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SU(1,1) ANALYSIS OF DUAL RESONANCE MODELS

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ABSTRACT

The representations of the non compact group SU(1, 1) are discussed with regard to applications to dual resonance models. The Gliozzi operators are constructed from a standard differential representation of SU(1, 1). We point out that the delicate limiting procedure appearing in the recent literature has its group theoretical basis in the fact that SU(1, 1), unlike its compact counterpart SU(2), has no non-trivial unitary spin zero representation. We further note that the vertex appearing in the model effectively transforms as the spin $-\alpha_0$ representation of the continuous class, exceptional interval, of SU(1, 1). The N point dual amplitude then appears as the coupling of N such vertices to the identity. Finally, we discuss the classification of the states in the model under the group. A complete classification in terms of SU(1, 1) is shown to break down at $\alpha(s) = 8$ on and below the 4th daughter trajectory.

I. INTRODUCTION

The N-point generalization of the Veneziano formula[1] was shown by Koba and Nielsen[2] to possess an SL(2, C) symmetry. In their work, they exhibit the N-point amplitude as an ordered integration over points on the unit circle which are in one-to-one correspondance with the external scalar mesons, i.e.

$$A_N = \frac{1}{C} \int_0^{2\pi} d\phi_1 \ldots \int_0^{2\pi} d\phi_N \prod_{i=1}^{N-1} \Theta(\phi_i - \phi_{i+i}) \quad \times$$

$$\prod_{i=1}^{N} |z_i - z_{i+1}|^{\alpha_0 - 1} \prod_{i<j} |z_i - z_j|^{k_i \cdot k_j} \qquad (1.1)$$

where $z_i = e^{i\phi_i}$ and (1.2)

$$C = \int_0^{2\pi} d\phi_a \int_0^{2\pi} d\phi_{a+1} \int_0^{2\pi} d\phi_{a+2} \frac{\Theta\left(\phi_a - \phi_{a+1}\right)\Theta\left(\phi_{a+1} - \phi_{a+2}\right)}{|z_a - z_{a+1}|\,|z_{a+1} - z_{a+2}|\,|z_{a+2} - z_a|} .$$

As a result of the SL(2, C) symmetry, it is possible to fix any three points and then integrate over the remaining N-3 variables. A convenient choice for these points is $z_{N-1} = 1$, $z_N = -i$, $z_1 = i$, corresponding to the configuration of Figures 1-a and 1-b. The remaining variables are then integrated over the first quadrant. Any multiperipheral dual configuration can be obtained by mapping any three consecutive points[3] into the positions 1, -i, and i, leaving the order unchanged. Should non-consecutive points be mapped onto the above locations, one arrives at non-multiperipheral configurations of the amplitude. These transformations form a discrete subgroup of SU(1, 1), the continuous subgroup of SL(2, C) that maps the unit circle onto itself. Its properties were first studied in detail by V. Bargmann;[4] we summarize them in Appendix A.

In the next section, we discuss in detail the group-theoretical content of the scalar dual vertices. Section III is devoted to the problem of classifying the intermediate states of the dual amplitude under SU (1, 1).

II. GROUP PROPERTIES

As shown by Fubini and Veneziano,[5,6] there appears in the N-point function an operator which ostensibly transforms as a scalar under SU(1, 1). However, SU(1, 1) has no non-trivial scalar Unitary Irreducible Representation (UIR), as noted in Appendix A. It is the purpose of this section to try to clarify the connection between the group SU(1, 1) and the various operators appearing in the factorization of the amplitude.[7]

We start by writing an operator function in a basis belonging to a spin J representation $D_J^{(+)}$ of the algebra

$$F_\rho(z) = \sum_{m=o}^{\infty} a_\rho^{(m)} \; |J, -J, m) = \sum_{m=o}^{\infty} a_\rho^{(m)} \left[\frac{(m-1-2J)!}{m!}\right]^{1/2} z^{-J+m} , \quad (2.1)$$

where we have used equations (A-9) and (A-13). Similarly, a corresponding operator function expanded in the conjugate basis $D_J^{(-)}$ is given by

$$\tilde{F}_\rho(z) = \sum_{m=o}^{\infty} a_\rho^{(m)\dagger} \,|\, J, J, -m) = \sum_{m=o}^{\infty} a_\rho^{(m)\dagger} \left[\frac{(m-1-2J)!}{m!} \right]^{1/2} z^{J-m}, \quad (2.2)$$

where we take $a_\rho^{(m)}$ and $a_\rho^{(m)\dagger}$ to be harmonic oscillator operators[7] satisfying

$$\left[a_\rho^{(m)}, a_\sigma^{(\ell)\dagger} \right] = g_{\rho\sigma} \delta_{\ell m} \quad (2.3)$$

we use the metric $(+ + + -)$. Let $-2J = \epsilon$ where ϵ is a small positive real number. Expanding to first order in ϵ, we obtain

$$F_\rho(z) = (1 + \frac{\epsilon}{2} \ell nz)\left(\frac{q_{o\rho}}{2} + i \frac{p_{o\rho}}{\epsilon} \right) + \sum_{m=1}^{\infty} a_\rho^{(m)} \frac{z^m}{\sqrt{m}} \quad (2.4a)$$

$$\tilde{F}_\rho(z) = (1 - \frac{\epsilon}{2} \ell nz)\left(\frac{q_{o\rho}}{2} - i \frac{p_{o\rho}}{\epsilon} \right) + \sum_{m=1}^{\infty} a_\rho^{(m)\dagger} \frac{z^{-m}}{\sqrt{m}} \quad (2.4b)$$

where $q_{o\rho}$ and $p_{o\rho}$ are the canonical coordinates corresponding to the "zero" mode. These expressions are not well defined. However, the factor $\frac{1}{\epsilon} p_{o\rho}$ appearing in both forms disappears in their sum

$$Q_\rho(z) = F_\rho(z) + \tilde{F}_\rho(z) \quad (2.5a)$$

$$= q_{o\rho} + i\, p_{o\rho} \ell nz + \sum_{m=1}^{\infty} \frac{a_\rho^{(m)} z^m + a_\rho^{(m)\dagger} z^{-m}}{\sqrt{m}} \quad (2.5b)$$

We recognize the scalar operator of references (5, 6) which is a direct sum of two operators, each irreducible under SU(1, 1).

It is possible to construct a representation of the SU(1, 1) algebra in the space of the operators (2. 3) by sandwiching the differential expressions (A-8) between the operators F or $\tilde{F}$ and then using the orthogonality properties of the basis states.

The result is:

$$\hat{L}_o = (F|\, L_o \,|F) = \sum_{m=o}^{\infty} (m-J) a^{(m)\dagger} \cdot a^{(m)} \qquad (2.6a)$$

$$\hat{L}_+ = (F|\, L_+ \,|F) = \sum_{m=o}^{\infty} \left[\frac{(m-2J)(m+1)}{2}\right]^{1/2} a^{(m+1)\dagger} \cdot a^{(m)} \qquad (2.6b)$$

$$\hat{L}_- = (F|\, L_- \,|F) = \sum_{m=o}^{\infty} \left[\frac{(m-2J)(m+1)}{2}\right]^{1/2} a^{(m)\dagger} \cdot a^{(m+1)} \qquad (2.6c)$$

These results have been obtained by keeping J arbitrary <u>both</u> in equations (2. 1) and (A-8). When we let $J = -\frac{1}{2}\epsilon$ and expand to first order in ϵ, we find (dropping the carets from now on) that the operators (2. 6) become respectively

$$L_o = \frac{1}{2} p_o^2 + \sum_{m=1}^{\infty} m a^{(m)\dagger} \cdot a^{(m)} \qquad (2.7a)$$

$$L_+ = \frac{1}{\sqrt{2}} \left[i p_o \cdot a^{(1)\dagger} + \sum_{m=1}^{\infty} \sqrt{m(m+1)}\, a^{(m+1)\dagger} \cdot a^{(m)} \right] \qquad (2.7b)$$

$$L_- = \frac{1}{\sqrt{2}} \left[-i p_o \cdot a^{(1)} + \sum_{m=1}^{\infty} \sqrt{m(m+1)}\, a^{(m)\dagger} \cdot a^{(m+1)} \right]. \qquad (2.7c)$$

These are the operators formed by Gliozzi[8] in connection with the Ward-like identities. This means that the representation (2. 7) of the SU(1, 1) algebra is the relevant one when dealing with the amplitude (1.1). Under these operators, our fundamental operators $F_\rho(z)$ transform as follows [taking $-2J = \epsilon << 1$] :

$$\left[L_o, F_\rho(z)\right] = -z \frac{d}{dz} F_\rho(z) \tag{2.8a}$$

$$\left[L_+, F_\rho(z)\right] = -\frac{z}{\sqrt{2}} \left(z\frac{d}{dz} + \frac{\epsilon}{2}\right) F_\rho(z) \tag{2.8b}$$

$$\left[L_-, F_\rho(z)\right] = -\frac{z^{-1}}{\sqrt{2}} \left(z\frac{d}{dz} - \frac{\epsilon}{2}\right) F_\rho(z) \tag{2.8c}$$

from which

$$\left[L_+, [L_-, F_\rho(z)]\right] + \left[L_-, [L_+, F_\rho(z)]\right] - \left[L_o, [L_o, F_\rho(z)]\right] = \frac{\epsilon}{2}\left(1 - \frac{\epsilon}{2}\right) F_\rho(z) \tag{2.9}$$

The same commutation relations hold when $F_\rho(z)$ is replaced by $\tilde{F}_\rho(z)$. As stated above, this means that $Q_\rho(z)$ transforms reducibly under the Gliozzi operators (2.7). It follows from (2.8) that under finite transformations, we have

$$e^{i\xi\cdot L} F_\rho(z)\, e^{-i\xi\cdot L} = |\alpha^* - \beta z|^{-\epsilon} F_\rho\left(\frac{\alpha z - \beta^*}{-\beta z + \alpha^*}\right) \tag{2.10}$$

where $e^{i\xi\cdot L}$ is the representation of the group element h, corresponding to the matrix (A-1) of Appendix A.

Next we would like to consider the modified vertex describing the absorption or emission of an on shell scalar meson of momentum k_ρ

$$V(k, z) \equiv e^{ik\cdot \tilde{F}(z)}\, e^{ik\cdot F(z)} , \tag{2.11}$$

the form of which becomes suggestive in the quart-antiquark interpretation of the emitted meson. It follows directly[9] from equations (2.8) that as $\epsilon \to 0$

$$[L_o, V(k, z)] = - z \frac{d}{dz} V(k, z) \tag{2.12a}$$

$$[L_\pm, V(k, z)] = - \frac{z^{\pm 1}}{\sqrt{2}} \left(z \frac{d}{dz} \pm \frac{k^2}{2} \right) V(k, z) \tag{2.12b}$$

where we have used

$$\lim_{\epsilon \to 0} \epsilon \sum_{m=o}^{\infty} \frac{(m + \epsilon - 1)!}{m!} = 1 \tag{2.13}$$

Then, as a consequence of Equations(2.12),

$$\left[L_+ [L_-, V(k, z)]\right] + \left[L_-, [L_+, V(k, z)]\right] - \left[L_o, [L_o, V(k, z)]\right] = \frac{k^2}{2} (1 - \frac{k^2}{2}) V(k, z) \tag{2.14}$$

and

$$e^{i\xi \cdot L} V(k, z) e^{-i\xi L} = |\alpha^* - \beta z|^{-k^2} V\left(k, \frac{\alpha z - \beta^*}{\alpha^* - \beta z}\right) \tag{2.15}$$

The last equation has been derived by taking z to be on the unit circle. We take k^2 to be real. It can then be seen from equation (A-18) that V(k, z) formally transforms according to an UIR of the continuous class, exceptional interval with $\sigma = \alpha_o - \frac{1}{2}$ (or $\frac{k^2}{2} - \frac{1}{2}$ if we take equation (2.11) to hold for off-mass shell mesons). Then $-\alpha_o$ is the SU(1, 1) "spin" of the vertex operator.

We might now ask how one might couple N such vertices to make an SU(1, 1) invariant. This problem is solved by V. Bargmann (see Equation A-17) for the case N = 2. An obvious generalization to N > 2 is

$$A_N = \prod_{j=1}^{N} \int_0^{2\pi} d\phi_j \left[1 - \cos(\phi_j - \phi_{j+1}) \right]^{\frac{\alpha_o - 1}{2}} V(k_j, z_j) \qquad (2.16)$$

where $z_i = e^{i\phi_i}$ and the ϕ's are defined cyclically ($\phi_{N+1} = \phi_1$). The invariance of A_N under the group SU(1, 1) is evident from equation (2.15) since

$$\prod_{j=1}^{N} d\phi_i \left[1 - \cos(\phi_j - \phi_{j+1}) \right]^{\frac{\alpha_o - 1}{2}} \left| \alpha^* - \beta e^{i\phi_j} \right|^{-2\alpha_o} = \prod_{j=1}^{N} d\phi_i' \left[1 - \cos(\phi_j' - \phi_{j+1}') \right]^{\frac{\alpha_o - 1}{2}} \qquad (2.17)$$

where

$$e^{i\phi_j'} = \frac{\alpha e^{i\phi_j} - \beta^*}{\alpha^* - \beta e^{i\phi_j}} \ . \qquad (2.18)$$

In order to make contact with the N-point Veneziano function, we note the following

$$1 - \cos(\phi_j - \phi_i) = \frac{1}{2} \left| z_j - z_i \right|^2 \qquad (2.19)$$

and

$$< 0 \left| \prod_{j=1}^{N} V(k_j, z_j) \right| 0> = \prod_{\ell<j} \left| z_\ell - z_j \right|^{k_\ell \cdot k_j} e^{i \frac{\pi}{2} \sum_{\ell<j} k_\ell \cdot k_j \, \epsilon_{\ell j}} e^{\frac{Nk^2}{2\epsilon}} \tag{2.20}$$

where we have used conservation of momentum $\left(\sum_{i=1}^{N} k_{i\rho} = 0 \right)$, and

$$\epsilon_{ij} = \begin{cases} +1 & \phi_\ell > \phi_j \\ -1 & \phi_\ell < \phi_j \end{cases} \tag{2.21}$$

The last factor in Equation (2.20) can be absorbed by a suitable redefinition of the V's. The phase appearing in (2.20) can also be absorbed in V provided that we impose an ordering condition on the angles, $\phi_\ell > \phi_{\ell+1}$ or $\phi_\ell < \phi_{\ell+1}$, corresponding to a cyclic or anticyclic ordering of the momenta k_ℓ respectively. Since SU(1,1) does not change the ordering of points on a circle, modifying Equation (2.16) by the product of the appropriate Θ - functions does not alter its invariance properties. We then obtain

$$< 0 \left| A_N \right| 0> = \prod_{i=1}^{N} \int_0^{2\pi} d\phi_i \left| z_i - z_{i+1} \right|^{\alpha_0 - 1} \Theta(\phi_i - \phi_{i+1}) \prod_{i<j} \left| z_i - z_j \right|^{k_i \cdot k_j} \tag{2.22}$$

which is the Koba-Nielsen form of the Veneziano amplitude (equation 1.1) apart from the invariant factor given by equation (1.2).

Finally, we wish to point out that in the absence of any ordering condition on the angles, the phase factors in Equation (2.20) play a special role in the case $\alpha_0 = 1$. Indeed, we can then show explicitely that for N = 4, the unordered form of the amplitude (2. 22) is proportional to the sum of the three beta functions[10]

$$<0 | A_N | 0> = 2C \left[B(-\alpha_s, -\alpha_t) + B(-\alpha_s, -\alpha_u) + B(-\alpha_t, -\alpha_u) \right] \qquad (2.23)$$

where C is the constant appearing in Equation (1. 2), and the numerical factor stems from the double-counting of each inequivalent permutation of the external legs. This equation would not hold in the absence of the phase factors. One can further speculate that this is also true for $N > 4$, and that the unordered amplitude $<0 | A_N | 0>$ represents twice the sum over all inequivalent permutations of the external legs, i. e., the total dual amplitude.

III. CLASSIFICATION OF THE STATES

The spectrum of the intermediate states appearing in the factorization of the multi-Veneziano amplitude (1. 1) is well-known. It consists of particles whose mass m_n, satisfies $\alpha(m_n^2) = n$, $n = 0, 1, \ldots$; the enormous degeneracy present in the theory arises from the fact that for a given n, the contributing states are those eigenstates of the partition operator.

$$R_o = \sum_{m=1}^{\infty} m a^{\dagger(m)} \cdot a^{(m)} \qquad (3.1)$$

with eigenvalues less than n. This situation is somewhat alleviated in the case $\alpha_o = 1$ where only the states with eigenvalue n contribute. In terms of the Gliozzi operator, the contributing states satisfy

$$L_o \left| \Psi_{phys} \right> = (\alpha_o - \ell) \left| \Psi_{phys} \right> \qquad (3.2)$$

where $\ell = 0, 1, 2, \ldots$. Hence, for a given α_o, the spectrum of L_o is bounded from above.

We now wish to build representations of the Gliozzi algebra, and start by considering the state

$$\left| k, 0 \right> = e^{ik \cdot q_o} \left| 0 \right> \qquad (3.3)$$

which is normalizable and annihilated by L_-. Thus, in the manner of Appendix A, one is tempted to generate all the states by repeated application of L_+

$$\left| k, j \right> = \gamma_j(k^2)(L_+)^j \left| k, 0 \right> \qquad (3.4)$$

where $\gamma_j(k^2)$ is a normalization constant

$$\gamma_j(k^2) = \left[\frac{j!}{2^j} \frac{(k^2 + j - 1)!}{(k^2 - 1)!} \right]^{-1/2} \qquad (3.5)$$

In addition,

$$L^2 \left| k, j \right> = \frac{k^2}{2} \left(1 - \frac{k^2}{2} \right) \left| k, j \right> \qquad (3.6)$$

so that the lowest states ($\frac{k^2}{2} = \alpha_o$) have the same SU(1, 1) spin as V(k, z). The physical states are those for which $j = \alpha(s) - \ell$ with $\ell = 0, 1, \ldots, \alpha(s)$. Unfortunately, these states are always the spin zero projection of the full tensor states so that they certainly do not represent the whole spectrum. Furthermore, it can be seen that for j sufficiently large, the states (3.4) are linear combinations of occupation number states which cannot all be accounted for by SU(1, 1) quantum numbers alone. This point will be clarified. In order to construct states with spin, introduce the operators

$$R_+ = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \sqrt{n(n+1)}\; a^{(n+1)^\dagger} . a^{(n)} \tag{3.7a}$$

$$R_- = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \sqrt{n(n+1)}\; a^{(n)^\dagger} . a^{(n+1)} \tag{3.7b}$$

which form a SU(1, 1) algebra with the partition operator R_o. We wish to classify the intermediate states in terms of UIR's of this algebra.

First we note that the number operator

$$N = \sum_{n=1}^{\infty} a^{\dagger (n)} . a^{(n)} \tag{3.8}$$

commutes with R_+, R_-, and R_o although it is not simply related to the Casimir operator

$$R^2 = R_+R_- + R_-R_+ - R_o^2 . \tag{3.9}$$

It follows from Schur's lemma that we can construct different UIR's as eigenstates of N. We thus define the basis states

$$| k;n, o> = \prod_{\rho=0}^{3} \frac{\left(a_\rho^{(1)^\dagger}\right)^{n_\rho}}{\sqrt{n_\rho !}} \, | k; 0 > \tag{3.10}$$

where the ket on the left hand side stands for all partition states of n into four integers $\left(n = \sum_{\rho=0}^{3} n_\rho \right)$. Since R_- annihilates the states (3.10) we can form UIR's by repeated application of R_+.

$$| k; n, m > \; = \; N(n, m) \, (R_+)^m \, | k; n, 0 > \tag{3.11}$$

These form an orthonormal set if we choose

$$N\,(n, m) \; = \; \left[\frac{2^m (2n-1)!}{m!\,(2n+m-1)!} \right]^{1/2} . \tag{3.12}$$

By construction we have

$$R_o | k; n, m > \; = \; (n + m) \, | k; n, m > \tag{3.13a}$$

$$L_o | k; n, m > \; = \; (\frac{k^2}{2} + n + m) \, | k; n, m > \tag{3.13b}$$

$$N | k, n, m > \; = \; n \, | k; n, m > \tag{3.13 c}$$

$$R^2 | k; n, m > \; = \; n(1 - n) \, | k; n, m > \; . \tag{3.13d}$$

Again, the states contributing to the pole $\alpha(s)$ = integer are those for which

$$0 \le n + m \le \alpha(s) \; = \; \alpha_o - \frac{k^2}{2} \tag{3.14}$$

The kets (3.11) are in one to one correspondance with the occupation number states up to $\alpha(s) = 4$. Here unfortunately the huge degeneracy in the model begins to complicate the SU(1,1) classification. This is easy to see since (3.11) contains only one state with $R_o = 4, N = 2$ namely

$$| k, 2, 2 > \; = N(2, 2) \, (\sqrt{3} \, a^{(3)^\dagger} a^{(1)^\dagger} + a^{(2)^\dagger} a^{(2)^\dagger}) \, | k\,0 > \tag{3.15}$$

whereas there are clearly two independent occupation number states with these quantum numbers. However, the state

$$N\ (2,2)\ (a^{(3)\dagger} a^{(1)\dagger} - \sqrt{3}\, a^{(2)\dagger} a^{(2)\dagger})\ |\, k\ 0 > \qquad (3.16)$$

is orthogonal to (3.15) and is annihilated by R_-. Hence, (3.16) serves as the lowest state of a new UIR of SU(1,1) with Casimir invariant $R^2 = 4(1-4)$. This procedure is satisfactory; however, only up to $\alpha(s) = 8$ when further trouble sets in. At this level there are two extra independent occupation number states leading to two representations degenerate in R^2, R_o and N. Beyond this point, the SU(1,1) of Eqs. (3.1) and (3.7) becomes progressively more inadequate for classification of states suggesting that either a larger group is at work or that many states (good states as well as ghosts) are decoupled in the model from physical scalar meson states. The above anomaly only affects states lying on or below the fourth daughter trajectory. All those on the parent and first three daughters (as well as all states below $\alpha(s) = 8$) can be unambiguously classified under the SU(1,1) scheme. In view of the simple connection between the R algebra and the Gliozzi algebra, one might hope that such a classification will lead to general statements about the norms of decoupled states.

IV. CONCLUSION

In this paper, we have discussed the multi-Veneziano amplitude in terms of the group SU(1, 1) which is responsible for the duality transformations. We have shown that the intercept of the trajectory played a fundamental role as it could be likened to the SU(1, 1) spin (its relation to the SU (1, 1) subgroup of the Lorentz group that acts on the external momenta is still not clear).[11] The vertices describing the emission (or absorption) of a scalar particle satisfy transformation properties under the Gliozzi algebra which allow for the existence of an invariant form, the v. e. v. of which is shown to be the multi-Veneziano formula. As this construction depends crucially on the mass of the emitted (absorbed) scalars, it does not allow for duality off the mass-shell. This suggests that the transformation properties of the three-reggeon vertex be investigated in the hope of building dual amplitudes for particles with spin. This point is currently under study. In addition, when $\alpha_o = 1$, the most general invariant is seen to represent the total amplitude (at least for N = 4) with all non-equivalent permutations included. Our approach suggests this might be true in general, thus, providing an attractive closed form for the total N-point amplitude. Finally, it is puzzling that the SU(1, 1) classification of the occupation number states in the theory fails even though we have many different UIR's of the algebra at our disposal (the vector space is highly reducible). The value of the Casimir operator obeys a certain spectrum, thus suggesting the existence of a higher group, perhaps $0(3,1)$ or $0(4,2)$. On the other hand, it might just be that the theory cannot distinguish between all different occupation number states, in which case the degeneracy will be greatly reduced.

APPENDIX A

Definitions: The group SU(1, 1) is the group of all two-dimensional unimodular pseudounitary matrices. A general element h of SU(1, 1) is in one-to-one correspondance with the matrix

$$h \rightarrow \begin{pmatrix} \alpha^* & \beta^* \\ \beta & \alpha \end{pmatrix} , \quad |\alpha|^2 - |\beta|^2 = 1 \qquad \text{(A-1)}$$

where α and β are arbitrary complex numbers (* denotes complex conjugation). Thus, h can be specified by three real parameters. Correspondingly, the Lie algebra of SU(1, 1) contains three linearly independent elements L_o, L_1 and L_2 which are identified in the defining non-unitary representation of SU(1, 1) in terms of the Pauli matrices as follows

$$L_o = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ; \; L_1 = \frac{1}{2}i\sigma_2 = \frac{1}{2}i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ;$$
$$L_2 = -\frac{1}{2}i\sigma_1 = -\frac{1}{2}i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(A-2)}$$

These obey the commutation relations

$$[L_o, L_1] = iL_2 \; ; \; [L_o, L_2] = -iL_1 \; ; \; [L_1, L_2] = -iL_o \qquad \text{(A-3)}$$

The Casimir invariant of the above algebra is the quadratic operator L^2

defined by

$$L^2 = L_1^2 + L_2^2 - L_o^2 \tag{A-4}$$

Finally, every element h of SU(1, 1) may be expressed as a product

$$e^{i\mu L_o}\, e^{i\zeta L_2}\, e^{i\nu L_o} \tag{A-5}$$

corresponding to the decomposition

$$\begin{pmatrix} \alpha^* & \beta^* \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{pmatrix} \begin{pmatrix} \cosh \zeta/2 & \sinh \zeta/2 \\ \sinh \zeta/2 & \cosh \zeta/2 \end{pmatrix} \begin{pmatrix} e^{i\nu/2} & 0 \\ 0 & e^{-i\nu/2} \end{pmatrix} \tag{A-6}$$

Although the parameters μ, ζ, and ν are not uniquely defined by the above relations, it suffices to specify their ranges as follows

$$0 \le \zeta < \infty \;\; ; \;\; -2\pi \le \mu, \nu < 2\pi \tag{A-7}$$

We now consider the representations of SU(1, 1). Each Unitary Irreducible Representation (UIR) can be characterized by the value of the Casimir invariant L^2, and the spectrum of eigenvalues of L_o. We first summarize the representations of the algebra. For that purpose we introduce a convenient realization of the commutation relations (I-3) in terms of differential operators

$$L_o = z\frac{d}{dz}$$

$$L_+ = \frac{1}{\sqrt{2}}(L_1 + iL_2) = \frac{z}{\sqrt{2}}\left(z\frac{d}{dz} - J\right) \tag{A-8}$$

$$L_- = \frac{1}{\sqrt{2}}(L_1 - iL_2) = \frac{z^{-1}}{\sqrt{2}}\left(z\frac{d}{dz} + J\right)$$

from which $L^2 = -J(J+1)$ is automatically a c-number. Representations of the algebra can be constructed in the complex z-plane in terms of eigenstates of L_o

$$|J,k,m) = N(J,k,m)\, z^{k+m} \tag{A-9}$$

with the normalization N chosen so as to insure

$$(J,k,m\,|\,J',k',m') = \delta_{m,m'}\,\delta_{k,k'}\,\delta_{J,J'} \tag{A-10}$$

Clearly

$$L_o|\,J,k,m) = (k+m)\,|\,J,k,m) \tag{A-11}$$

where m is an integer and k is an auxiliary variable further defining the representation.

There are three types of representations:

a) Representations bounded below for which $K = -J$ so that $L_-\,|J,-J,0) = 0$. All states are obtained by repeated application of L_+ on $|\,J,-J,0)$; the spectrum of eigenvalues of L_o is $-J$, $-J+1$, $-J+2, \ldots$.

b) Representations bounded above for which $k = J$ so that $L_+\,|J,J,0) = 0$. All states are obtained by repeated application of L_- on $|J,J,0)$; the spectrum of eigenvalues of L_o is J, $J-1$, $J-2, \ldots$

c) Unbounded representations for which $k \neq \pm J$. The spectrum of eigenvalues of L_o is unbounded, increasing in integer steps.

Further restriction to unitary representations demands that J be real and negative for bounded representations, and J be real or $J = -\frac{1}{2} + $ is

$(0 \leq s < \infty)$ for the unbounded representations. In addition, the requirement $L_+ = (L_-)^\dagger$ when taken between the states (A-9) yields a condition on the normalizations

$$\frac{|N(J,k,m)|^2}{|N(J,k,m-1)|^2} = \frac{(k+m-1-J)^*}{(k+m+J)} \qquad \text{(A-12)}$$

from which

$$N(J, \pm J, m) = \left[\frac{(-2J+|m|-1)!}{|m|!}\right]^{1/2} \qquad \text{(A-13)}$$

for the bounded representations. We note that $N(J, \pm J, 0) = \sqrt{(-2J-1\,!}$ diverges as $J \to 0$. Hence, the scalar UIR of SU(1, 1) is not well-defined for bounded representations and must be considered as the limit of $-J$ small but finite. Similarly, in the case of continuous representations, we must have

$$-1 < \mathrm{Re}J < 0 \qquad \text{(A-14)}$$

The norm of the states is again not well-defined at the endpoints. In constructing UIR's of the Lie group as opposed to the algebra, the further restriction $-1 < \mathrm{Re}J < -\frac{1}{2}$ arises. The single valued UIR's of SU(1, 1) have been determined by V. Bargmann.[4] The restriction to single-valued representations implies that k is either an integer or a half-integer.

The different classes of UI R's are the following:

A) Continuous class, integral case, non-exceptional interval, denoted by C_J^0:

$$J = -\frac{1}{2} - is;\quad 0 \leq s < \infty;\quad k = 0,\ m = 0, \pm 1, \pm 2, \ldots$$

B) Continuous class, half-integral case, denoted by $C_J^{1/2}$:

$$J = -\frac{1}{2} - is; \quad 0 \le s < \infty; \quad k = \frac{1}{2}, \quad m = 0, \pm 1, \pm 2, \ldots$$

C) Continuous class, exceptional interval, denoted by C_J^0:

$$J = -\frac{1}{2} - \sigma; \quad 0 < \sigma < \frac{1}{2}; \quad k = 0, \quad m = 0, \pm 1, \pm 2, \ldots$$

D) Discrete class, positive m, denoted by $D_J^{(+)}$:

$$J < 0; \quad k = -J, \quad m = 0, 1, 2, \ldots$$

E) Discrete class negative m, denoted by $D_J^{(-)}$:

$$J < 0; \quad k = J, \quad m = 0, -1, -2, \ldots$$

We now proceed to summarize the various realizations of the above UIR's.

i) UIR's of the class C_J^0 non-exceptional interval can be realized in a Hilbert space H of square integrable functions on the unit circle. Elements of H correspond to functions $f(\phi)$ of the real variable ϕ varying in the range $0 \le \phi < 2\pi$. The inner product of an element f with an element g is given by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \; f^*(\phi) \; g(\phi) \tag{A-15}$$

If h is an element of SU(1, 1) corresponding to the matrix (A-1) there corresponds a unitary operator U(h) such that

$$\left.\begin{aligned} \left[U(h)f\right](\phi) &= \left|\alpha^* - \beta e^{i\phi}\right|^{-1-2is} f\left(\Psi_h(\phi)\right) \\ e^{i\Psi_h(\phi)} &= \frac{\alpha e^{i\phi} - \beta^*}{\alpha^* - \beta e^{i\phi}} \quad 0 \le \phi, \; \Psi_h(\phi) < 2\pi \end{aligned}\right\} \tag{A-16}$$

The forms of the generators L_o, L_1, L_2 can be obtained by properly specializing the element h.

ii) UIR's of the class C_J^0 exceptional series may be constructed explicitely in Hilbert spaces H_σ consisting of a certain class of functions $f(\phi)$ on the unit circle $0 \leq \phi < 2\pi$ the scalar product of two elements f, g is given by

$$\left.\begin{aligned} (f,g)_\sigma &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \, f^*(\phi_1)\, L_\sigma(\phi_1-\phi_2) g(\phi_2) \\ L_\sigma(\phi_1-\phi_2) &= (2\pi)^{1/2} \frac{\Gamma(\sigma+1/2)}{2^\sigma \Gamma(\sigma)} \left[1-\cos(\phi_1-\phi_2)\right]^{\sigma-1/2} \end{aligned}\right\} \tag{A-17}$$

The unitary operator U(h) representing the element h corresponding to (A-1) acts as follows

$$\left[U(h)\, f\right](\phi) = \left|\alpha^* - \beta e^{i\phi}\right|^{-1-2\sigma} f\left[\Psi_h(\phi)\right]$$

$$e^{i\Psi_h(\phi)} = \frac{\alpha e^{i\phi} - \beta^*}{\alpha^* - \beta e^{i\phi}} \tag{A-18}$$

$$0 \leq \phi, \Psi_h(\phi) < 2\pi$$

Again, the forms of the generators L_o, L_1, L_2 can be obtained by a convenient choice of h.

iii) UIR's of the class $D_J^{(+)}$ $J = 1, 2, \ldots$ may be realized via unitary transformations in a Hilbert space H_J of analytic functions of a complex variable z. Elements of H_J correspond to functions f(z) which are analytic and free of singularities in the open unit circle $|z| < 1$.

The scalar product is defined as follows

$$(f, g)_J = \frac{-2J-1}{\pi} \oint d^2z \, (1 - |z|^2)^{-2J-2} \, f^*(z) g(z) \tag{A-19}$$

where the integration extends over the interior of the unit circle. The unitary operator U(h) corresponding to (A-1) acts as follows

$$\left[U(h)f\right](z) = (\alpha^* + i\beta z)^{2J} \, f\left(\frac{\alpha z - i\beta^*}{\alpha^* + i\beta z}\right) \tag{A-20}$$

from which the forms for the generators can be found by specializing h.

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[9] Equations (2.12) are equivalent to equations 2.6 and 2.7 of Ref. 6. Our V is related to the U of Fubini and Veneziano by

$$V(k_i z_i) = e^{\frac{k_i^2}{2\epsilon}} \; U_i(z_i)$$

[10] It was noted by Fairlie and Jones, Nucl. Phys. B15, 323 (1970) that the sum of the inequivalent permutations of the four point function could be written for $\alpha_o = 1$

$$A_{st} + A_{tu} + A_{us} = \int_{-\infty}^{+\infty} dx \, |x|^{-1-\alpha(s)} \, |1-x|^{-1-\alpha(t)}$$

The absolute values of x and 1-x arise naturally from the phase factors in equation (2.20).

[11] See in this respect a recent Johns Hopkins preprint by G. Domokos et al., Fully-Reggeized Scattering Amplitudes, to be published.

FIGURE CAPTIONS

Fig. 1a. Multiperipheral configuration for the N point function

Fig. 1b. Standard association of external particles with points on the unit circle. The amplitude can be factorized into N-3 integrations over the projections of $z_2, z_3, \ldots z_{n-2}$ onto the real axis.

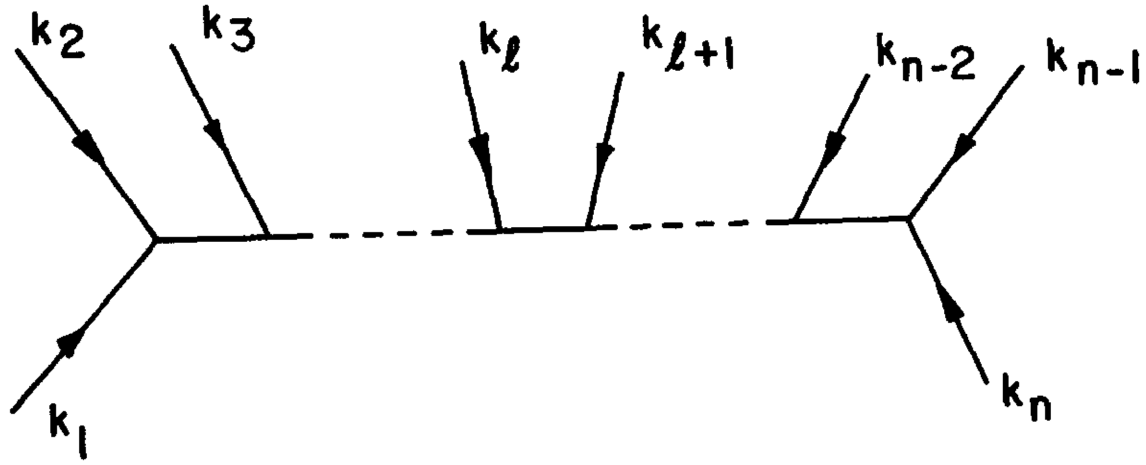

Fig. 1-a

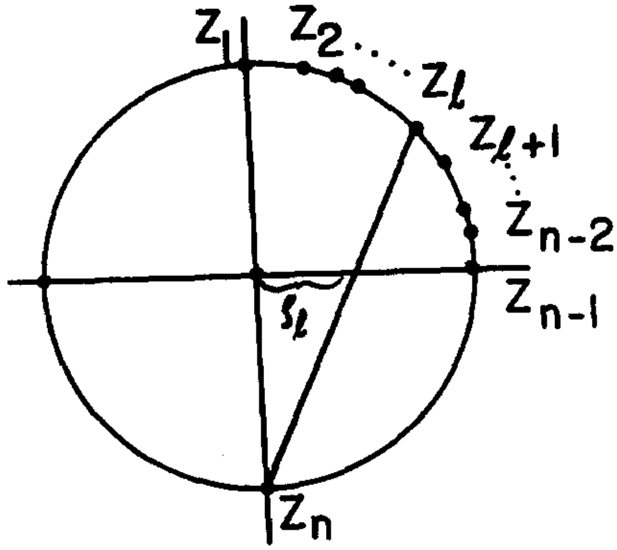

Fig. 1-b