national accelerator laboratory

THY - 10
1971

The Veneziano Transform and the High Energy Behavior of Dual Resonance Amplitudes

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ABSTRACT

The Veneziano Beta function transform technique is generalized to the case in which there are several asymptotic variables. The multiple transform of a dual tree amplitude is found to be again a dual tree amplitude apart from a well defined redefinition of the trajectories. The asymptotic behavior of dual amplitudes are discussed, with particular emphasis on the analytic structure in the complex angular momentum-like plane.

March 1971

I. INTRODUCTION

Recently there has appeared a number of fascinating experimental predictions resulting from the high energy behavior of dual resonance amplitudes.[1,2,3,4,5,6] Some of these results are remarkably similar (in some cases identical) to those obtained [7,8,9] from general arguments based on analyticity requirements in the complex angular momentum and helicity planes.

The asymptotic behavior of the dual amplitudes have for the most part been investigated by means of either steepest descent methods,[1] Mellin transforms,[10] or a variety of exponentiation procedures.[11] A simple and direct approach to asymptopia which provides contact with analyticity in the J-plane and is somehow natural for the dual amplitudes should be useful. Such an approach was proposed and applied two years ago by Veneziano [12] in the study of the J-plane structure of the four-point function. Here we shall generalize his method to the situation where there are several asymptotic variables. We shall illustrate the method by applying it to high-energy limits of dual tree graphs recovering of course previous results, but in addition noting some new unsuspected features of the dual amplitudes. In particular we find: (1) That it is no more difficult to compute a high energy limit in the multi-peripheral configuration than any other. (2) We have examined the double Regge Vertex in the case where the Reggeons have intercepted unity. We find that the unsignatured vertex vanishes when both momentum transfers tend to zero.

This result is quite analogous to that already obtained[2,3,4] for the triple Regge vertex involving three "Pomerons".[1]

We hope this new line of work will be helpful in comparing asymptotic results from dual amplitudes with those obtained from general analyticity requirements.

II. THE VENEZIANO TRANSFORM

The most convenient variable in which to study the analytic structure of the dual amplitude in the J-plane is not necessarily J itself. Here we use the variable τ, which is associated with the Veneziano transform.[12,13]

$$V^{(1)}(\alpha_s, X) = \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} d\tau \; \tilde{V}^{(1)}(\tau, X)\, B(-\tau, -\alpha_s) \tag{2.1}$$

$$(0 < \varepsilon < 1)$$

$$\tilde{V}^{(1)}(\tau, X) = \frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty} d\alpha_s \; V^{(1)}(\alpha_s, X)\, B(\tau+1, \alpha_s+1) \tag{2.2}$$

$$(0 < \eta < 1).$$

B denotes the Euler beta function, and $\alpha_s = \alpha_o + \alpha' s$ the <u>linear</u> s-channel parent trajectory with slope α', which for convenience we set to one. X indicates collectively the remaining variables.

If we chose for $V(\alpha_s, X)$ the usual s-t Veneziano four-point function, equation (2.1) would be the analogue of the Sommerfeld-Watson transform, and $\tilde{V}(\tau, X)$ the analytically continued t-channel partial wave amplitude.

It is the analytical structure of $\tilde{V}(\tau, X)$ which is central to Regge Pole theory - thus for example the leading singularities of τ, as we shall see, govern the asymptotic behavior when α_s, the transformed variable, becomes large.

We define a multiple transform with respect to n-variables which will eventually become asymptotic:

$$V^{(n)}(\alpha_{s_1}, \dots, \alpha_{s_n}; X) = \left(\frac{1}{2\pi i}\right)^n \int_{-\varepsilon_n - i\infty}^{-\varepsilon_n + i\infty} d\tau_n \cdots \int_{-\varepsilon_1 - i\infty}^{-\varepsilon_1 + i\infty} d\tau_1 \quad (2.3)$$

$$\cdot \tilde{V}^{(n)}(\tau_1, \dots, \tau_n; X)\, B(-\tau_1, -\alpha_{s_1}) \cdots B(-\tau_n, -\alpha_{s_n})$$

$$(0 < \varepsilon_i < 1, \quad i = 1, \dots, n)$$

$$\tilde{V}^{(n)}(\tau_1, \ldots, \tau_n ; X) = \left(\frac{1}{2\pi i}\right)^n \int_{-\eta_n - i\infty}^{-\eta_n + i\infty} d\alpha_{s_n} \cdots \int_{-\eta_1 - i\infty}^{-\eta_1 + i\infty} d\alpha_{s_1} \cdot \tag{2.4}$$

$$\cdot V^{(n)}(\alpha_{s_1}, \ldots, \alpha_{s_n} ; X) B(\tau_1 + 1, \alpha_{s_1} + 1) \cdots B(\tau_n + 1, \alpha_{s_n} + 1)$$

$$(0 < \eta_i < 1, \; i = 1, \ldots, n).$$

Before proceeding to the case of interest, i.e., when $V^{(n)}$ is a dual amplitude, we quote an important result of Ademollo and Del Giudice.[13]

If,

$$V^{(1)}(\alpha_s ; X) = \int_0^1 dy \; y^{-\alpha_s - 1} f(y; X) \tag{2.5}$$

then,

$$\tilde{V}^{(1)}(\tau ; X) = \int_0^1 dy' (1 - y')^{\tau} f(y'; X) \tag{2.6}$$

Here f(y,X) is not an explicit function of α_s.

We write the dual amplitude $V^{(n)}$ as:

$$V^{(n)}(\alpha_{s_{i_1}}, \ldots, \alpha_{s_{i_n}} ; X) =$$

$$\int dV \; u_{i_1}^{-\alpha_{i_1} - 1} u_{i_2}^{-\alpha_{i_2} - 1} \cdots u_{i_n}^{-\alpha_{i_n} - 1} \tag{2.7}$$

$$\cdot \prod_j u_{\alpha_j}^{-\alpha_j - 1},$$

where $u_{i_1} \ldots u_{i_n}$ are the Chan variables[14] associated with channel variables which will eventually become asymptotic, u_{α_j} are the remaining Chan variables, X the fixed channel variables and $d\mathcal{V}$ is the cyclic symmetric volume element. Finally we recall the duality constraint equations requires

$$u_j = 1 - \prod_{\alpha_k} u_{\alpha_k}^{(j)} \tag{2.8}$$

where $u_{\alpha_k}^{(j)}$ are all the Chan variables dual to u_j.

Inserting Eq. (2. 7) into Eq. (2. 4) and using repeatedly Eq. (2. 6) we obtain:

$$\tilde{V}^{(n)}(\tau_{\alpha_{i_1}}, \ldots, \tau_{\alpha_{i_n}}; X) = \int d\mathcal{V} \prod_j u_j^{-\bar{\alpha}_j - 1} \tag{2.9}$$

where u_j are the usual Chan variables, however, the Regge trajectories are modified according to:

$\bar{\alpha}_j = \alpha_j$ if the j^{th} channel is not dual to channels $i_1, \ldots i_n$ nor identical to any of the transformed channels.

$= \alpha_j + \sum_k \tau_{\alpha_{i_k}}$, where the summation over τ's are associated with channels that are dual to j, and j is not a transformed channel. (2. 10)

$= \left(\sum_k \tau_{\alpha_{i_k}}\right) + 1$ where again the summation over τ extends over all channels dual to j associated with $\tau_{\alpha_{i_k}}$ apart from j itself which is now associated with one of the $\tau_{\alpha_{i_1}} \ldots \tau_{\alpha_{i_n}}$. If the sum is empty - we replace it by zero.

We see from Eq. (2.9) the pleasant feature that the transformed dual amplitude is again a dual amplitude with modified Regge trajectories given by a simple rule, Eq. (2.10).

Finally to extract the asymptotic limit we insert Eq. (2.9) into Eq. (2.10) and as usual let $-\alpha_{s_i}$ $(i = 1, \ldots n)$ approach infinity along a ray in the complex α_{s_i} plane. If we desire the leading asymptotic behavior we rewrite Eq. (2.3) as:

$$\lim_{-\alpha \to \infty} V^{(n)}(\alpha_{s_{i_1}}, \ldots, \alpha_{s_{i_n}}; X) \longrightarrow \left(\frac{1}{2\pi i}\right)^n \int_{-\varepsilon_{i_n} - i\infty}^{-\varepsilon_{i_n} + i\infty} d\tau_{\alpha_{i_n}} \cdots \int_{-\varepsilon_{i_1} - i\infty}^{-\varepsilon_{i_1} + i\infty} d\tau_{\alpha_{i_1}} \cdot \tag{2.11}$$

$$\cdot \tilde{V}^{(n)}(\tau_{i_1} \cdots \tau_{i_n}; X)\,(-\alpha_{s_{i_1}})^{\tau_{i_1}} \cdots (-\alpha_{s_{i_n}})^{\tau_{i_n}} \cdot$$

$$\cdot \Gamma(-\tau_{i_1}) \cdots \Gamma(-\tau_{i_n}), \qquad (0 < \varepsilon_{i_j} < 1,\ j = 1, \ldots, n)$$

and study the singularity structure of the integrand. Eq. (2.11) is the analogue of the multiple Sommerfeld-Watson transform.

At present we have not been specific with regard to the collective variable X, nor the particular channel in which we expect Regge-like singularities.

It appears to us that a simple rule leads quickly to the desired leading asymptotic results for a large class of high energy limits.

The rule. We require among the independent Chan variables those which correspond to channels whose corresponding Regge singularities we desire. We call these the chosen variables. This is of course dictated by the particular asymptotic limit.[2] We then send the choosen variables to zero thus exposing the anticipated singularities. (All variables dual to the chosen variables tend to one).

We find that this rule does lead to previous results, however we caution the reader that we have not yet taken proper care of signature factors. These arise from treating definite inequivalent permutation of the dual amplitudes,[3] and assigning suitable phases to these amplitudes. Thus at times one might find spurious poles in our amplitudes - which necessitate a more thorough analysis with regard to the proper inclusion of signature, before ascribing any special physical significance to them. thus it is perhaps more reasonable to define in the first place a signatured multiple-transform. We hope to return to this question in a future note. We shall touch on this point briefly when we discuss the double Regge vertex.

We illustrate the multiple Veneziano transform technique by considering: (1) the double Regge limit for the five-point function, (2) the triple and double Regge limit of the six-point function in the multiperipheral and semi-multiperipheral configuration respectively.

For the latter the kinematics are appropriate for the pionization limit of the single particle distribution.

III. APPLICATIONS

Before proceeding we list here three basic relations which will be used repeatedly:[15]

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds \quad x^{-s}\,\Gamma(-s)\,\Gamma(a+s)\,\Gamma(a-c+1+s) = \tag{3.1}$$

$$x^{a}\,\Gamma(a)\,\Gamma(a-c+1)\,\Psi(a,c,x)$$

$$\left(-\mathrm{Re}\,a < \gamma < \min(0, 1-\mathrm{Re}\,c),\ -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}\right)$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\ \Gamma(z+\alpha)\,\Gamma(z+\beta)\,\Gamma(\gamma-z)\,\Gamma(\delta-z) = \tag{3.2}$$

$$\Gamma(\alpha+\delta)\,\Gamma(\gamma+\beta)\,B(\alpha+\gamma, \beta+\delta)$$

$$(\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta, \mathrm{Re}\,\gamma, \mathrm{Re}\,\delta > 0)$$

$$\psi(z) = \psi(1+z) - \frac{1}{z} \tag{3.3}$$

$\Psi(a,c,x)$ is the confluent hypergeometric function B is the Euler beta function, and $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$.

The Five-Point Function

We compute the double Regge limit of the five-point amplitude. All external particles considered in this paper are taken to be identical I = 0 scalar particles. (See Fig. 1a).

The three asymptotic variables are:

$$s_{04} = (p_0 + p_4)^2 \,, \quad s_{12} = (p_1 + p_2)^2 \qquad (3.4)$$

$$s_{23} = (p_2 + p_3)^2$$

we take the limit such that $(-\alpha_{04})$, $(-\alpha_{12})$, and $(-\alpha_{23})$ become infinite whereas the ratio

$$K_{04} = \frac{(-\alpha_{12})(-\alpha_{23})}{(-\alpha_{04})} \qquad (3.5)$$

and α_{10} and α_{34} are kept fixed. Here α_{10} and α_{34} are the linear trajectories associated with the channel energies:

$$s_{01} = (p_0 + p_1)^2 \,, \quad s_{34} = (p_3 + p_4)^2 \qquad (3.6)$$

For later use we record the linear relation between the cosine of Toller angle, ω, and $s_{04}/(s_{12})(s_{23})$:

$$\frac{s_{04}}{s_{12}\ s_{23}} = \frac{\left(-\alpha_0 - s_{01} - s_{34} + \sqrt{s_{01} s_{34}}\ \cos\omega\right)}{\lambda\left(s_{01}, s_{34}, -\alpha_0\right)} \tag{3.7}$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$$

Using Eq. (2.9) and (2.10) we obtain for the triply transformed amplitude:

$$\widetilde{B}^{(3)}_5 \left(\alpha_{04}, \alpha_{12}, \alpha_{23}; \alpha_{10}, \alpha_{34}\right) =$$

$$\int d\upsilon \quad u_{01}^{-\alpha_{01} + \tau_{04} + \tau_{12} - 1} \quad u_{34}^{-\alpha_{34} + \tau_{04} + \tau_{23} - 1} \tag{3.8}$$

$$u_{12}^{\tau_{23}} \quad u_{23}^{\tau_{12}} \quad u_{04}^{0} \,.$$

We remark that it is quite simple to read off this result from the dual diagram. (See Fig. 1b). We have indicated by the solid line the asymptotic channels. We have also indicated by a dotted line a particular channel, (34). We note that the dotted line crosses channels (04) and (23). Thus the exponent of u_{34} must be shifted by a positive amount, $\tau_{04} + \tau_{23}$. Similarly for u_{01}. The asymptotic channel (04) has no asymptotic line crossing it, hence the exponent zero. Channels (23) and (12) are asymptotic thus we deleted the exponents $-\alpha_{12}-1$, and $-\alpha_{23}-1$, and replace them by the exponents τ_{23} and τ_{12} respectively, since (23) crosses (12) and (12) crosses (23).

Clearly we are interested in the singularities in the channels (01) and (34) hence we choose u_{01} and u_{34} as independent Chan variables, and obtain using Eq. (2.11):

$$\lim_{-\alpha\to\infty} B_5^{(3)} = \left(\frac{1}{2\pi i}\right)^3 \int d\tau_{04}\, d\tau_{23}\, d\tau_{12}\, d\nu \; u_{01}^{-\alpha_{01}+\tau_{04}+\tau_{12}-1} \; u_{34}^{-\alpha_{34}+\tau_{04}+\tau_{23}-1}$$
$$u_{12}^{\tau_{23}} \; u_{23}^{\tau_{12}} \; (-\alpha_{04})^{\tau_{04}} \; (-\alpha_{12})^{\tau_{12}} \; (-\alpha_{23})^{\tau_{23}} \cdot \tag{3.9}$$
$$\cdot \Gamma(-\tau_{04})\, \Gamma(-\tau_{12})\, \Gamma(-\tau_{23}) \,.$$

We perform first the τ_{12} integration. The leading singularity to the left of our contour is the pole located at:

$$\tau_{12} = \alpha_{01} - \tau_{04} \tag{3.10}$$

which occurs when u_{01} tends to zero. It's residue is

$$\lim_{-\alpha\to\infty} B_5^{(3)} = \left(\frac{1}{2\pi i}\right)^2 \int d\tau_{04}\, d\tau_{23}\, d\nu \; u_{34}^{-\alpha_{34}+\tau_{04}+\tau_{23}-1} \cdot \tag{3.11}$$
$$\cdot\, u_{23}^{\alpha_{01}-\tau_{04}} \; \Gamma(\tau_{04}-\alpha_{01})\, \Gamma(-\tau_{04})\, \Gamma(-\tau_{23}) \,\cdot$$
$$\cdot\, (-\alpha_{12})^{\alpha_{01}} \left(\frac{-\alpha_{04}}{-\alpha_{12}}\right)^{\tau_{04}} (-\alpha_{23})^{\tau_{23}} \,.$$

Similarly one does the τ_{23} integration and obtains:

$$\lim_{-\alpha\to\infty} B_5^{(3)} = \frac{1}{2\pi i}(-\alpha_{12})^{\alpha_{01}}(-\alpha_{23})^{\alpha_{34}}\int d\tau_{04}$$

$$\left[\frac{(-\alpha_{12})(-\alpha_{23})}{(-\alpha_{04})}\right]^{-\tau_{04}}\Gamma(-\alpha_{01}+\tau_{04})\Gamma(-\alpha_{34}+\tau_{04})\Gamma(-\tau_{04}) \tag{3.12}$$

We recognize the integral expression in Eq. (3.12) as related to the integral representation of the confluent hypergeometric Ψ, function. Using Eq. (3.1) and (3.5) we obtain:

$$\lim_{-\alpha\to\infty} B_5^{(3)} = (-\alpha_{12})^{\alpha_{01}}(-\alpha_{23})^{\alpha_{34}}\Gamma(-\alpha_{01})\Gamma(-\alpha_{34})\cdot$$

$$\cdot K_{04}^{-\alpha_{01}}\Psi(-\alpha_{01},\ \alpha_{34}-\alpha_{01}+1,\ K_{04}) \tag{3.13}$$

We stress that Eq. (3.13) is valid only in the limit where $\alpha_{34} \neq \alpha_{01}$. As we see, from the integrand of Eq. (3.12), when $\alpha_{34} = \alpha_{01}$ we have a sequence of double poles arising from the factor $\Gamma^2(-\alpha_{01} + \tau_{04})$.[4] We shall return to this important limit after we demonstrate by means of our transform technique, the "factorizability" of the double Regge vertex.

The Six-Point Function, Multiperipheral Configuration

We first compute the triple Regge limit of the six-point amplitude in the multi-peripheral configuration (see Fig. 2a).

The six asymptotic variables are:

$$s_{05} = (p_0+p_5)^2,\quad s_{12} = (p_1+p_2)^2,\quad s_{23} = (p_2+p_3)^2 \tag{3.14}$$

$$s_{34} = (p_3+p_4)^2,\quad s_{13} = (p_1+p_2+p_3)^2,\quad s_{24} = (p_2+p_3+p_4)^2$$

We take the limit where $-\alpha_{05}$, $-\alpha_{12}$, $-\alpha_{23}$, $-\alpha_{34}$ $-\alpha_{13}$, $-\alpha_{24}$ become infinite whereas the ratios,

$$K_1 = \frac{(-\alpha_{12})(-\alpha_{23})}{(-\alpha_{13})} \quad , \quad K_2 = \frac{(-\alpha_{23})(-\alpha_{34})}{(-\alpha_{24})} \tag{3.15}$$

$$K_3 = \frac{(-\alpha_{12})(-\alpha_{23})(-\alpha_{34})}{(-\alpha_{05})}$$

are kept fixed. Furthermore we make use of a factorization condition[16]

$$K_3 = K_1 K_2 \tag{3.16}$$

Again one easily obtains the transformed amplitude using Eq. (2.9) and (2.10):

$$\tilde{B}_6^{(6)} = \int dV \; u_{01}^{-\alpha_{01}+\tau_{05}+\tau_{13}+\tau_{12}-1} \cdot u_{02}^{-\alpha_{02}+\tau_{05}+\tau_{13}+\tau_{23}-1} \cdot u_{03}^{-\alpha_{03}+\tau_{05}+\tau_{24}+\tau_{34}-1} \cdot f(u_\alpha) \, . \tag{3.17}$$

We have choosen u_{01}, u_{02}, u_{03}, as the appropriate independent Chan invariables. u_α denotes the remaining Chan variables, and $f(u_\alpha)$ is a function which of course tends to unity as the three chosen variables approach zero.

Applying the multiple transform, Eq. (2.11), we find upon sending u_{01}, u_{02}, and u_{03} to zero and picking up the residues of the leading singularities in the τ_{12}, τ_{23}, and τ_{34} complex planes we are left with:

$$\lim_{-\alpha\to\infty} B_6^{(6)} = \left(\frac{1}{2\pi i}\right)^3 \int d\tau_{05}\, d\tau_{13}\, d\tau_{24}\, (-\alpha_{12})^{\alpha_{01}-\tau_{05}-\tau_{13}} \cdot$$
$$\cdot (-\alpha_{23})^{\alpha_{02}-\tau_{05}-\tau_{13}-\tau_{24}} (-\alpha_{34})^{\alpha_{03}-\tau_{05}-\tau_{24}} \cdot \tag{3.18}$$
$$\cdot \Gamma(\tau_{05}+\tau_{13}-\alpha_{01})\, \Gamma(\tau_{05}+\tau_{13}+\tau_{24}-\alpha_{02}) \cdot$$
$$\cdot \Gamma(\tau_{05}+\tau_{24}-\alpha_{03})\, (-\alpha_{05})^{\tau_{05}} (-\alpha_{13})^{\tau_{13}} (-\alpha_{24})^{\tau_{24}} \cdot$$
$$\cdot \Gamma(-\tau_{05})\, \Gamma(-\tau_{13})\, \Gamma(-\tau_{24}),$$

Changing variables such that:

$$X = \tau_{05}+\tau_{13} \quad , \quad y = \tau_{05}+\tau_{24} \tag{3.19}$$
$$z = \tau_{05}+\tau_{13}+\tau_{24} .$$

We have using, Eq. (3.16):

$$\lim_{-\alpha\to\infty} B_6^{(6)} = (-\alpha_{12})^{\alpha_{01}} (-\alpha_{23})^{\alpha_{02}} (-\alpha_{34})^{\alpha_{03}} \cdot$$
$$\cdot \left(\frac{1}{2\pi i}\right)^3 \int dx\, dy\, dz\; \Gamma(X-\alpha_{01})\, \Gamma(y-\alpha_{03}) \tag{3.20}$$
$$\cdot \Gamma(z-\alpha_{02})\, \Gamma(x-z)\, \Gamma(y-z)\, \Gamma(z-x-y)$$
$$\cdot \kappa_1^{-x}\, \kappa_2^{-y} .$$

We may now easily perform the z integration using Eq. (3.2) and obtain:

$$\lim_{-\alpha\to\infty} B_6^{(6)} = (-\alpha_{12})^{\alpha_{01}} (-\alpha_{23})^{\alpha_{02}} (-\alpha_{34})^{\alpha_{03}} \cdot \Gamma^{-1}(-\alpha_{02}) \frac{1}{2\pi i} \int \Gamma(-x)\Gamma(x-\alpha_{02})\Gamma(x-\alpha_{01}) \cdot K_1^{-x} dx \cdot \frac{1}{2\pi i} \int \Gamma(-y)\Gamma(y-\alpha_{02})\Gamma(y-\alpha_{03}) K_2^{-y} dy. \quad (3.21)$$

Using Eq. (3.1) we have:

$$\lim_{-\alpha\to\infty} B_6^{(6)} = (-\alpha_{12})^{\alpha_{11}} (-\alpha_{23})^{\alpha_{02}} (-\alpha_{34})^{\alpha_{03}} \cdot \Gamma(-\alpha_{01}) V(\alpha_{01}, \alpha_{02}, K_1)\Gamma(-\alpha_{02}) \cdot V(\alpha_{02}, \alpha_{03}, K_2)\Gamma(-\alpha_{03}) \quad (3.22)$$

where

$$V(\alpha_{oi}, \alpha_{oi+1}, K_i) = K_i^{-\alpha_{oi}} \Psi(-\alpha_{oi}, \alpha_{oi+1} - \alpha_{oi} + 1, K_i) \quad (3.23)$$

This result is manifestly factorizable in the sense defined by Bardacki and Ruegg[11] and of course identical with their result. We note again that Eq. (3.22) is valid only in the limit in which the momentum transfer α_{oi}, α_{oi+1} are unequal.

<u>The Double Regge Vertex When $\alpha_i = \alpha_{oi+1}$.</u>

We see from the previous analysis that the occurrence of double poles when $\alpha_{oi} = \alpha_{oi+1}$ in the intergrands of Eq. (3.21) requires special attention. From Eq. (3.23) we realize that this is equivalent to the classical problem of defining the $\Psi(a, b, x)$ function interms of the Φ function when the argument b is a positive integer.

We quote the results of[15] of that analysis:

$$\lim_{\alpha_{oi} \to \alpha_{oi+1}} V(\alpha_{oi}, \alpha_{oi+1}, K_i) = K_i^{-\alpha_{oi}} \frac{(-1)}{\Gamma(-\alpha_{oi})} \Big\{$$

$$\Phi(-\alpha_{oi}, 1, K_i) \log K_i + \frac{1}{\Gamma(-\alpha_{oi})} \sum_{r=0}^{\infty} \Gamma(-\alpha_{oi}+r) \cdot \tag{3.24}$$

$$\left[\psi(-\alpha_{oi}+r) - 2\psi(1+r) \right] \frac{K_i^r}{r!} \Big\}$$

where $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$ and $\Phi(a, b, X)$ is defined by the confluent hypergeometric series:

$$\Phi(a,b,x) = 1 + \frac{a}{b} x + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \tag{3.25}$$

$$+ \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{x^3}{3!} + \ldots$$

It is amusing to observe the behavior of Eq. (3.24) for the case in which all Regge intercepts are taken to unity i.e.,

$$\alpha_{oi} = 1 + s_{oi} \quad , \quad \alpha_{oi+1} = 1 + s_{oi+1} \tag{3.26}$$

and the momentum transfer s_{oi} and s_{oi+1} approach zero. We have:

$$\lim_{\substack{\alpha_{oi} \to \alpha_{oi+1} \\ \alpha_{oi+1} \to 1}} V(\alpha_{oi}, \alpha_{oi+1}, K_i) = -K_i^{-\alpha_{oi}} \left\{ \frac{1}{\Gamma(1)} - \frac{K_i}{\Gamma(1)} \right\} \tag{3.27}$$

Here we have used repeatedly Eq. (3. 3). Finally we note the kinematical constraint Eq. (3. 7):

$$\lim_{\substack{\alpha_{oi} = \alpha_{oi+1} \\ \alpha_{oi} \to 1}} K_i = 1 \tag{3.28}$$

Hence,

$$\lim_{\substack{\alpha_{oi} \to \alpha_{oi+1} \\ \alpha_{oi} \to 1}} V\left(\alpha_{oi}, \alpha_{oi+1}, K_i\right) = 0 \tag{3.29}$$

If we identify the trajectories α_{oi}, and α_{oi+1} with the Pomeranchuk, (here it is in fact an I = 0 even signature trajectory) we appear to resolve[5] the Finkelstein Kajantie [17] paradox concerning multiple Pomeranchuk exchange. Such a resolution was proposed some time ago by Verdiev, Kancheli, Matinyan, Popova and Ter-Martirosyan.[18]

Before concluding that this result would remain for the physical amplitude one must first properly include the signature factor. One might naively expect the usual multiplicative factor, $(1 + e^{-i\pi\alpha_{oi}}) \cdot (1 + e^{-i\pi\alpha_{oi+1}})$, which produce the proper non-sense correct signature decoupling, and do not modify our result. However, Drummond, Landshoff, and Żakrzewski [19] argue in a somewhat model independent way that the physical double-Regge Vertex involves both the expected modification but in addition a remainder which is related to a definite discontinuity of the vertex in the complex κ - plane.

The effect of this addition is to alter radically the physical vertex at wrong signature points. In their analysis they have neglected a class of diagrams which appear to contribute to a quite different asymptotic limit (e.g., that in which lines 1 and 2 are interchanged in the five-point function). It appears to us that this class of diagrams may in fact play an important role both with regard to signature and the appearance of double poles.

Six-Point Function, Semi-Multiperipheral Configuration

We next examine the six-point function in the kinematical configuration appropriate for the pionization limit of the single particle distribution. (See Fig. 3a).

We define the kinematic variables:

$$\begin{aligned} &s_{aa'} = (p_a - p_{a'})^2 , \quad s_{bb'} = (p_b - p_{b'})^2 , \quad s_M = (p_a + p_b - p_1)^2 \\ &s_{a1} = (p_a - p_1)^2 \quad s_{a'1'} = (p_{a'} - p_{1'})^2 , \quad s_{1b} = (p_b - p_1)^2 \\ &s_{1'b'} = (p_{b'} - p_{1'})^2 , \text{ etc.} \end{aligned} \tag{3.30}$$

We consider the high energy limit in which $-\alpha_M$, $-\alpha_{a1}$, $-\alpha_{a'1'}$, and $-\alpha_{b1}$, $-\alpha_{b'1'}$, approach infinity, yet keeping the ratios

$$\kappa = \frac{(-\alpha_{a'1'})(-\alpha_{b'1'})}{(-\alpha_M)} < 0 , \quad \frac{-\alpha_{a1}}{-\alpha_{a'1'}} = \frac{-\alpha_{b1}}{-\alpha_{b'1'}} \rightarrow 1 \tag{3.31}$$

fixed.

As before it is straight forward to obtain the multiple transform:

$$\tilde{B}_6^5 = \int d\nu \; u_{aa'}^{-\alpha_{aa'}+\tau_{a1}+\tau_M+\tau_{a'1'1}-1} \; u_{bb'}^{-\alpha_{bb'}+\tau_{b1}+\tau_M+\tau_{b'1'1}-1} \cdot$$
$$\cdot\, u_{aa'1'}^{-\alpha_{aa'1'}+\tau_{a1}+\tau_M+\tau_{b'1'1}-1} \; u_{bb'1'}^{-\alpha_{bb'1'}+\tau_{b1}+\tau_M+\tau_{a'1'1}-1} \tag{3.32}$$
$$\cdot\, u_{a1b}^{0} \cdot f'(u_\alpha) \,.$$

We choose $u_{aa'}$, $u_{bb'}$, and $u_{aa'1'}$ as independent variables. Applying the multiple transform, we find upon sending $u_{aa'}$ and $u_{bb'}$ to zero and picking up the residues of the leading singularities in the $\tau_{a'1'}$ and $\tau_{b'1'}$ complex planes we obtain:

$$\lim_{-\alpha\to\infty} B_6^{(5)} = \left(\frac{1}{2\pi i}\right)^3 \int d\tau_{a1}\, d\tau_{b1}\, d\tau_M\, d\nu \cdot$$
$$\cdot\, (-\alpha_{a'1'})^{\alpha_{aa'}-\tau_{a1}-\tau_M} \, (-\alpha_{b'1'})^{\alpha_{bb'}-\tau_{b1}-\tau_M} \cdot \tag{3.33}$$
$$\cdot\, \Gamma(\tau_{a1}+\tau_M-\alpha_{aa'})\, \Gamma(\tau_{b1}+\tau_M-\alpha_{bb'}) \cdot$$
$$\cdot\, \Gamma(-\tau_M)\, \Gamma(-\tau_{b1})\, \Gamma(-\tau_{a1}) \cdot$$
$$\cdot\, (-\alpha_{a1})^{\tau_{a1}} (-\alpha_{b1})^{\tau_{b1}} (-\alpha_M)^{\tau_M} \cdot$$
$$\cdot\, u_{aa'1'}^{-\alpha_{aa'1'}+\tau_{a1}-\tau_{b1}+\alpha_{bb'}-1} \; u_{bb'1'}^{-\alpha_{bb'1'}+\tau_{b1}-\tau_{a1}+\alpha_{aa'}-1} \,.$$

Recognizing that the remain Chan variables integrate to the Beta function and using Eq. (3.31) and Eq. (3.2) twice we obtain:

$$\lim_{-\alpha\to\infty} B_6^{(5)} = (-\alpha_{a'1'})^{\alpha_{aa'}} (-\alpha_{b'1'})^{\alpha_{bb'}} \left(\frac{1}{2\pi i}\right) \int d\tau_M \int_0^1 du$$

$$\cdot\, u^{-\alpha_{aa'1'}-1} (1-u)^{-\alpha_{bb'1'}-1} \left(\frac{(-\alpha_{a'1'})(-\alpha_{b'1'})}{(-\alpha_M)(1-u)u}\right)^{-\tau_M}, \quad (3.34)$$

Using Eqs. (3.1) and (3.31) we have:

$$\lim_{-\alpha\to\infty} B_6^{(5)} = (-\alpha_{a'1'})^{\alpha_{aa'}} (-\alpha_{b'1'})^{\alpha_{bb'}} \cdot$$

$$\cdot\, \Gamma(-\alpha_{aa'}) \cdot \Gamma(-\alpha_{bb'}) \cdot \int_0^1 du \; u^{-\alpha_{aa'1'}-1} \cdot$$

$$\cdot\, (1-u)^{-\alpha_{bb'1'}-1} \; V\left(\alpha_{aa'}, \alpha_{bb'}, \frac{K}{u(1-u)}\right), \quad (3.35)$$

which agrees with that of Refs. 3 and 4.

Finally we note that in the physical limit in which the trajectories functions $\alpha_{aa'}$, $\alpha_{bb'}$ are identical we must use Eq. (3.24) for V.

IV. CONCLUSION

As we have seen the multiple transform technique permits a rather careful examination of the J-plane singularity structure of the dual -tree amplitudes. We have discussed briefly the standard Regge limits. However, it appears to us that these limits are quite interlocked with certain fixed angle limits.[20] We believe the transformed technique can also be of use in evaluating these limits.

Although we have only considered the leading singularities, we believe it is not an impossible problem to probe the lower lying trajectories and reinvestigate the interesting factorization[21] questions which arise. Furthermore it might prove of interest to study the high energy effects of trajectories with non zero Toller quantum numbers, which are of course present[21] at the daughter level.

Finally we should mention that the dual loop amplitudes[22,23,5] can also be subjected to multiple transform techniques. We have verified this for the planar single loop amplitudes.[22] Their transforms are again the planar loop with shifts in the Regge trajectories analogous to Eq. (2.10) . We suspect that this pleasant feature will be maintained for all dual loop amplitudes.

ACKNOWLEDGMENTS

We are grateful for the helpful criticism of G. Veneziano. We also acknowledge useful discussions with D. Freedman, S. Fubini, C. E. Jones, F. E. Low, J. H. Weis, and J. Young.

FOOTNOTES

[1] We note that the vanishing of the triple Regge vertex occurs when the intercepts of the three parent trajectories are at unity, and all momentum transfer variables tend to zero, see for example Eq. (3.22) of Ref. 2.

[2] The collective variable X denotes those channel variables which are held fixed. Of course depending on the asymptotic limit there may emerge definite ratio's of asymptotic variables which are kept fixed.

[3] For an interesting general discussion of the non-trivial relations which hold between different inequivalent permutations of a dual amplitudes, see E. Plahte, Nuovo Cimento 66A, 713 (1970). A beautiful application of the Plahte relations in the context of inclusive reactions may be found in Ref. 4.

[4] The presence of double poles when $\alpha_{34} = \alpha_{01}$, in the double Regge limit of the five-point production amplitude had been noted in a model calculation involving sums of ladder graphs by W. J. Zakrzewski, Nuovo Cimento, 60A, 263 (1969).

[5] We note that for unit intercept the dual amplitudes are quite likely free of ghosts (M. A. Virasoro, Phys. Rev. D1, 2933 (1970)).
It is intriguing that a possible threat to unitarity appears suppressed in exactly the Virasoro case. We note, however, that we still have a problem due to the presence of the undesirable Tachyon pole.

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[23] See e. g., for multiloop amplitudes, C. Lovelace, Phys. Rev. Letters, **32B**, 703 (1970), V. A. Allessandrini, CERN preprint 1215 (1970).

FIGURE CAPTIONS

Fig. 1 a) Five-point function,

b) Dual diagram for five-point function. The solid internal lines indicates the asymptotic channels. The dashed interior line indicates a particular fixed energy channel. (34).

Fig. 2 a) The six-point function in the multiperipheral configuration.

b) The dual diagram for the six-point function. The solid interior lines indicate the asymptotic channels.

Fig. 3. a) The six-point function in the semi-multiperipheral configuration.

b) The dual diagram for the six-point function. The solid interior lines indicate the asymptotic channels.

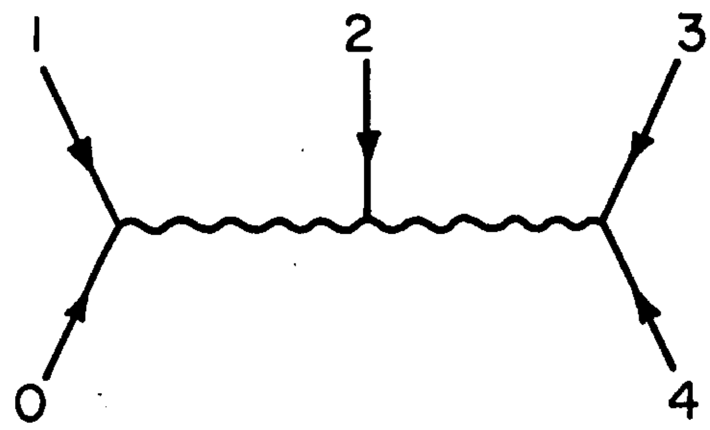


Fig. 1 (a)

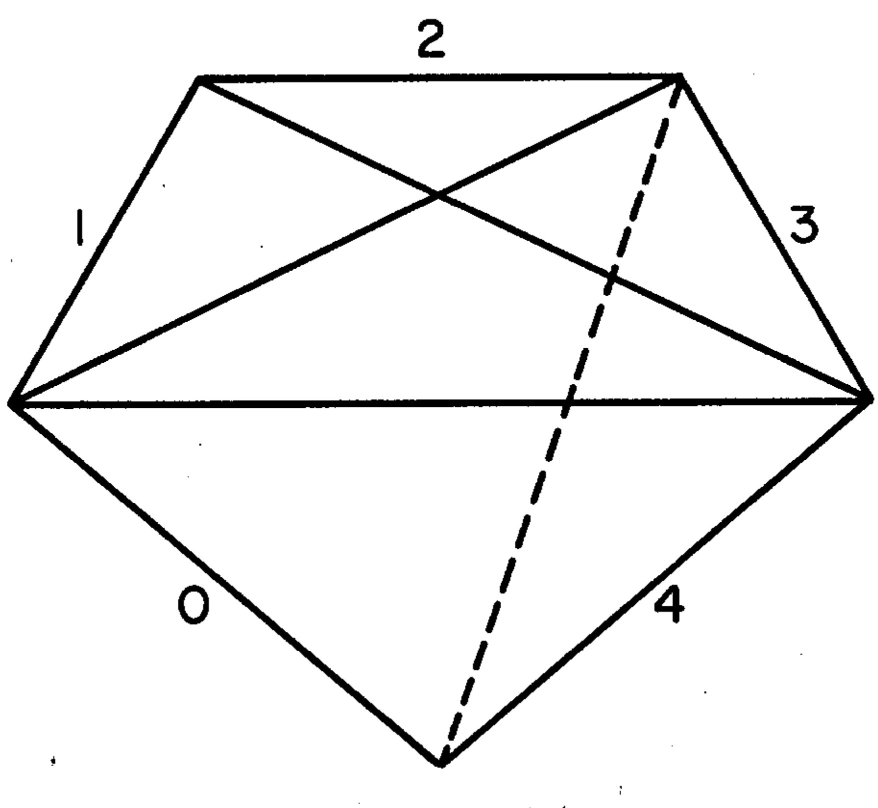


Fig. 1 (b)

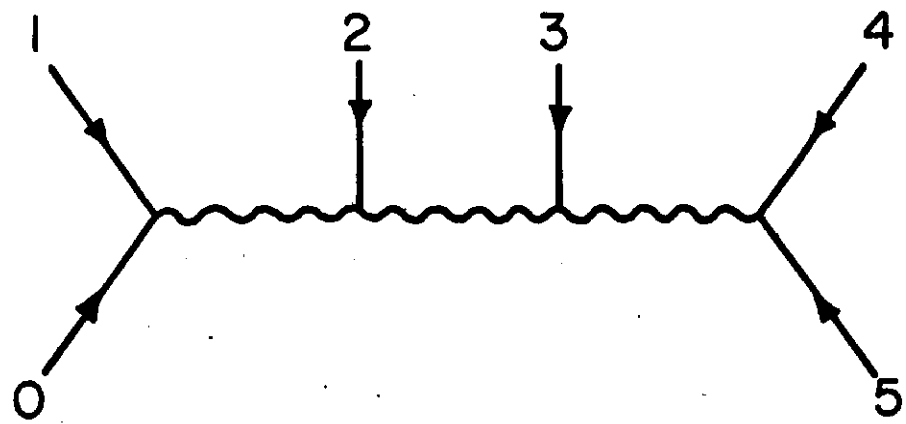
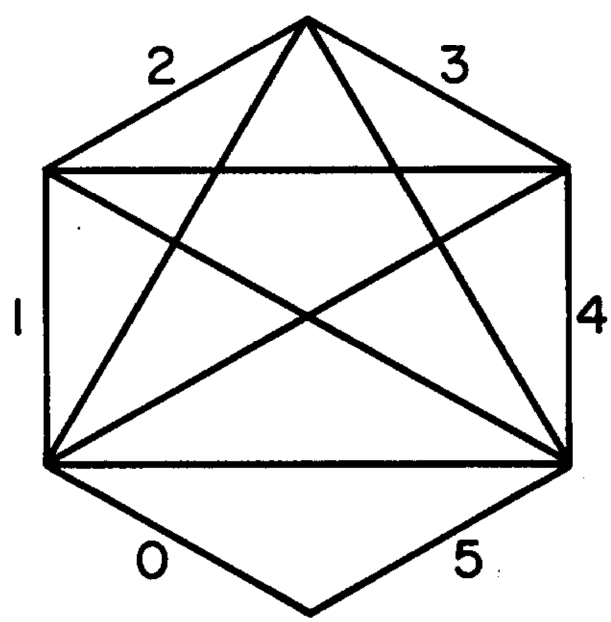


Fig. 2 (a)



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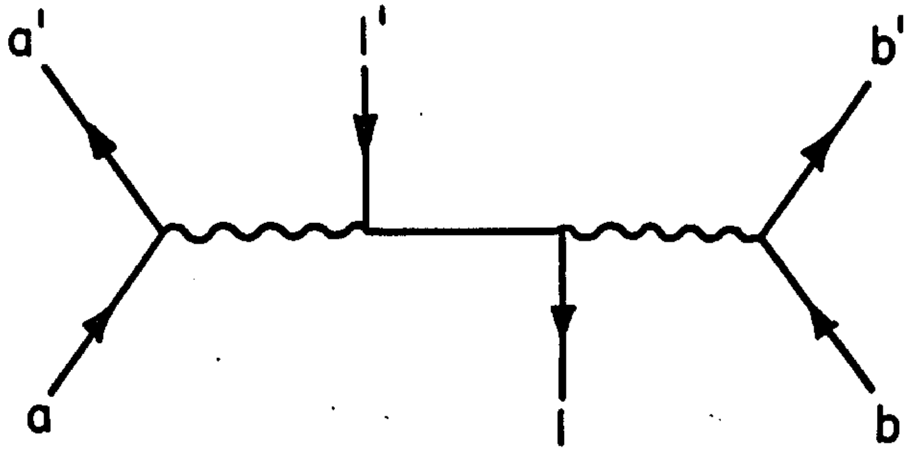

Fig. 3 (a)

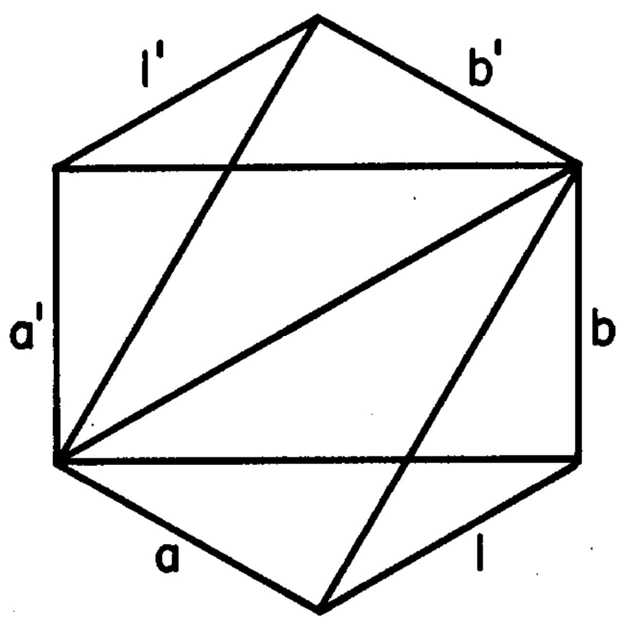

Fig. 3 (b)