirement that A_N be SU(1,1) invariant imposes severe restrictions on the kernel function $K_N(\{z\})$. In fact, given the transformation properties (27) of the dual vertices, we have been able to find such a kernel only when all the external particles had the same SU(1,1) spin J_s, which explains the previous requirement. We now show how to build K_N up to any SU(1,1) invariant function.

It is easy to see by using the projective invariance of the vacuum and inserting $T^\dagger T$ between the vacuum and $V(k_1, \{\lambda\}_1, z_1)$ and pushing T to the right by means of Eq. (27) that any such kernel must obey

$$dz_1 \ldots dz_N K_N(\{z\}) \prod_{i=1}^{N} |\alpha^* + \beta^* z_i|^{2J_s^{(i)}} = dz_1' \ldots dz_N' K_N(\{z'\}) , \qquad (31)$$

where

$$z_i' = \frac{\alpha z_i + \beta}{\alpha^* + \beta^* z_i} \qquad i = 1, 2, \ldots, N. \qquad (32)$$

From the last equation, it is straightforward to see that

$$\frac{dz_i'}{z_i'} = \frac{dz_i}{z_i} \frac{1}{|\alpha^* + \beta^* z_i|^2} , \qquad (33)$$

as well as

$$\frac{z_{i+1}' - z_i'}{z_{i+1} - z_i} = \frac{1}{(\alpha^* + \beta^* z_{i+1})(\alpha^* + \beta^* z_i)} . \qquad (34)$$

We find a solution to Eq.(31) when all $J_s^{(i)}$ are equal, say

$$J_s^{(i)} \equiv J_s \qquad i = 1, \ldots N, \tag{35}$$

namely

$$K_N(z_1, \ldots, z_N) = \frac{1}{z_1 \ldots z_N} \prod_{i=1}^{N} |z_{i+1} - z_i|^{-J_s - 1}, \tag{36}$$

where we have defined $z_{N+1} \equiv z_1$. We point out that this solution is not unique as it can be multiplied by any SU(1,1) scalar function of the z_i's. In particular, we can put an ordering condition on the arguments of the z_i's according to the order in which the vertices appear. As we shall show by example, this condition is necessary for the factorization of the amplitude. Hence one factorizable amplitude is given by

$$A_N(k_1, \ldots, k_N) = \int \prod_{i=1}^{N} \frac{dz_i}{z_i} |z_{i+1} - z_i|^{-J-1} \theta(\arg z_{i+1} - \arg z_i) \left\langle 0 \right| \prod_{i=1}^{N} V(k_i, \{\lambda\}_i; z_i) \left| 0 \right\rangle \delta^{(4)} \left(\sum_1^N k_i \right). \tag{37}$$

Since the integrand is invariant under a three parameter group, it is really a function of N-3 variables. So far we have not said how duality comes about. The fact is that all the vertices we shall consider give rise to cyclic invariant amplitudes. All we can say is that the covariance under SU(1,1) does not seem to be sufficient to insure cyclic invariance. It may be that covariance of the vertices under the Virasoro algebra is a requirement for it.[13] However, in the following we shall not concern ourselves with such highbrow considerations; rather we aim to show in detail how the various ideas discussed above come into being when one considers specific vertices which obey our criteria.

IV. CONSTRUCTION OF THE N-POINT VENEZIANO FUNCTION

In order to give content to the preceding section, we give in great detail the derivation of the N-point function for external scalar particles using as a starting point the dual vertex for the absorption of a scalar particle. We first observe that

$$\left[L_0,\ F_\rho(z)\right] = -z\frac{d}{dz}F_\rho(z) \tag{38a}$$

$$\left[L_\pm,\ F_\rho(z)\right] = -\frac{z^{\pm 1}}{\sqrt{2}}\left(z\frac{d}{dz} \pm \frac{1}{2}\epsilon\right)F_\rho(z) \tag{38b}$$

where we have used representation (18) for the generators and the commutation relations (12). This means that $F_\rho(z)$ transforms with an SU(1, 1) spin $J_s = -\epsilon/2$. The same holds for $\tilde{F}_\rho(z)$. This is a very important point, and it will be used for functions for which J_s is not infinitesimal when we want to add extra quantum numbers to the model.

Introduce the vertex for the absorption of a scalar particle[14]

$$V(k_\mu, \{0\}; z) \equiv V_0(k, z) = e^{-\frac{k^2}{2\epsilon}}\ e^{ik\cdot\tilde{F}(z)}\ e^{ik\cdot F(z)} \tag{39}$$

where the factor appearing in front cancels the infinity that appears in $\tilde{F}_\rho$ and F_ρ. In fact we can rewrite it as

$$V_0(k_\mu; z) = \ :e^{ik\cdot Q(z)}: \tag{40}$$

where the normal ordering : : only applies to the periodic modes. In order to see if this is a suitable dual vertex, we must check its commutation relations with the SU(1, 1) generators.

Since we now approach the realm of detailed calculations, it is good to quote a well-known and probably forgotten identity: if A and B are any two operators, then

$$e^{A} B e^{-A} = B + [A, B] + \frac{1}{2!} \Big[A, [A, B]\Big] + \ldots, \tag{41}$$

where the other terms are left to the imagination of the reader. Then, it is easy to check that

$$\Big[L_0, V_0(k, z)\Big] = - z \frac{d}{dz} V_0(k, z) \tag{42a}$$

$$\Big[L_\pm, V_0(k, z)\Big] = - \frac{z^{\pm 1}}{\sqrt{2}} \left(z \frac{d}{dz} \pm \frac{1}{2} k^2 \right) V_0(k, z) \tag{42b}$$

where we have used the mathematically ambiguous[15] form

$$\lim_{\epsilon \to 0} \epsilon \sum_{m=0}^{\infty} \frac{(m + \epsilon - 1)!}{m!} = 1. \tag{43}$$

Nevertheless, the end result is the same whether or not one chooses to calculate using a representation where ϵ is not yet equal to zero. The use of Eq. (43) yields consistent results and we shall keep with the use of the representation (18) for the generators.

This means that for a scalar particle $J_s = - \frac{1}{2} k^2$.

Introduce the trajectory function

$$\alpha(x) = \alpha_0 + \frac{1}{2} x \tag{44}$$

which means that $J_s = -\alpha_0$, the intercept of the mother trajectory. Now that we have a respectable vertex we can try to calculate an amplitude for the absorption of any number of scalars.

Consider the vacuum expectation value of N scalar vertices, the computation of which is made easy by realizing that the commutator between F and $\tilde{F}$ is a c-number, namely

$$\left[F_\rho(z_j),\ \tilde{F}_\sigma(z_1)\right] = g_{\rho\sigma}\left(\frac{1}{\epsilon} - \ln|z_j - z_1| - \frac{1}{2} i\pi\phi_{j1}\right) \tag{45}$$

where

$$\phi_{j1} = \begin{cases} +1 & \arg z_j > \arg z_1 \\ -1 & \arg z_j < \arg z_1 . \end{cases} \tag{46}$$

Needless to say the last equation is obtained by using Eq. (11) and noting that

$$\frac{z_1 - z_j}{\sqrt{z_1 z_j}} = -i\phi_{1j}\,|z_1 - z_j| \tag{47}$$

where z is a point on the unit circle, as well as the expansion for the logarithm

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} . \tag{48}$$

We can then use the identity

$$e^A e^B = e^B e^A e^{[A, B]}, \tag{49}$$

which holds only when $[A, B]$ is a c-number. It then follows from Eqs. (45) and (46) that

$$\langle 0 | \prod_{i=1}^{N} V_0(k_i, z_i) | 0 \rangle = e^{-\frac{1}{2\epsilon}\sum_{i=1}^{N} k_i^2} \, e^{-\sum_{j<l} \left[k_j \cdot F(z_j), k_l \cdot \tilde{F}(z_l) \right]} \tag{50a}$$

$$= e^{-\frac{1}{2\epsilon}\sum_{i=1}^{N} k_i^2 - \frac{1}{\epsilon}\sum_{j<l} k_j \cdot k_l + \frac{i\pi}{2}\sum_{j<l} k_j \cdot k_l \phi_{jl}} \prod_{j<l} |z_j - z_l|^{k_j \cdot k_l}. \tag{50b}$$

By noting that

$$\sum_{i<j} k_i \cdot k_j = \frac{1}{2}(k_1 + \dots + k_N)^2 - \frac{1}{2}\sum_{i=1}^{N} k_i^2 , \tag{51}$$

we get rid of the infinite factor by using in addition conservation of momentum. The phase factor can be absorbed only if ϕ_{jl} does not change sign, which shows the need for the ordering condition on the angles. As stated before we can form an amplitude only when all the SU(1, 1) spin are equal, i.e. in this case only when all the scalar particles have the same mass. In this case the amplitude is given by

$$A_N(k_i, \dots, k_N) = \int \prod_{i=1}^{N} \left\{ \frac{dz_i}{z_i} \, |z_{i+1} - z_i|^{\alpha_0 - 1} \, \theta(\arg z_{i+1}) \right\} \prod_{j<l} |z_j - z_l|^{k_j \cdot k_l} \tag{52}$$

which is, up to a factor, the Koba-Nielsen[2] form. Note the disappearance of the kernel when $\alpha_0 = 1$. We have stated above that, due to the invariance of the integrand under a three parameter group, three integration variables are superfluous. So as to give meaning to this statement, we now proceed to show how the above reduces to the well-known B-function in the case N = 4 (see Fig. 2).

Introduce the anharmonic ratio

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \tag{53}$$

which is real when all the z's are on the unit circle. Then

$$(1-x) = \frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_2-z_4)} \ . \tag{54}$$

Together with the use of the kinematical relations

$$\left.\begin{aligned} k_1 \cdot k_2 &= k_3 \cdot k_4 = -\alpha(s) - \alpha_0 \\ k_1 \cdot k_4 &= k_2 \cdot k_3 = -\alpha(t) - \alpha_0 \\ k_1 \cdot k_3 &= k_2 \cdot k_4 = \alpha(s) + \alpha(t) \end{aligned}\right\} \tag{55}$$

we obtain

$$\prod_{j<l} |z_j - z_l|^{k_j \cdot k_l} = x^{-\alpha(s)}(1-x)^{-\alpha(t)}(|z_1-z_2||z_3-z_4||z_2-z_3||z_1-z_4|)^{-\alpha_0} \ . \tag{56}$$

The last factor is cancelled by the measure, leaving us with

$$\int \frac{dz_1 dz_3 dz_4}{(z_1-z_3)(z_3-z_4)(z_4-z_1)} \theta(\arg z_1 - \arg z_3)\, \theta(\arg z_3 - \arg z_4)\, \theta(\arg z_4 - \arg z_1)$$

$$\times \int_0^1 dx\, x^{-1-\alpha(s)} (1-x)^{-1-\alpha(t)}. \qquad (57)$$

Hence all the kinematics are contained in one integrand while a three-fold integration separates. Furthermore it can be checked that the range of integration of x is from 0 to 1 only because of the θ-functions. Thus we have

$$A_4(k_1, \ldots, k_4) = \left(\int dH\right) \int_0^1 dx\, x^{-1-\alpha(s)} (1-x)^{-1-\alpha(t)}. \qquad (58)$$

The differential dH is known as the Haar measure.[16] It is an infinite constant and should be divided out of the amplitude. Calculations performed for an arbitrary number of legs show it to be independent of the number of external particles. It acquires a certain physical significance when one realizes that it is equal to the three-point function between scalar particles and could thus be interpreted as a bare coupling constant. This interpretation, however, is not consistent with factorization of an amplitude with many external particles.

We should emphasize that the ordering condition is necessary for factorization in the sense that it allows for the correct range of x. The amplitude (58) factorizes and corresponds to Fig. 2. We should, in order to obtain the full amplitude, add all the inequivalent penetrations of the external legs, as shown in Fig. 3.

When $\alpha_0 = 1$, however, the amplitude we calculate by omitting the ordering conditions is automatically equal to the sum of all the inequivalent penetrations. This remarkable property is true independent of the number of legs.[17]

We went through this calculational section to acquaint the reader with the mathematical techniques used in dual resonance models. Although the calculations were performed using the scalar vertex, much of the "meat" is the same when considering amplitudes (or equivalently vertices) with more complicated external particles. We feel that it is always good to give meaning to the abstruse concepts of the previous section by showing explicitely how they lead to familiar results.

V. GAUGE CONDITIONS

In constructing the scalar vertrices, it seems that we got more than we bargained for. Indeed, using the representation (19) for the Virasoro operators (of which the SU(1, 1) generators form a subset), we find the following commutation relations[18]

$$\left[L_{\pm n},\ V_0(k;z)\right] = -\frac{1}{\sqrt{2}}\, z^{\pm n}\left(z\frac{d}{dz} \pm n\frac{k^2}{2}\right) V_0(k;z) \qquad (59)$$

which means that the scalar vertices are covariant under a much more general algebra. The physical meaning of this peculiar equation becomes clear in the case $\alpha_0 = k^2/2 = 1$. We can then rewrite the commutator as

$$\left[L_{\pm n},\ V_0^{\alpha_0=1}(k,z)\right] = -\frac{1}{\sqrt{2}}\, z\frac{d}{dz}\left[z^{\pm n}\, V_0^{\alpha_0=1}(k;z)\right] \qquad (60)$$

remembering that z is on the unit circle, this means that the commutator is a perfect differential. In the case $\alpha_0 = 1$, the amplitude can be written as

$$A_N(k_1\ldots,k_N) = \langle 0|\prod_{i=1}^{N} \mathcal{V}(k_i)|0\rangle\, \delta^{(4)}\left(\sum_1^N k_i\right) \qquad (61)$$

where

$$\mathcal{V}(k_i) = \int_0^{2\pi} \frac{dz_i}{z_i}\, V_0^{\alpha_0=1}(k_i, z_i). \qquad (62)$$

Then, using the periodicity condition, we obtain

$$\left[L_{\pm n},\ V(k)\right] = 0. \qquad (63)$$

Since $L_{-n}|0\rangle = 0$, it is clear that the above means that there are subsidary conditions between the states appearing in the factorization of the model. A quick look back at the explicit representation of the L_m Eq. (21) shows that in the rest frame, they relate the "ghost-like" mode introduced by $a_0^{(n)}$ $(n = 1, \dots)$ which gives rise to negative norm states to other states in the theory. In fact explicit calculations show that they act as "ghost compensators" in such a way that there does not seem to be any negative norm state when $\alpha_0 = 1$ although there is a tachyon at the nonsense point of the mother trajectory.[19] We note that such a tachyon will appear whenever the mother trajectory has positive intercept as it does in the real world for the ρ! We wish to stress that although this condition for ghost elimination so far holds only when the mother trajectory has unit intercept, the DRM's are the only model to have such a mechanism. Indeed it may be argued that a similar mechanism will always be needed for relativistic theories of strong interactions where an infinite number of negative norm states will be introduced by the nature of the Lorentz metric.

At this stage of our understanding of the dual resonance model, it is well to review what we have. First of all, the model we have considered has no internal quantum numbers so that we are really dealing (at best) with what is hopefully the skeleton of a strong interaction theory. The nature of the commutation relations (12a) introduces negative norm states which are compensated only when the mother trajectory has unit intercept, thereby introducing a tachyon (which we would have had in any case in any channel involving the ρ trajectory). The model has an additional difficulty because the J_s of the dual vertices depends on the mass of the particle; since they have to be the same to obtain SU(1, 1) invariance, it is not clear how to go off mass shell and keep SU(1, 1).[20] This problem is, of course, acutely felt when one wants to introduce electromagnetic interactions.

Although the above remarks make it clear that the model must be improved, we have found a surprising group theoretical structure which seems to be at the origin of all the esthetic properties of the model. In addition we have found the existence of a gauge-like algebra, which seems to eliminate unwanted negative norm states. It is clear then that at least one of these features must be kept in devising more physical dual models.

In order to gain more familiarity with the SU(1, 1) aspect of the bare model, we will try to build dual vertices for the excited states of the theory. Then we will try to add quantum numbers to the bare theory and will examine several models that were recently proposed.

VI. DUAL VERTICES FOR EXCITED STATES

We now try to construct dual vertices for the excited particles that appear in the bare dual model. We start by constructing the vertex for the emission of a vector particle. Recall that P_μ is like a generalized momentum, which suggests the form[11]

$$V_\mu(k,\ j=1;\ z) = e^{ik\cdot\tilde{F}(z)}\ P_\mu(z)\ e^{ik\cdot F(z)} \tag{64}$$

which is written in a normal ordered form and where we disregard the factor $e^{-k^2/2\epsilon}$ from now on. Using the representation (18) for the generators, we find that

$$\left[L_0,\ V_\mu(k,\ j=1;\ z)\right] = -z\frac{d}{dz}V_\mu(k,\ 1;\ z) \tag{65a}$$

$$\left[L_\pm,\ V_\mu(k,1;z)\right] = -\frac{z^{\pm1}}{\sqrt{2}}\left[z\frac{d}{dz} \pm \left(1+\frac{k^2}{2}\right)\right]V_\mu(k,1;z) + \frac{z^{\pm1}}{2\sqrt{2}}k_\mu V_0(k,z). \tag{65b}$$

There are two important things to notice about the last equation. First, as written $V_\mu(k,1;z)$ is not covariant under SU(1,1) because of the second term appearing in Eq. (65b) and that this extra term is along k_μ and appears with the same sign for both L_+ and L_-. The only way to get rid of this term is to put a spin 1 projection operator! Thus the requirement of covariance under SU(1,1) forces the addition of the projection operator. The dual vertex for a vector meson is then

$$V_\mu(k,\ z) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) e^{ik\cdot\tilde{F}(z)}\ P_\nu(z)\ e^{ikF(z)}. \qquad (66)$$

The second thing to notice is that the J_s of this vector dual vertex is

$$J_s = - \left(1 + \frac{k^2}{2} \right) = - \left(1 - \frac{M_v^{\ 2}}{2} \right). \qquad (67)$$

Using our criterion (4), this vector meson will interact in a dual way with scalars only if

$$\frac{M_s^{\ 2}}{2} = -1 + \frac{M_v^{\ 2}}{2} \qquad (68)$$

where M_s and M_v are the masses of the scalar and the vector particles, respectively. This means that our vector meson lies on the first recurrence of the mother trajectory. Similarly, we can find the dual vertex for a particle of spin two.

Introduce the notation

$$\hat{P}_\mu(z) \equiv \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) P_\nu(z). \qquad (69)$$

The obvious choice is

$$V_{\mu\nu}(k,\ j=2;\ z) = e^{ik\tilde{F}(z)} : \hat{P}_\mu(z)\ \hat{P}_\nu(z) : e^{ik\cdot F(z)}. \qquad (70)$$

Explicit calculation shows that as it stands this vertex transforms covariantly under SU(1, 1) with a dual spin

$$J_s^{(2)} = -\left(2 + \frac{k^2}{2} \right) . \tag{71}$$

However, before interpreting it as the dual vertex for a spin two particle, we must subtract the traces. It turns out that this procedure is enforced by requiring covariance under $L_{\pm 2}$!![21] Hence, as hinted at in the previous section, the Virasoro operators play the role of projecting the vertices into definite spin states. The relation (71) shows that the spin 2 particle we are talking about lies on the second recurrence of the mother trajectory. It is rather straightforward to generalize our procedure to take into account all the states on the mother trajectory. (See Fig. 4.)

It turns out that dual vertices can be written for some of the daughter states that appear in the theory, as has been shown by Fubini and collaborators.[22] This construction is relevant only where the propagator is diagonal, i.e. $\alpha_0 = 1$. We just quote the result for the spin 1 daughter which has a dual vertex

$$V_\mu^D(k, z) = e^{ik\cdot\tilde{F}(z)} : \left(z \frac{d}{dz} + \frac{2}{k^2} k\cdot P \right) \hat{P}_\mu + k_\mu \{ \cdots \} : e^{ik\cdot F(z)} . \tag{72}$$

This vertex has $J_s^D = -(2 + k^2/2)$ so that it lies under the spin 2 state on the mother trajectory. We note, however, that it has a component along k_μ so that in this case covariance under SU(1,1) is not sufficient to eliminate this troublesome component. Indeed we need covariance under $L_{\pm 2}$ to handle it satisfactorily. However, the problem is technically very complicated and as of this writing not entirely solved, i. e. although one finds all the possible daughter vertrices covariant under SU(1, 1) it is very hard to separate the correct linear continuation which are also covariant under $L_{\pm 2}$, although considerable effort in this direction has been spent.

We wish to point out, however, that the spin 1 part of this daughter vertex can be rewritten as a perfect differential when $\alpha_0 = 1$, so that the first daughter decouples according to the mechanism outlined in the previous section. Actually this phenomenon occurs all along the first daughter trajectory when $\alpha_0 = 1$ so that it decouples entirely from the problem. This was first pointed out by Di Vecchia and Del Giudice[19] by a close analysis of the spectrum.

The main conclusion of this section is that the dual daughter vertices for definite spin states have not all been constructed as they must be covariant under the Virasoro algebra.[23] This problem seems hopelessly complicated at the moment, and we have nothing to add to it; rather we turn our attention to the inclusion of internal quantum numbers in the bare model.

VII. DUAL MODELS WITH ADDITIONAL QUANTUM NUMBERS

As stated earlier, another direction of research is to incorporate additional degrees of freedom into the bare model without upsetting the group theoretical properties under at least SU(1,1). In this section we describe three such models in their chronological appearance in the literature. These are the models proposed by Bardacki-Halpern[24] (I), Clavelli[25] (II), and Neveu and Schwarz[26] (III). Their common feature is that they start by introducing new operators as coefficients of the basis functions of the bounded representation of SU(1,1). They all have a G-parity operator and display a spectrum which, although not yet the physical one, shows great improvement over that of the bare model. The last model (III) has a new feature which is responsible for decoupling the tachyon appearing on the mother trajectory although another tachyon appears in the model. These statements will be clarified by considering the models in detail.

A. Bardacki-Halpern Model

The new degrees of freedom are introduced through the quark-like operator function

$$\psi_{(r)} = \sum_{m=0}^{\infty} b_r^{(m)} \left| -\tfrac{1}{2}, m\right)_+ + d_r^{(m)\dagger} \left| -\tfrac{1}{2}, m\right)_- \qquad r = 1,2,3, \tag{73}$$

where the notation for the states is that of Section II and the coefficient obeys the following anticommutation relations

$$\left\{b_r^{(m)}, d_s^{(n)}\right\} = \left\{b_r^{(m)\dagger}, d_s^{(n)}\right\} = 0$$
$$\left\{b_r^{(n)}, b_s^{(m)\dagger}\right\} = \delta_{rs}\,\delta_{nm} = \left\{d_r^{(n)}, d_s^{(m)\dagger}\right\} \qquad r,s = 1,2,3 \qquad m,n = 0,1,\ldots \tag{74}$$

The point of this construction is that one can define new SU(1,1) and Virasoro operators

$$L_{-m}' = -\frac{i}{4\pi}\int_0^{2\pi} \frac{dz}{z}\, z^{-m} : \left[\psi_{(r)}, z\frac{d}{dz}\psi_{(r)}\right] : \tag{75}$$

such that the quark-like functions $\psi_{(r)}$ transform under these new operators with a SU(1,1) spin - 1/2. It follows that, if we define the new SU(1,1) operators as the sum of those appearing in Section II and the operators defined by Eq. (75)

$$L_{-m}^{T} = L_{-m} + L'_{-m}, \tag{76}$$

then the following vertices transform covariantly under $L_{\pm m}^{T}$, L_0^{T}:

(a) a vertex without internal quantum number "Pomeron" vertex which is just the scalar vertex of the bare theory

$$V^P(k,z) = V_0(k,z) \text{ with } J_s^P = -\frac{k^2}{2} \tag{77a}$$

(b) a vertex representing a quark-like state

$$V_{(r)}^{Q}(k,z) = \psi_{(r)}(z)V_0(k,z) \text{ with } J_s^Q = -\frac{1}{2} - \frac{k^2}{2} \tag{77b}$$

(c) a meson-like vertex

$$V_\alpha^M(k,z) = :\psi^\dagger(z)\lambda_\alpha\psi(z): V_0(k,z) \text{ with } J_s^M = -1 - \frac{k^2}{2} \qquad (77c)$$

where the λ_α are the SU(3) matrices.

As implied by the definition (75), the operators L_{-m}^T satisfy among themselves a Virasoro algebra. In addition, since the above quoted vertices are also covariant under the Virasoro operators, there is a decoupling scheme at work where $J_s = -1$ which conveniently yields a mass zero meson. In this case, the spectrum is shown in Fig. 5. The model suffers from certain diseases, namely the lack of half integer spaced trajectories, existence of tachyons and exotic "quark" states.

B. Clavelli Model

In this case the new degrees of freedom are introduced by a scalar function belonging to $D_J^{(+)}$ and $D_J^{(-)}$ with $J = -\frac{1}{4}$

$$H(z) = \sum_{m=0}^{\infty} b^{(m)}\,|-\tfrac{1}{4}, m)_- + d^{(m)\dagger}\,|-\tfrac{1}{4}, m)_+ \qquad (78)$$

where

$$\left\{b^{(m)}, d^{(n)}\right\} = \left\{b^{(m)}, d^{(n)\dagger}\right\} = 0$$
$$\left\{b^{(m)}, b^{(n)\dagger}\right\} = \left\{d^{(m)}, d^{(n)\dagger}\right\} = \delta_{n,m} \quad n,m = 0,1,\ldots \qquad (79)$$

The new SU(1,1) generators are introduced by sandwiching the representation (5) of the generators between the states H in the same way as was done earlier for the bare model. However, it does not seem to be possible to build the Virasoro operators.

As should be obvious by now, the H(z) transform covariantly under these new SU(1,1) operators with $J_S = -\frac{1}{4}$. The dual vertex for a pseudo-scalar meson is

$$V^M(k,z) = :H^\dagger(z)\,H(z):V_0(k,z), \tag{80}$$

which has

$$J_s = -\frac{1}{2} - \frac{1}{2}k^2. \tag{81}$$

There is a G-parity operator

$$G = e^{i\pi\sum_{m=0}^{\infty}\left(b^{(m)\dagger}\cdot d^{(m)} + d^{(m)\dagger}\cdot b^{(m)}\right)}. \tag{82}$$

The obvious choice is to take the dual vertex $V^M(k,z)$ to represent the pion. Then one fixes J_s so that its mass vanishes through Eq. (81). The spectrum of states one obtains this way is shown in Fig. 6. It has the virtue of having a ρ trajectory with the correct intercept, i.e., half integer spacing between meson trajectories. Although many particles have their correct mass value, (π, ρ, A_1), the model has no room for abnormal parity coupling $(\omega, A_2, \ldots)$. In addition, negative norm states appear on the fifth trajectory. Another model was considered by the same author to include SU(3) breaking by introducing the quark-like function

$$H_{(r)}(z) = \sum_{m=0}^{\infty} b_r^{(m)}|\eta_r, m)_- + d_r^{(m)}|\eta_r, m)_+ \qquad r = 1,2,3 \tag{83}$$

where the η_r act as the breaking parameters. However, the model leads to $\pi^o \eta$ degeneracy.

C. Neveu-Schwarz Model

The authors consider the function

$$H_\mu(z) = \sum_{m=0}^{\infty} c_\mu^{(m)\dagger} \,|\, -\tfrac{1}{2}, m)_- + c_\mu^{(m)} \,|\, -\tfrac{1}{2}, m)_+ \tag{84}$$

where $c_\mu^{(n)}$ and $c_\mu^{(n)\dagger}$ are four-vector operators obeying the anticommutation relations

$$\left\{ c_\mu^{(n)}, c_\rho^{(m)\dagger} \right\} = \delta_{nm} g_{\mu\rho} \tag{85}$$

from which one can build the Virasoro operators

$$L_{-m}^{(c)} = \frac{1}{4\pi} \int_0^{2\pi} \frac{dz}{z}\, z^{-m} : H(z)\; z \frac{d}{dz} H(z) : \tag{86}$$

which obey the usual algebra.

The new operators are the sum, as in the previous models

$$L_{-m}^{T} = L_{-m}^{(a)} + L_{-m}^{(c)}. \tag{87}$$

It is no wonder that under these operators, $H_\mu(z)$ transforms covariantly with $J_s = -1/2$. At this point, a new feature of this particular model emerges. Since H_μ is a four vector, it can be coupled to P_μ, which leads us to consider the operators[27]

$$G_{-m} = \frac{1}{\sqrt{2}\,\pi} \int_0^{2\pi} \frac{dz}{z}\, z^{+m} H_\mu(z) P^\mu(z) \qquad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots \tag{88}$$

which satisfy

$$\left[L_m^T, G_n\right] = \left(\frac{m}{2} - n\right) G_{n+m} \tag{89}$$

$$\left\{G_n, G_m\right\} = -2 L_{n+m}^T . \tag{90}$$

As we shall see later these new operators act as additional decouplers when $\alpha_0 = 1$. The dual pion vertex is

$$V^\pi(k, z) = k \cdot H(z) V_0(k, z) \quad \text{which has } J_s = -\frac{1}{2} - \frac{k^2}{2} , \tag{91}$$

which can be rewritten as

$$V^\pi(k,z) = -\frac{z^{-\frac{1}{2}}}{\sqrt{2}} \left[G_{\frac{1}{2}}, V_0(k, z)\right] \tag{92}$$

then it follows that

$$\left\{G_{\frac{1}{2}}, V^\pi(k,z)\right\} = -\sqrt{2}\, z^{-\frac{1}{2}}\left[L_1, V_0(k, z)\right] \tag{93a}$$

$$= -\sqrt{2}\, z^{\frac{1}{2}}\left(\frac{d}{dz} + \frac{k^2}{2}\right) V_0(k,z). \tag{93b}$$

Since $\alpha_0 = 1$, $k^2/2 = 1/2$ so that in fact we have

$$\left\{G_{\frac{1}{2}}, V^\pi(k, z)\right\} = -\sqrt{2}\, z \frac{d}{dz}\left[z^{\frac{1}{2}} V_0(k, z)\right] . \tag{94}$$

We have a perfect differential on the right-hand side of Eq. (94). Such a perfect differential eliminates an integration variables in the amplitude and this eliminates a propagator, thus giving zero for the amplitude. These are the new gauges introduced by Neveu and Schwarz, and they serve to decouple the tachyon lying on the mother trajectory. More detail is to be found in Ref. 26.

The virtues of this model are quite remarkable since it allows for abnormal parity couplings, the first dual factorizable model to do so. Also since there are two decoupling schemes at work, it is not likely that negative norm states will appear in the model.

Although the discovery of these new gauges allows for the construction of a more "real life" model (see Fig. 7), it is clear that one still has a long way to go. It should be noted that by adding a fifth mode[28] to the Neveu Schwarz model, one can eliminate the tachyon,[29] but the price is the loss of half integer spacing between meson trajectories and an increase in the ω-ρ mass difference.

VIII. CONCLUSION

These lectures have been delivered with the aim of familiarizing the reader with what seems to many to be an exotic field of physics; we hope they have been successful in this respect. For the sake of completeness, however, we should point out the existence of more fundamental approaches to dual theories that have been sparked by Y. Nambu[30] and H. B Nielsen[31] as well as many other people.[32] The most exciting aspect of these works is the understanding of the Virasoro algebra as a gauge group, not unlike that found in general relativity.[33] There is little doubt that if duality has anything to do with strong interactions,[34] these can be considered as the strong gauges, pretty much on the same footing as the electromagnetic gauge for electromagnetic interactions.

In summary we can say that we are at the beginning of an understanding of duality in terms of strong interactions and that the theories we discussed are necessarily very elementary, but there are group theoretical concepts that seem to transcend any given dual model--when we understand their origin we shall undoubtedly be able to build more satisfactory (in the sense that they reproduce the observed spectrum) dual models.

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FIGURE CAPTIONS

Fig. 1. Dual N-point factorizable amplitude.

Fig. 2. Four point amplitude.

Fig. 3. Total dual amplitude for four external scalars.

Fig. 4. Particle spectrum in the $\alpha_0 = 1$ case of the bare dual model. The dotted line means that the particles lying on it are decoupled from the rest.

Fig. 5. Particle spectrum in the Bardacki-Halpern model.

Fig. 6. Particle spectrum in the Clavelli model.

Fig. 7. Particle spectrum in the Neveu-Schwarz model.

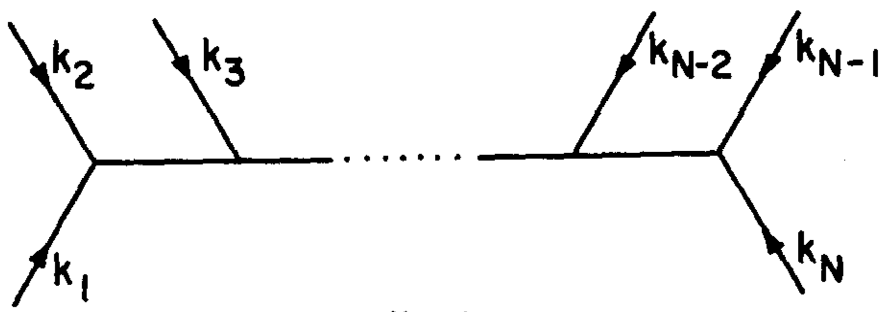


Fig.1

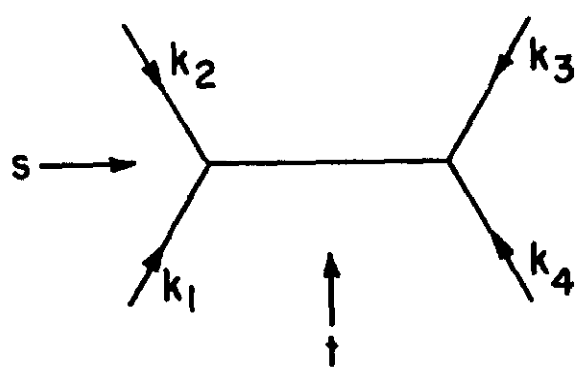
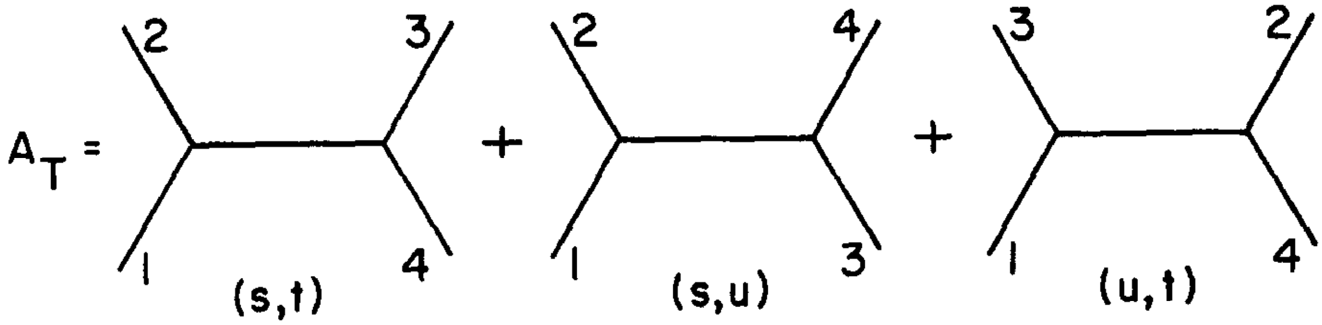
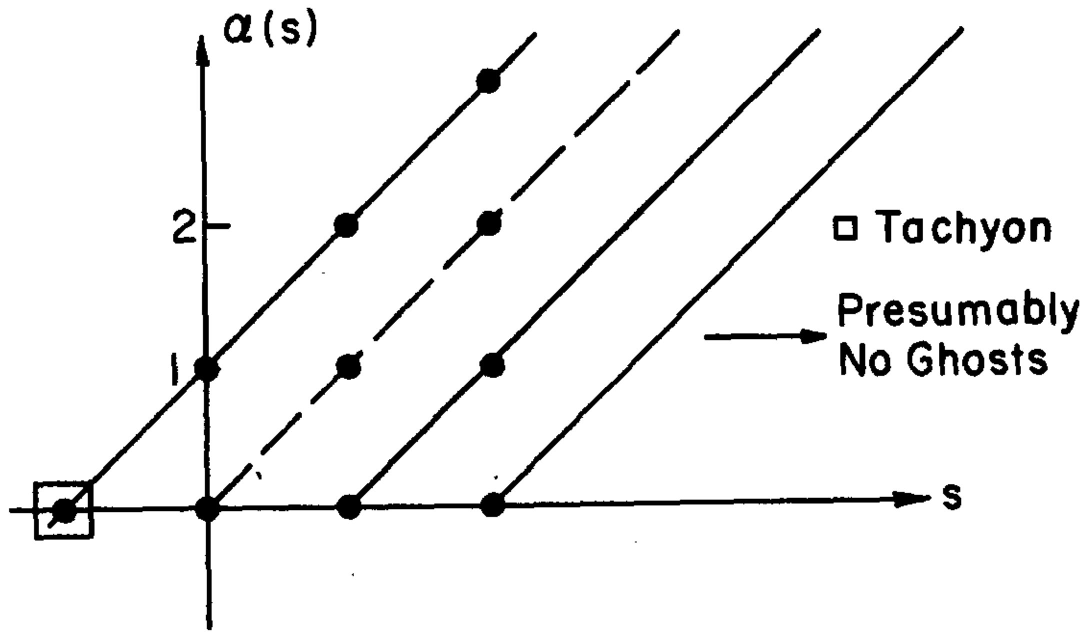


Fig.2



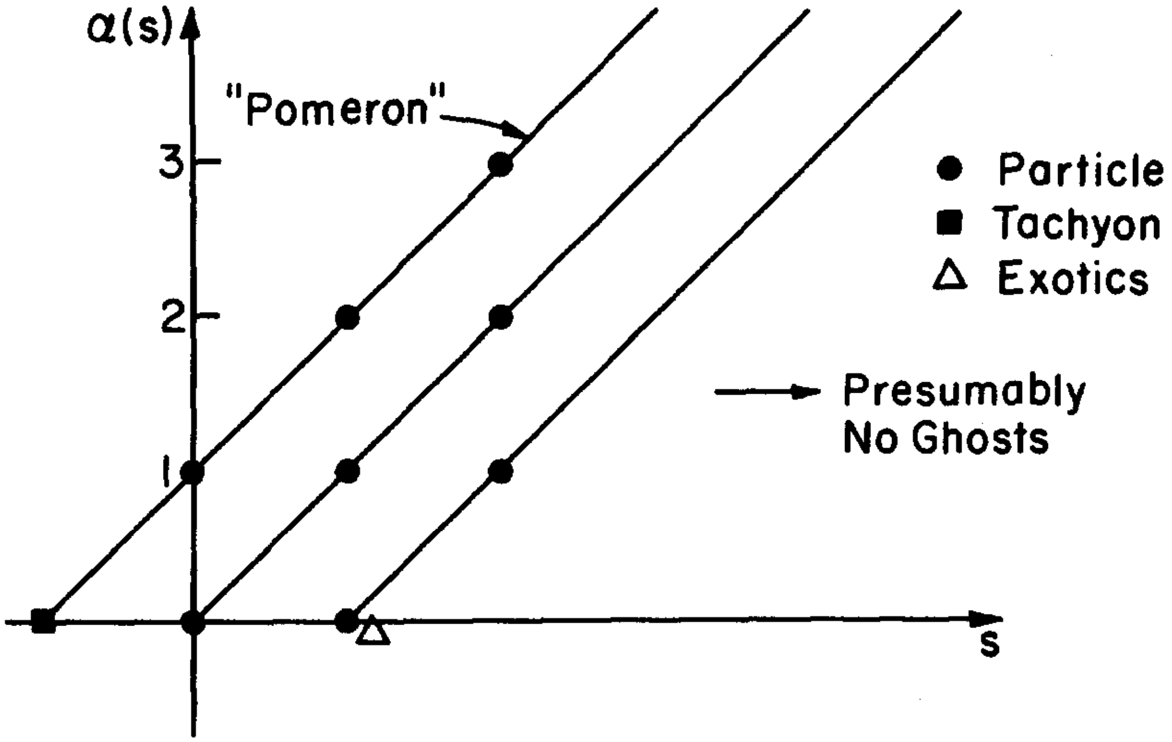
A_T =
2
3
1
4
(s,t)
+
2
4
1
3
(s,u)
+
3
2
1
4
(u,t)

Fig.3



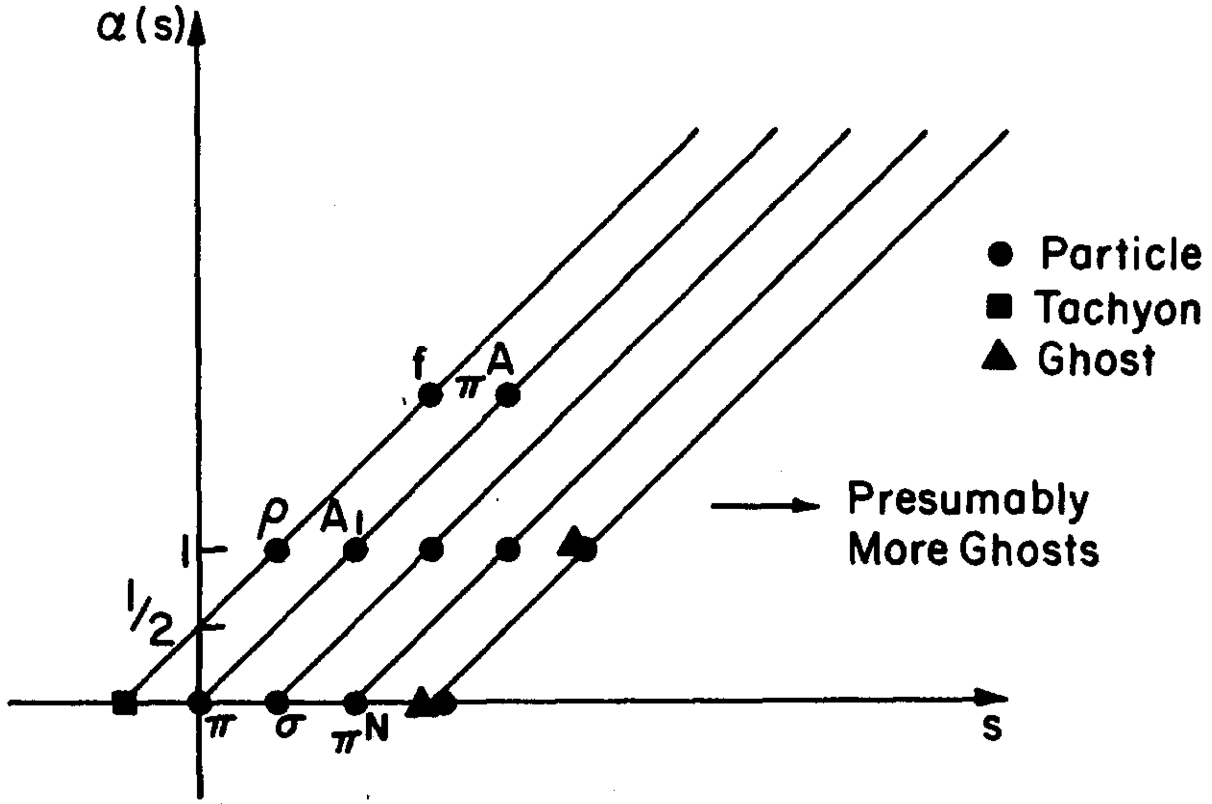
α(s)
2
1
s
Tachyon
Presumably
No Ghosts

Fig. 4



α(s)
"Pomeron"
3
2
1
s
Particle
Tachyon
Exotics
Presumably
No Ghosts

Fig.5



α(s)
f
π A
ρ
A1
1
1/2
π
σ
π N
s
Particle
Tachyon
Ghost
Presumably
More Ghosts

Fig.6

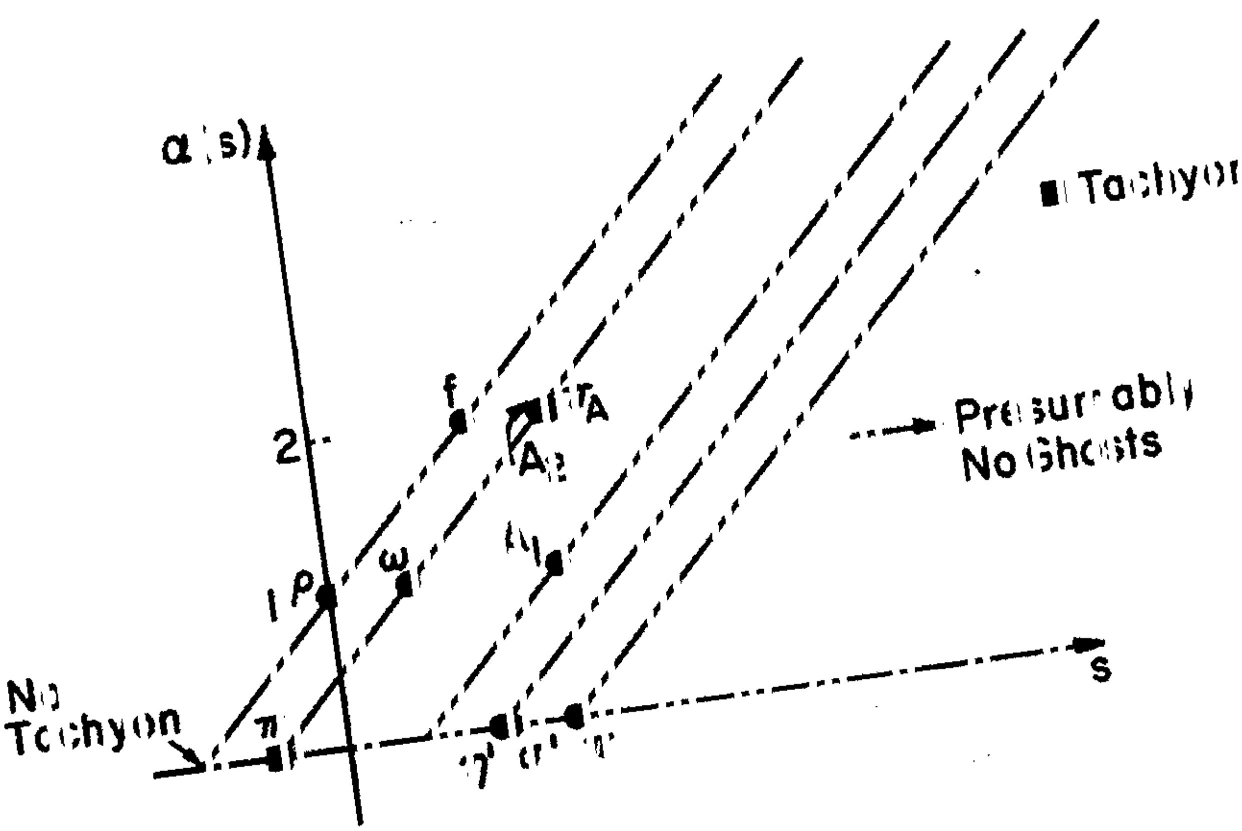


Fig.7