p-70/6-T
THY - 6

GROUP THEORETICAL CONSTRUCTION OF DUAL AMPLITUDES

L. Clavelli and P. Ramond
National Accelerator Laboratory, Batavia, Illinois 60510

September 1970

ABSTRACT

We present general group theoretical requirements for the construction of factorizable, dual N-point functions. Dual amplitudes for the scattering of arbitrary numbers of spinning particles are built as an example of this approach.

Although the SU(1, 1) invariance of the Veneziano N-point function has long been recognized,[1] it has not yet been systematically exploited in the construction of new dual amplitudes. In the following, we would like to generalize the group theoretic structure of the multi-Veneziano function, discussed earlier by the authors,[2] to include external particles of different types. As an example of our techniques, we give a simple, factorizable dual amplitude for the absorption of arbitrary numbers of high spin particles.

For the purpose of this paper, we regard duality as a purely group theoretic concept, implying SU(1, 1) invariance but not necessarily any particular asymptotic behavior (e. g., Regge). Basic to our description are the three generators of SU(1, 1), L_o, L_+ and L_- which satisfy the algebra

$$[L_o, L_\pm] = \pm L_\pm \quad [L_+, L_-] = -L_o. \tag{1}$$

We present the following minimal set of group theoretical conditions for the construction of dual amplitudes.

(i) Associate with the absorption of a particle with momentum k_μ, spin j, j_3, internal quantum numbers $\{\lambda\}$, a vertex operator $V(k_\mu, j, j_3, \{\lambda\}; z)$, where $z = e^{-i\tau}$ is a complex variable on the unit circle.

(ii) Require that $V(k_\mu, j, j_3, \{\lambda\}; z)$ transforms under SU(1,1) as a spin J_s representation, that is to say

$$[L_o, V] = -z \frac{d}{dz} V \tag{2a}$$

$$[L_\pm, V] = \frac{-1}{\sqrt{2}} z^{\pm 1}\left(z \frac{d}{dz} \mp J_s\right) V, \tag{2b}$$

where we take J_s to be in general a function of the Casimir operators of the Lorentz and internal symmetry groups

$$J_s = J_s(m^2, j, c^\lambda). \tag{3}$$

(iii) Under the Lorentz and internal symmetry groups, V is required to transform in the same way as the field of the absorbed particle. This insures the correct selection rules at each vertex.

(iv) Any number of external particles 1, 2, ..., ℓ, ... can interact in a dual manner only if they have the same SU(1,1) spin, i.e.,

$$J_s\left(m_1^2, j_{(1)}, c^\lambda_{(1)}\right) = J_s\left(m_\ell^2, j_{(\ell)}, c^\lambda_{(\ell)}\right) \text{ for all } \ell . \tag{4}$$

This implies relations among the quantum numbers of different particles.

(v) Then the factorizable dual amplitude for the scattering of particles 1, 2, ..., ℓ, ..., N in that order, is given by

$$A_N = \frac{1}{C} < 0 \left| \oint \prod_{\ell=1}^{N} \left\{ \frac{dz_\ell}{z_\ell} \left| z_\ell - z_{\ell+1} \right|^{-1-J_s} \theta(\arg z_{\ell+1} - \arg z_\ell) \times \right. \right.$$

$$\left. \times V(k_{\ell\mu}, j_{(\ell)}, j_{3(\ell)}, \{\lambda\}_{(\ell)}; z_\ell) \right\} \left| 0 > \right. \tag{5}$$

where C is the integrated Haar measure, and the contour is taken around the unit circle. This type of expression was shown to be SU(1,1) invariant in Ref. (2). If V is a matrix in the internal group space, the vacuum expectation value must be defined to include the trace of the product of the vertices.

We now proceed to illustrate these rules by considering several examples of dual amplitudes. The best known example of a vertex satisfying criteria (i) - (iv) is the one for scalar isoscalar particles

$$V(k_\mu, 0, 0; z) = e^{\frac{-k^2}{2\epsilon}} e^{ik\cdot F^+(z)} e^{ik\cdot F(z)} \tag{6}$$

where F(z) satisfies[2,3]

$$[L_o, F_\rho(z)] = -z \frac{d}{dz} F(z) \tag{7a}$$

$$[L_\pm, F_\rho(z)] = -\frac{z^{\pm 1}}{\sqrt{2}} \left(z \frac{d}{dz} \pm \frac{\epsilon}{2}\right) F(z) \tag{7b}$$

This vertex obeys Eqs. (2) with $J_s = -\frac{k^2}{2}$.

Its insertion in Eq. (5) yields the usual multi-Veneziano amplitude, provided (criterion (iv) that all the particles have the same mass. We will let the factor $e^{-k^2/2\epsilon}$ be understood from here on.

A simple extension of this vertex for isovector scalar particles

$$V(k_\mu, 0, I_k; z) = \sigma_k V(k_\mu, 0, 0; z) \tag{8}$$

leads to the Chan-Paton amplitude.[4]

We now apply our criteria to the construction of dual vertices for particles of higher spin. Our basic building blocks will be the Nambu field[5]

$$Q_\mu(z) = F_\mu(z) + F^+_\mu(z) \tag{9a}$$

$$= \sum_{m=0}^{\infty} \left[\frac{(m-1+\epsilon)!}{m!}\right]^{1/2} \left\{ a_\mu^{(m)} z^{m+\frac{\epsilon}{2}} + a_\mu^{(m)^+} z^{-m-\frac{\epsilon}{2}} \right\} \tag{9b}$$

and its conjugate momentum

$$P_\mu(z) = i [L_o, Q_\mu(z)] \tag{10a}$$

$$= -i \sum_{m=0}^{\infty} \left(m + \frac{\epsilon}{2}\right) \left[\frac{(m-1+\epsilon)!}{m!}\right]^{1/2} \left\{ a_\mu^{(m)} z^{m+\frac{\epsilon}{2}} - a_\mu^{(m)^+} z^{-m-\frac{\epsilon}{2}} \right\} \tag{10b}$$

where ϵ has to be taken to zero at the end of all calculations. It follows from Eqs. (10) and (7) that

$$[L_o, P_\mu(z)] = -z \frac{d}{dz} P_\mu(z) \tag{11a}$$

$$[L_\pm, P_\mu(z)] = -\frac{z^{\pm 1}}{\sqrt{2}} \left(z \frac{d}{dz} \pm (1 + \frac{\epsilon}{2}) \right) P_\mu(z) + \frac{i\epsilon}{2\sqrt{2}} z^{\pm 1} Q_\mu(z) \tag{11b}$$

As a first attempt at constructing a vertex for the absorption of a vector meson, we might consider $e^{ik\cdot F^{+}(z)}\, e^{ik\cdot F(z)}\, P_\mu(z)$. This indeed transforms irreducibly under the group but its lack of normal ordering leads to a divergent dual amplitude. From this stems the general necessity to consider only normal ordered vertices, as is common in field theory. On the other hand, the normal ordered vertex $e^{ik\cdot F^{+}(z)}\, P_\mu(z)\, e^{ik\cdot F(z)}$ does not transform irreducibly, and in fact the commutator with $L_\pm$ diverges if $P_\mu(z)$ and $F(z)$ are evaluated at the same point, i. e.

$$[\, L_+,\ e^{ik\cdot F^{+}(z)}\, P_\mu(z+\delta)\, e^{ik\cdot F(z)}] = -\frac{z}{\sqrt{2}}\left(z\frac{d}{dz} + 1 + \frac{k^2}{2}\right) e^{ik\cdot F^{+}(z)} P_\mu(z+\delta) e^{ik.F(z)}$$

$$+ k_\mu \frac{z}{\sqrt{2}}\left(\frac{1}{2} + \frac{2z}{\delta}\right) e^{ik\cdot F^{+}(z)}\, e^{ik\cdot F(z)} \qquad (12)$$

The undesirable extra term can be eliminated only if one takes the transverse part of $P_\mu(z)$

$$\hat{P}_\mu(z) = \left(g_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}\right) P_\rho(z) \qquad (13)$$

Then, a well-behaved, normal ordered vertex for vector mesons is

$$V_\mu(k;z) = e^{ik\cdot F^{+}(z)}\ \hat{P}_\mu(z)\ e^{ik.F(z)} \qquad (14)$$

and satisfies

$$[\, L_o,\ V_\mu(k,z)\,] = -z\frac{d}{dz}\ V_\mu(k,z) \qquad (15a)$$

$$[\, L_\pm,\ V_\mu(k,z)\,] = -\frac{z^{\pm 1}}{\sqrt{2}}\left(z\frac{d}{dz} \pm \left(1+\frac{k^2}{2}\right)\right) V_\mu(k,z). \qquad (15b)$$

Thus V_μ transforms as a $J_s = -1 - \frac{k^2}{2}$ representation of SU(1, 1).

According to our rules, for any fixed value of k^2, we can construct a dual vector meson N-point amplitude. However, if we wish to have a dual amplitude containing both scalar and vector mesons, criterion (iv) forces the mass relation

$$1 + \frac{(k_v)^2}{2} = \frac{(k_s)^2}{2}, \tag{16}$$

which fixes the vector meson as the first Regge recurrence of the ground state scalar.

As an example we will calculate the amplitude $S + V \to S + S$. Putting $\frac{(k_s)^2}{2} = \alpha_o$, we have from Eq. (5).

$$A_\mu = \frac{1}{C} \oint \prod_{\ell=1}^{4} \left\{ \frac{dz_\ell}{z_\ell} \left| z_\ell - z_{\ell+1} \right|^{\alpha_o - 1} \theta(\arg z_{\ell+1} - \arg z_\ell) \right\} \times$$

$$\times < 0 \mid V(k_1, z_1)\, V_\mu(k_2, z_2)\, V(k_3, z_3)\, V(k_4, z_4) \mid 0 > \tag{17a}$$

$$= \left(g_{\mu\rho} - \frac{k_{2\mu} k_{2\rho}}{k_v^2} \right) \frac{1}{C} \oint \prod_{\ell=1} \left\{ \frac{dz_\ell}{z_\ell} \left| z_\ell - z_{\ell+1} \right|^{\alpha_o - 1} \times \right.$$

$$\left. \times\ \theta(\arg z_{\ell+1} - \arg z_\ell) \right\} \prod_{i<j} \left| z_i - z_j \right|^{k_i \cdot k_j} \left(\sum_{\ell \neq 2} \frac{1}{2} k_{\ell\rho} \frac{z_\ell + z_2}{z_\ell - z_2} \right) \tag{17b}$$

$$= \left(g_{\mu\rho} - \frac{k_{2\mu} k_{2\rho}}{k_v^2} \right) \int_o^1 dx\, x^{-1-\alpha(s)} (1-x)^{-1-\alpha(t)} \left[k_{1\rho}(1-x) - k_{3\rho} x \right] \tag{17c}$$

$$= \int_o^1 dx\, x^{-1-\alpha(s)} (1-x)^{-1-\alpha(t)} \left[k_{1\mu}(1-x) - k_{3\mu} x + k_{2\mu}\left(\frac{1}{2} - x\right) \right] \tag{17d}$$

This amplitude coincides with the one proposed by Bouchiat, Gervais and Sourlas to represent a vector current interacting with three scalars. From the group theoretical point of view, however, it is clear that the amplitude more appropriately describes a vector meson since the mass must remain fixed for duality. Indeed in the present SU(1, 1) coupling scheme, a dual theory with currents must await the construction of a current vertex whose SU(1, 1) transformation properties are independent of its squared four-momentum.

Needless to say, it is trivial to calculate a dual amplitude involving any number of scalars and vectors using Eq. (5) and maintaining the mass relation (16).

We would now like to investigate dual vertices for higher spin particles. If we write symbolically

$$\hat{P}_{\mu}^{(n)}(a) \equiv \prod_{i=1}^{n} \hat{P}_{\mu_i}(z) \tag{18}$$

then

$$V_{\mu}^{(n)}(z) = e^{ik\cdot F^{+}(z)} : \hat{P}_{\mu}^{(n)}(z) : e^{ik\cdot F(z)} \tag{19}$$

transforms reducibly under the Lorentz group, containing spins up to $j_{max} = n$. Furthermore, under SU(1, 1)

$$[\, L_{\pm},\ V_{\mu}^{(n)}(z)\ ] = -\frac{z^{\pm 1}}{\sqrt{2}} \left(z\frac{d}{dz} \pm (n + \frac{k^2}{2}) \right) V_{\mu}^{(n)}(z) \tag{20}$$

i. e., $V_{\mu}^{(n)}$ transforms as a $J_s = -n - \frac{k^2}{2}$ representation. In order for the particles represented by the various $V_{\mu}^{(n)}$ to interact in a dual manner we must have

$$n + \frac{k_n^2}{2} = \alpha_o \quad n = o, 1, \ldots . \tag{21}$$

This mass relation identifies $V_\mu^{(n)}$ as representing particles at the (n + 1)th mass level. By operating on $V_\mu^{(n)}$ with the various spin projection operators, we can separate distinct vertices for particles of spin n, n-2, n-4,... etc. The odd daughters, however, cannot be constructed in this scheme since $V_\mu^{(n)}$ is transverse in all its indices. Furthermore, we have no way to distinguish degenerate states at the same spin.

Although we have investigated here only a very limited class of dual vertices, it seems clear that the group theoretical approach should prove useful in the consideration of internal symmetries, off shell currents, and the positive intercept problem. For example, the identities found by Virasoro[7] require only that $J_s = -1$, thereby perhaps avoiding the necessity of introducing an unphysical pole.

REFERENCES

[1] Z. Koba and H. B. Nielsen, Nuclear Physics B12, 517 (1969)

[2] L. Clavelli and P. Ramond, Phys. Rev. D Sept. 15, (1970)

[3] S. Fubini and G. Veneziano, Nuovo Cimento 67, 29 (1970)

[4] J. E. Paton and Chan Hong-Mo, Nuclear Phys. B10, 516 (1969)

[5] Y. Nambu, University of Chicago preprint EFI6964 (1969)

[6] C. Bouchiat, J. L. Gervais, N. Sourlas, Lett. al Nuovo Cimento 3, 767 (1970). See also R. Brower and J. Weis, Phys. Rev. 188, 2486 and 2495 (1969) and LRL preprint (1970).

[7] M. A. Virasoro, Phys. Rev. 1, 2933 (1970)